Equicontinuity and sensitivity

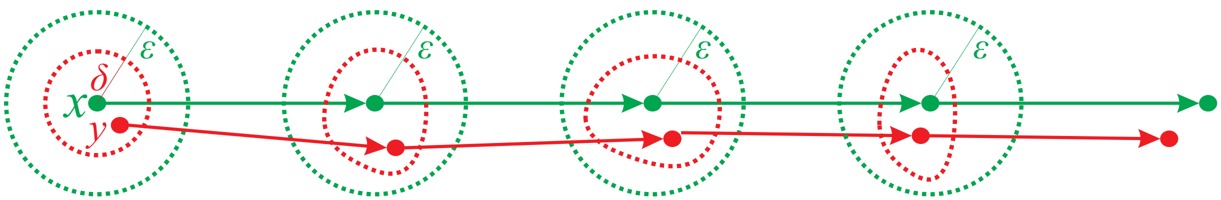
Equicontinuity points of a dynamical system are points whose orbits are well tracked by all other sufficiently close points. So the system is stable under small perturbations of equicontinuity points.

Generally: A family $\mathcal{F}$ of functions $f : X \longrightarrow Y$ between metric spaces X and Y is **equicontinuous** at $x \in X$ if the functions in $\mathcal{F}$ are continuous at x with with the same positive parameter $\delta > 0$ corresponding to any $\varepsilon > 0$:

$$(\forall \varepsilon > 0)\ (\exists \delta > 0)\ :\ d_X(x, y) < \delta \Longrightarrow (\forall f \in \mathcal{F})\ d_Y(f(x), f(y)) < \varepsilon.$$

For dynamical systems: A dynamical system $F : X \longrightarrow X$ is **equicontinuous** at $x \in X$ if the function family $\{F, F^2, F^3, \ldots\}$ is:

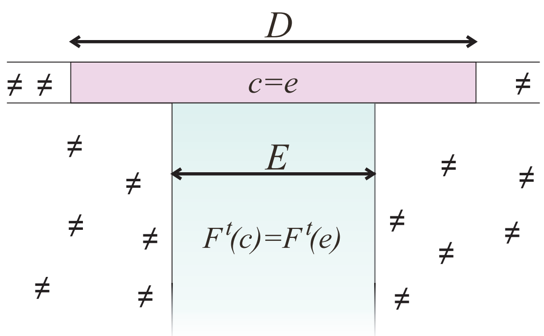
$$(\forall \varepsilon > 0)\ (\exists \delta > 0)\ :\ d(x, y) < \delta \Longrightarrow (\forall t \in \mathbb{N})\ d(F^t(x), F^t(y)) < \varepsilon.$$



For CA in terms of cylinders: A cellular automaton F is **equicontinuous** at configuration c (and c is an **equicontinuity point** of F) if:

For every finite $E \subseteq \mathbb{Z}^d$ there exists a finite $D \subseteq \mathbb{Z}^d$ such that for all e

$$e|_D = c|_D \Longrightarrow (\forall t \geq 0)\ F^t(e)|_E = F^t(c)|_E.$$



In other words: no matter how closely we want to track the orbit of c (in window E), we may choose any configuration e that is sufficiently close to c (identical in window D), and the orbit of e is guaranteed to track the orbit of c with the desired precision.

Let $\mathcal{E}_F$ (or simply $\mathcal{E}$) denote the set of equicontinuity points of F.

Example. Majority CA

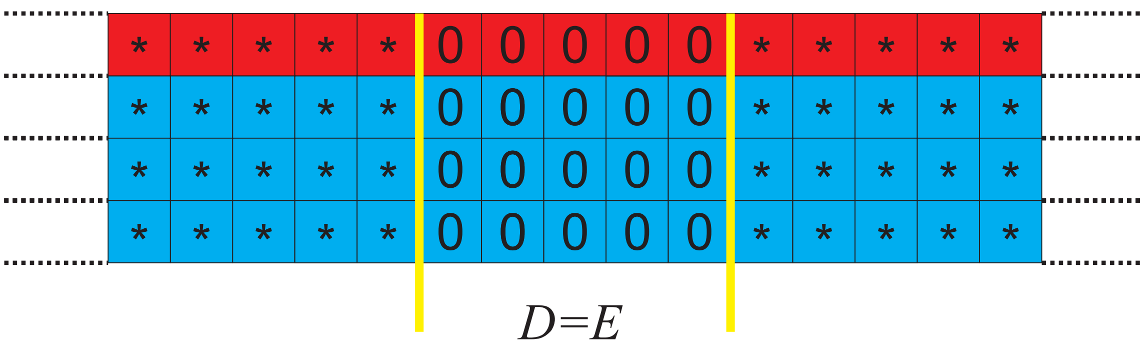
The majority CA is the elementary CA in which a cell changes to the state that is in the majority in its neighborhood (Wolfram number 232):

$$
\begin{array}{lcl}
111 & \longrightarrow & 1 \\
110 & \longrightarrow & 1 \\
101 & \longrightarrow & 1 \\
100 & \longrightarrow & 0 \\
011 & \longrightarrow & 1 \\
010 & \longrightarrow & 0 \\
001 & \longrightarrow & 0 \\
000 & \longrightarrow & 0
\end{array}
$$

Example. Majority CA

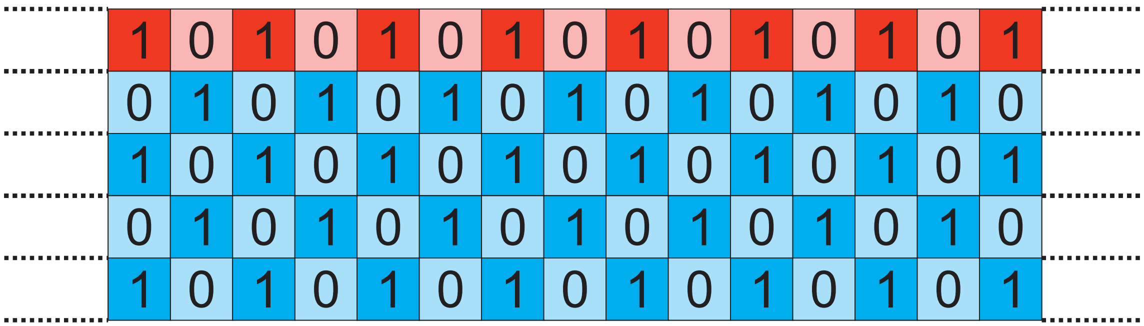
Configuration $c = \ldots 000 \ldots$ is an equicontinuity point: For any finite segment $E \subseteq \mathbb{Z}$ of length at least two we may choose $D = E$ and we have that

$$e|_D = c|_D \implies (\forall t \geq 0)\ F^t(e)|_E = F^t(c)|_E$$



Example. Majority CA

Configuration $c = \ldots 010101 \ldots$ is not an equicontinuity point. No matter what D you choose, flipping one bit outside of D will cause a perturbation that propagates everywhere:



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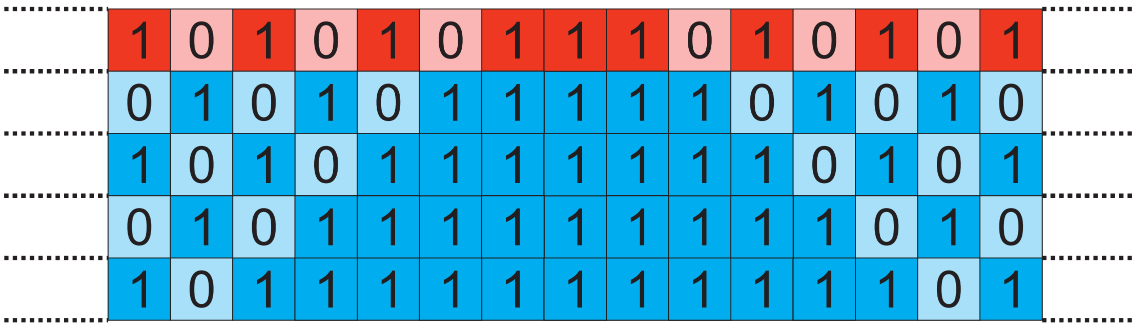

Equicontinuous CA

A CA F is **equicontinuous** if $\mathcal{E}_F = S^{\mathbb{Z}^d}$, i.e., all configurations are equicontinuity points.

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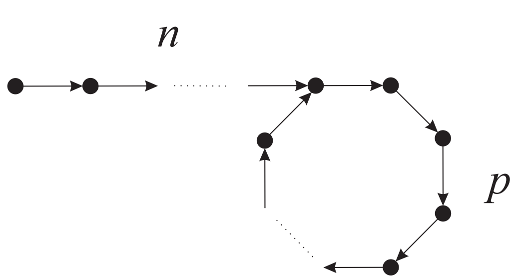
Recall that F is **periodic** if there exists a number $p \geq 1$ such that F^p is the identity function: $F^p(c) = c$ for all c. Every periodic CA is obviously reversible.

Recall that F **eventually periodic** if there exists number n (the pre-period) and $p \geq 1$ such that $F^{n+p} = F^n$.



Every periodic CA is also eventually periodic with pre-period $n = 0$.

Proposition. A CA F is equicontinuous if and only if F is eventually periodic.

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Proof. ($\Longleftarrow$)

If F is eventually periodic then for every $t \in \mathbb{N}$ the function F^t is the same as F^0 or F^1 or F^2 or $\ldots$ or F^{n+p}. Thus there is a finite set $N \subseteq \mathbb{Z}^d$ such that all F^t are defined by a local rule of neighborhood N.

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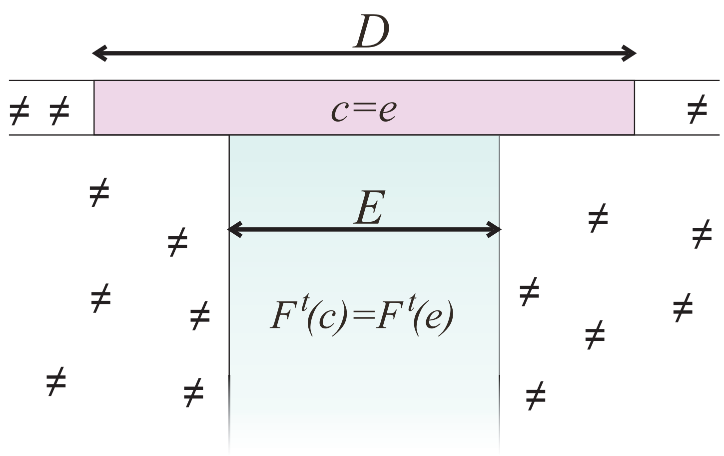
Thus equicontinuity follows: For any finite $E \subseteq \mathbb{Z}^d$ we may choose $D = E + N$, the set of all N-neighbors of cells in E. For all configurations c and all $t \in \mathbb{N}$ then $F^t(c)|_E$ is determined by $c|_D$.

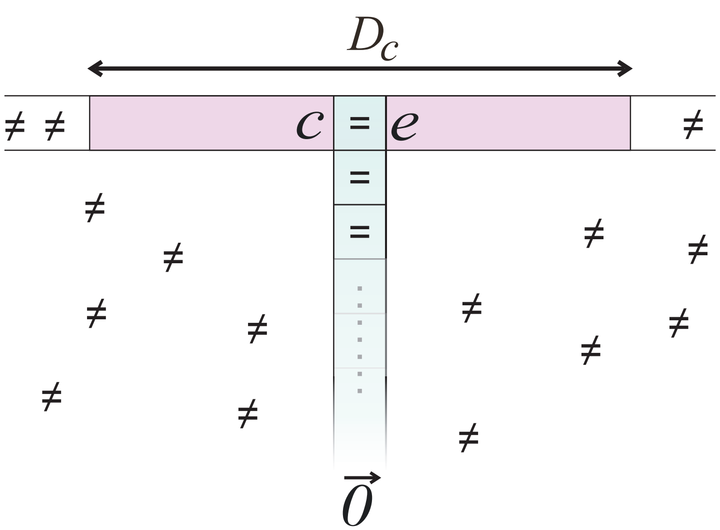


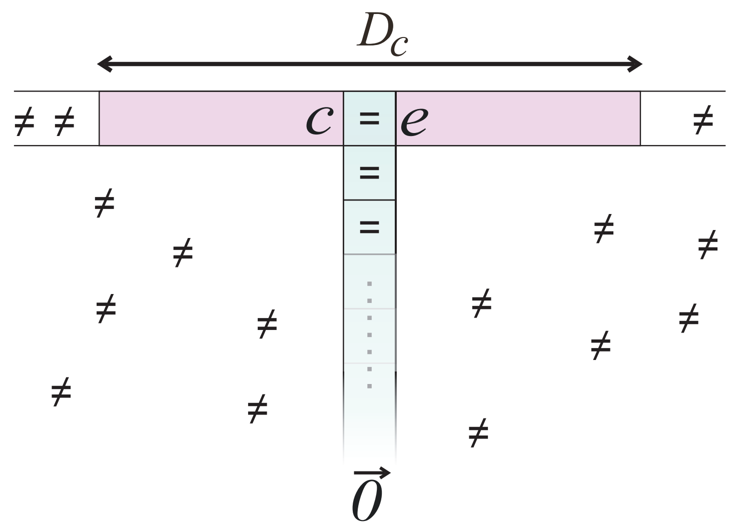
Proposition. A CA F is equicontinuous if and only if F is eventually periodic.

Proof. ($\Longrightarrow$)

Let F be an equicontinuous CA. Let $E = \{\vec{0}\}$, and for any $c \in S^{\mathbb{Z}^d}$ let finite $D_c \subseteq \mathbb{Z}^d$ be a set to satisfy the condition of equicontinuity for c and E:

$$e|_{D_c} = c|_{D_c} \implies (\forall t \geq 0)\ F^t(e)_{\vec{0}} = F^t(c)_{\vec{0}}.$$



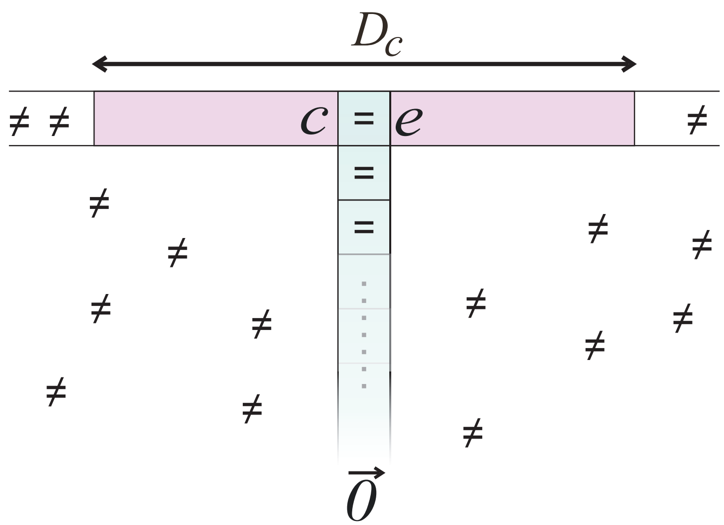


The cylinders

$$U_c = [c|_{D_c}]$$

over all configurations c cover the configuration space $S^{\mathbb{Z}^d}$ so that by compactness there is a finite subcover: configurations $c_1, \ldots, c_k$ such that

$$U_{c_1} \cup \cdots \cup U_{c_k} = S^{\mathbb{Z}^d}.$$



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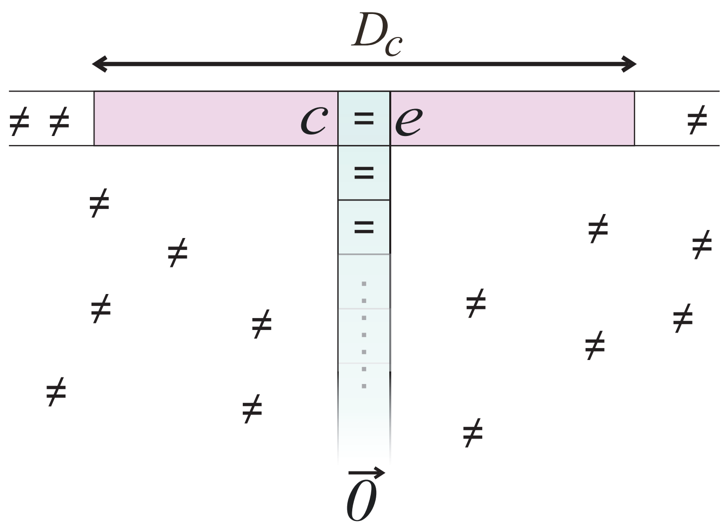
This means that the CA has at most k different temporal sequences (=**traces**)

$$\tau(c) = c_{\vec{0}}, F(c)_{\vec{0}}, F^2(c)_{\vec{0}}, \ldots$$

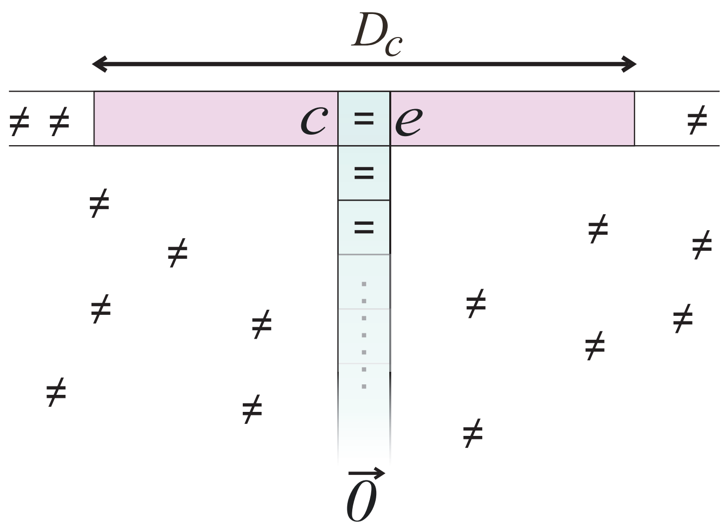
of states at cell $\vec{0}$: for every configuration c we have $\tau(c) = \tau(c_i)$ for some $i \in \{1, \ldots, k\}$.



But then every trace $\tau(c)$ is eventually periodic: As there are only finitely many different traces among

$$\begin{aligned} \tau(c) &= c_{\vec{0}},\ F(c)_{\vec{0}},\ F^2(c)_{\vec{0}},\ \ldots \\ \tau(F(c)) &= \phantom{c_{\vec{0}},\ } F(c)_{\vec{0}},\ F^2(c)_{\vec{0}},\ F^3(c)_{\vec{0}},\ \ldots \\ \tau(F^2(c)) &= \phantom{c_{\vec{0}},\ F(c)_{\vec{0}},\ } F^2(c)_{\vec{0}},\ F^3(c)_{\vec{0}},\ F^4(c)_{\vec{0}}, \ldots \\ &\vdots \end{aligned}$$

we must have $\tau(F^n(c)) = \tau(F^{n+p}(c))$, for some n and $p \geq 1$. So $\tau(c)$ is eventually periodic with pre-period n and period p.



A finite number of eventually periodic sequences has a **common pre-period** n and **period** $p \geq 1$: Choose n to be the maximum of the pre-periods of $\tau(c_i)$ and p to be a common multiple of the periods of $\tau(c_i)$.

Thus for every configuration c holds

$$F^{n+p}(c)_{\vec{0}} = F^n(c)_{\vec{0}}.$$

By translation invariance temporal sequences at other cells have the same pre-periods and periods also, so that $F^{n+p} = F^n$.

Corollary. A surjective CA is equicontinuous if and only if it is periodic.

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Proof.

It is enough to show that a surjective eventually periodic CA is periodic.

Let F be surjective and $F^{n+p} = F^n$. Let c be an arbitrary configuration. By surjectivity of F there exists e such that $F^n(e) = c$. But then

$$F^p(c) = F^{n+p}(e) = F^n(e) = c.$$

Thus F has period p.

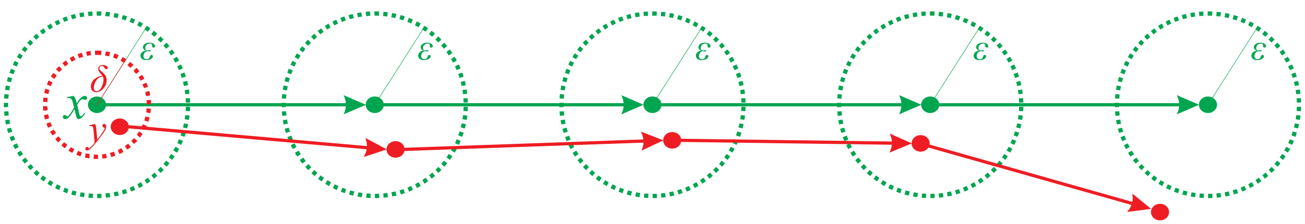
Sensitivity to initial conditions

Equicontinuous CA are very stable to changes in the initial configuration. In contrast, sensitive CA are very unstable on every initial configuration.

A dynamical system $F : X \longrightarrow X$ is **sensitive to initial conditions** (or simply **sensitive**) if

$$(\exists \varepsilon > 0)\ (\forall x \in X)\ (\forall \delta > 0)\ (\exists y \in B_\delta(x))\ (\exists t \geq 0)\ :\ d(F^t(c), F^t(e)) > \varepsilon.$$

(There is a positive number $\varepsilon > 0$, a **sensitivity constant**, such that for every point $x \in X$ there exist points y arbitrarily close to x whose trajectories diverge to distance $> \varepsilon$ from the trajectory of x.)

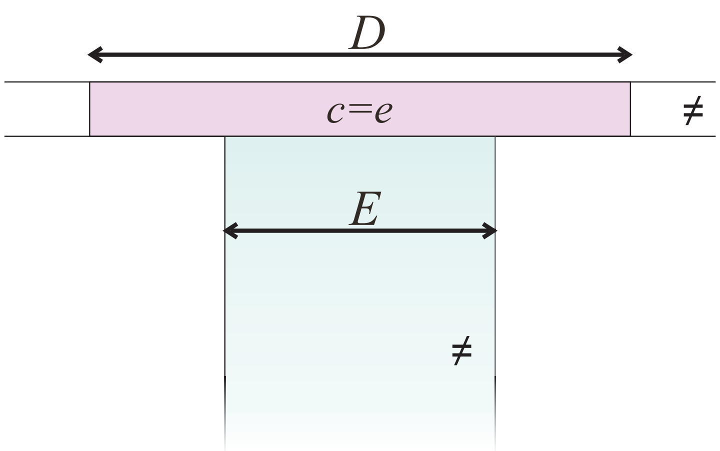


$$(\exists \varepsilon > 0)\ (\forall x \in X)\ (\forall \delta > 0)\ (\exists y \in B_\delta(x))\ (\exists t \geq 0)\ :\ d(F^t(c), F^t(e)) > \varepsilon.$$

For CA in terms of cylinders: A CA F is **sensitive** if there is a finite observation window $E \subseteq \mathbb{Z}^d$ such that any configuration may be changed arbitrarily far in such a way that the change propagates into E:

$$(\exists_{\text{finite}}\, E \subseteq \mathbb{Z}^d)\ (\forall c \in S^{\mathbb{Z}^d})\ (\forall_{\text{finite}}\, D \subseteq \mathbb{Z}^d)\ (\exists e \in S^{\mathbb{Z}^d})$$

$$e|_D = c|_D \text{ and } \exists t \geq 0 : F^t(e)|_E \neq F^t(c)|_E$$

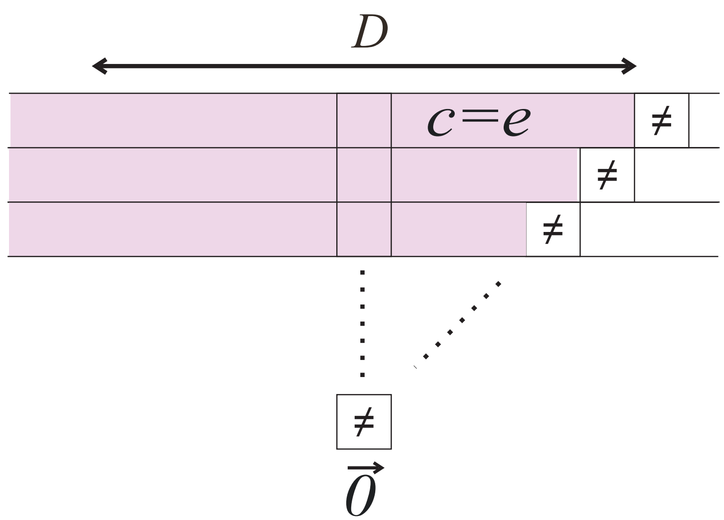


Set E is a **sensitivity constant** for F.

Example. XOR CA is sensitive

The single cell observation window $E = \{0\}$ works as the sensitivity constant.

For any configuration c and any finite $D \subseteq \mathbb{Z}$ flipping a single bit at a cell $n > 0$ that is to the right of D will cause a change that propagates to the left with speed one. Thus at time n the change will be seen inside E.



Thus XOR is sensitive.

A sensitive CA cannot have any equicontinuity points:

Proposition. If G is sensitive then $\mathcal{E}_G = \emptyset$.

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Proof.

G **sensitive:**

$$\begin{array}{l}
\exists \text{ finite } E \subseteq \mathbb{Z}^d \\
\quad \forall c \in S^{\mathbb{Z}^d} \\
\quad\quad \forall \text{ finite } D \subseteq \mathbb{Z}^d \\
\quad\quad\quad \exists e \in S^{\mathbb{Z}^d} \\
\quad\quad\quad\quad e|_D = c|_D \text{ and } \ \exists t \geq 0 : F^t(e)|_E \neq F^t(c)|_E
\end{array}$$

$\mathcal{E}_G = \emptyset$ **(no equicontinuity points):**

$$\neg \left(\begin{array}{l}
\exists c \in S^{\mathbb{Z}^d} \\
\quad \forall \text{ finite } E \subseteq \mathbb{Z}^d \\
\quad\quad \exists \text{ finite } D \subseteq \mathbb{Z}^d \\
\quad\quad\quad \forall e \in S^{\mathbb{Z}^d} \\
\quad\quad\quad\quad e|_D = c|_D \Longrightarrow (\forall t \geq 0) \ F^t(e)|_E = F^t(c)|_E.
\end{array} \right)$$

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Only the first two lines are in a different order. Clearly then sensitivity implies that there are no equicontinuity points. This holds in any dimension d.