Blocking words

Next we focus on **one-dimensional** CA. In this case we can prove more:

(1) The converse $(\mathcal{E}_G = \emptyset) \Longrightarrow G$ **sensitive** of the previous proposition holds.

(2) If $\mathcal{E}_G \neq \emptyset$ then $\mathcal{E}_G$ is residual (and hence dense).

These are based on **blocking words** that block the propagation of any changes to the initial configuration.

(An analogous, two-dimensional counter part "blocking rectangle" does not work the same: it does not split $\mathbb{Z}^2$ into disconnected areas, and so changes can circumvent the block.)

For any finite $E \subseteq \mathbb{Z}$ we say that a finite pattern $p \in S^D$ is **E-blocking** if

$$c, e \in [p] \implies \forall t \geq 0 \ : \ F^t(c)|_E = F^t(e)|_E.$$

(All configurations with pattern p in domain D have identical orbits in the observation window E.)

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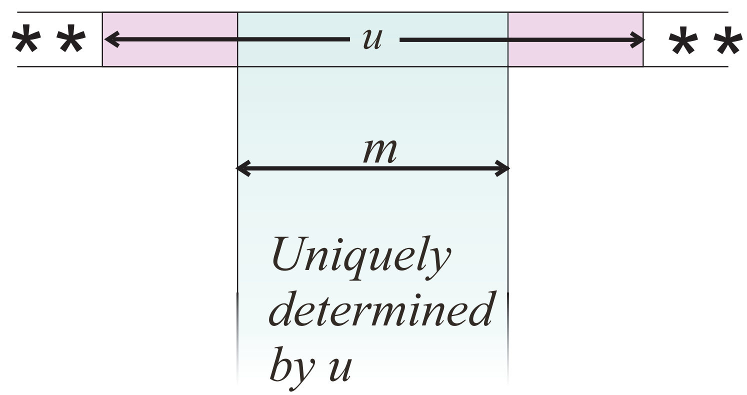
(All configurations with pattern p in domain D have identical orbits in the observation window E.)

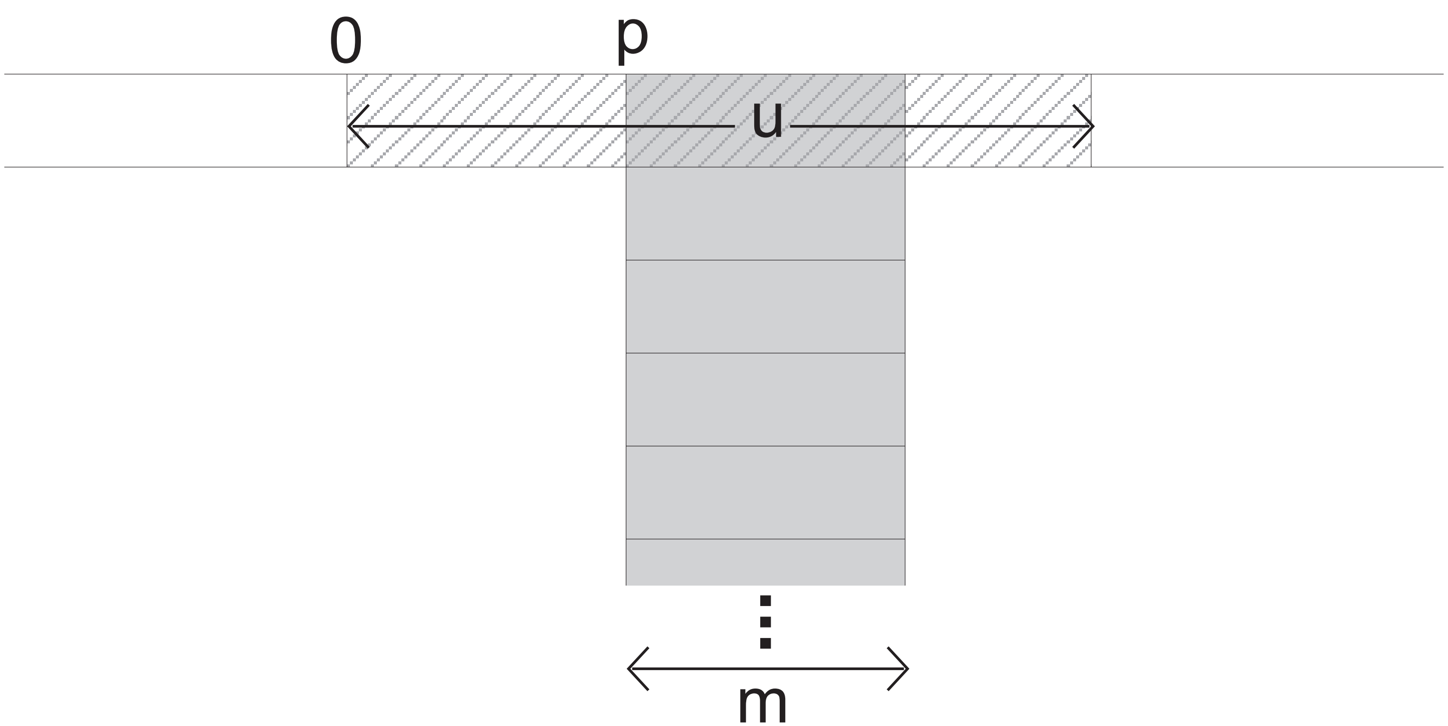
We focus on the cases where E and D are contiguous segments of cells: A word $u \in S^*$ is an **m-blocking word** for $m \in \mathbb{N}$ if u (viewed as a pattern on some contiguous segment) is E-blocking for some contiguous segment E of m cells:



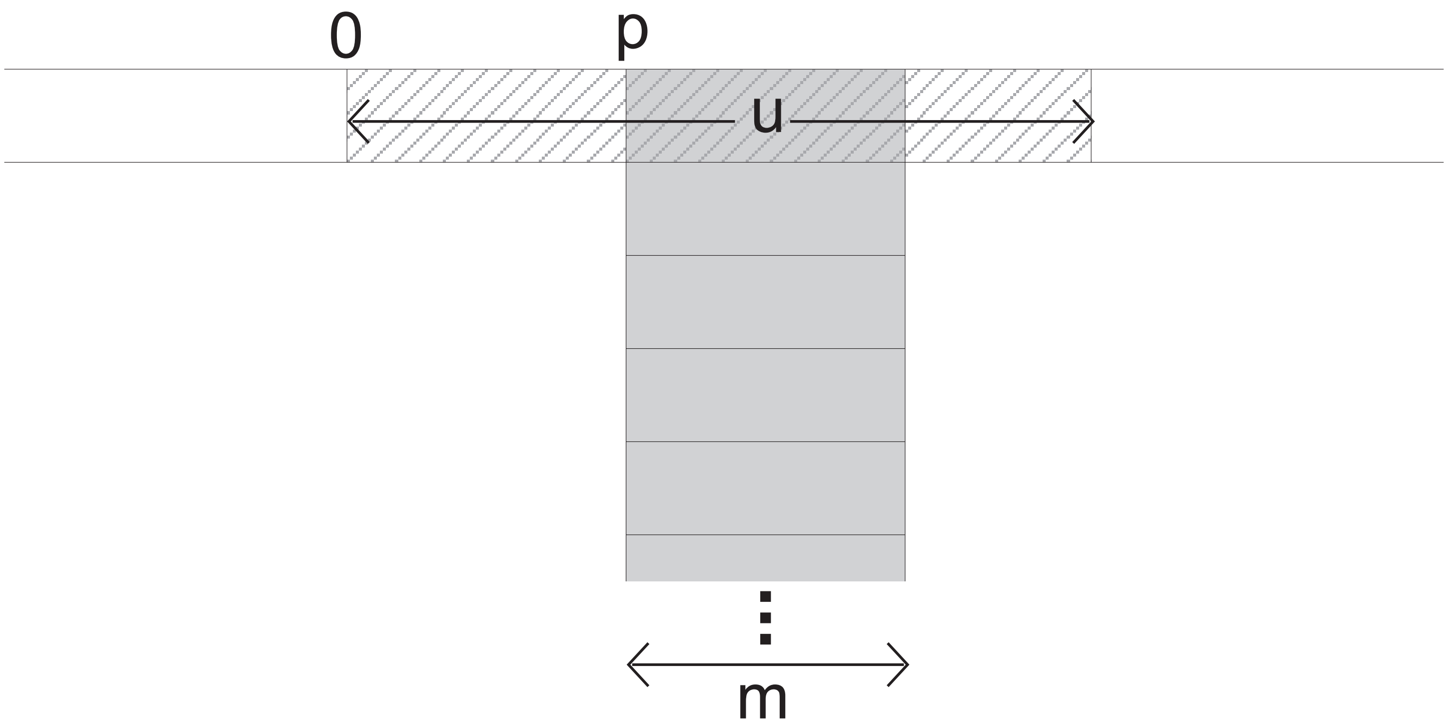
Precisely: Word u is m-blocking if there exists an offset $p \in \mathbb{Z}$ such that

$$(c|_D = u = e|_D) \implies (\forall t \geq 0)\ F^t(c)|_E = F^t(e)|_E,$$

where $D = \{0, \ldots, |u| - 1\}$ and $E = \{p, \ldots, p + m - 1\}$.



(The states in the gray area are independent of the states outside of the striped word u.)



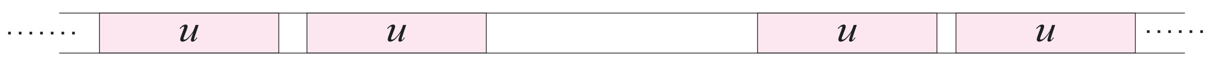
Remarks:

- $0 \leq p \leq |u| - m$
- If m-blocking word u is a subword of v then also v is m-blocking.
- If u is m-blocking then it is also k-blocking for all $k \leq m$.

Example. In the majority CA (elementary CA number 232) words 11 and 00 are both 2-blocking.

We are mostly interested in r-blocking words where r is a **neighborhood radius** of the CA. Such words prevent the propagation of any change from one side of the blocking word to the other side.

Lemma. Let F be a one-dimensional CA with neighborhood radius r. Suppose F has an r-blocking word u. Let c be any configuration that contains infinitely many copies of u to the right and to the left:



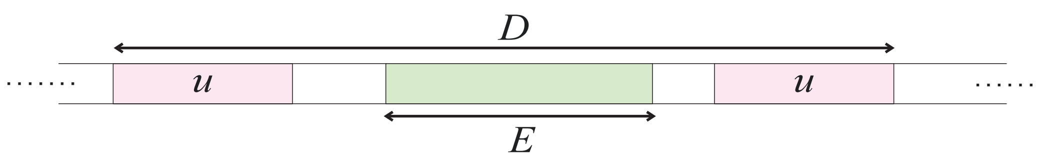
Such c is an equicontinuity point.

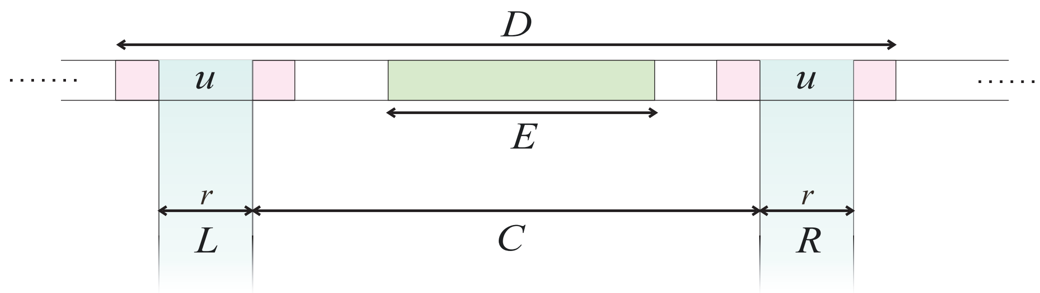
Proof.

Claim. If u is r-blocking then the configuration c

is an equicontinuity point.

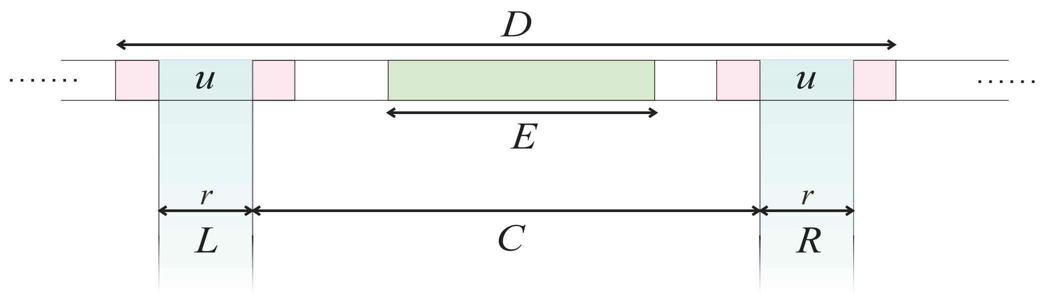
Proof. For any finite $E \subseteq \mathbb{Z}$ there is an occurrence of u completely on the left of E and an occurrence of u completely on the right of E. Let $D \subseteq \mathbb{Z}$ be a segment that begins with the left u and ends with the right u:





Consider any configuration e such that $e|_D = c|_D$. Because u is r-blocking there are segments L and R of length r completely on the left and on the right of E such that for all $t \geq 0$

$$F^t(e)|_L = F^t(c)|_L \text{ and } F^t(e)|_R = F^t(c)|_R.$$



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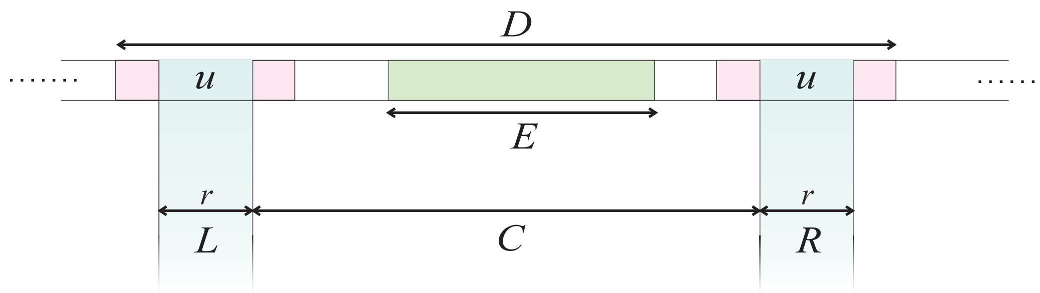
Let C be the segment between L and R. Because the local rule of the CA has radius r, we have for all $t \geq 0$ the implication

$$F^t(e)|_C = F^t(c)|_C \implies F^{t+1}(e)|_C = F^{t+1}(c)|_C.$$



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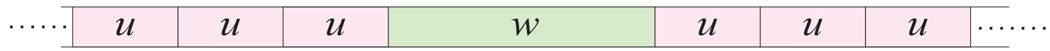
Because $F^0(e)|_C = F^0(c)|_C$, we see by mathematical induction on t that $F^t(e)|_C = F^t(c)|_C$ holds for all $t \geq 0$. As $E \subseteq C$ it follows that $F^t(e)|_E = F^t(c)|_E$ for all t, as needed.

Remark. If u is an r-blocking word then for any word w we have that the configuration

$$\ldots u \; u \; w \; u \; u \;\; \ldots$$

is an equicontinuity point. Thus the set of equicontinuity points is **dense**.



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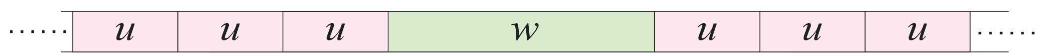

In the following we prove more: the set of equicontinuity points is **residual**.

Proposition. Let F be a one-dimensional CA, defined by a radius-r local rule. Then exactly one of the following two alternatives holds:

(1) $\mathcal{E}_F$ is residual. This is equivalent to the existence of an r-blocking word.

(2) $\mathcal{E}_F = \emptyset$. This is equivalent to sensitivity of F.

Proof.

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Proof. The following four implications are enough to prove the proposition:

$\exists$ r-blocking word

$\Longrightarrow \mathcal{E}_F$ residual

$\Longrightarrow \mathcal{E}_F \neq \emptyset$

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$\exists$ r**-blocking word** $\Longrightarrow \mathcal{E}_F$ **residual**

$\exists$ r-blocking word $\implies \mathcal{E}_F$ residual

For any finite $E \subseteq \mathbb{Z}$ denote

$U_E = \{c \in S^{\mathbb{Z}} \mid \exists_{\text{finite}} D \subseteq \mathbb{Z}$ such that $e|_D = c|_D \implies (\forall t \geq 0) F^t(e)|_E = F^t(c)|_E \}$

for the set of points that satisfy the equicontinuity condition w.r.t the observation window E.

- Equicontinuity points are those that belong to all U_E:

$$\mathcal{E}_F = \bigcap_{E \subseteq \mathbb{Z} \text{ finite}} U_E.$$

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$\exists$ r-blocking word $\implies \mathcal{E}_F$ residual

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- Since there is an r-blocking word, the set $\mathcal{E}_F$ is dense. As $\mathcal{E}_F \subseteq U_E$ the set U_E is also dense.

Conclusion: $\mathcal{E}_F$ is residual as a countable intersection of open dense sets U_E.

Proof. The following four implications are enough to prove the proposition:

$$\exists\ r\text{-blocking word}$$

$$\Longrightarrow \mathcal{E}_{\mathbf{F}} \text{ residual}$$

$$\Longrightarrow \mathcal{E}_F \neq \emptyset$$

$$\Longrightarrow F \text{ is not sensitive}$$

$$\Longrightarrow \exists\ r\text{-blocking word}$$

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$$\Longrightarrow \exists\ r\text{-blocking word}$$

The second implication is immediate from the **Baire theorem**: all residual sets are dense (hence non-empty).

Proof. The following four implications are enough to prove the proposition:

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$$\Longrightarrow \mathcal{E}_{\mathbf{F}} \neq \emptyset$$

$$\Longrightarrow \mathbf{F} \text{ is not sensitive}$$

$$\Longrightarrow \exists\ r\text{-blocking word}$$

The third implication is also known since we proved that

$$F \text{ sensitive} \qquad \Longrightarrow \qquad \mathcal{E}_F = \emptyset.$$

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F **is not sensitive** $\Longrightarrow \exists$ r**-blocking word**

F is not sensitive $\implies \exists\ r$-blocking word

Let $E \subseteq \mathbb{Z}$ be a segment of r consecutive cells.

Because F is not sensitive, set E does not work as a sensitivity constant. So there exists $c \in S^{\mathbb{Z}}$ and $D \subseteq \mathbb{Z}$ such that no change in c outside of D can propagate into E:

$$e|_D = c|_D \implies (\forall t \geq 0)\ F^t(e)|_E = F^t(c)|_E.$$

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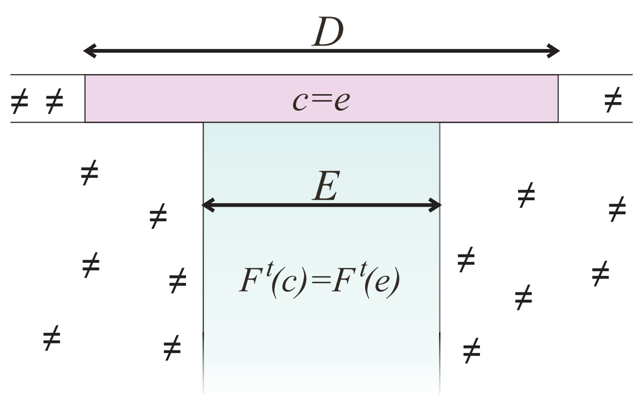
$$e|_D = c|_D \Longrightarrow (\forall t \geq 0)\ F^t(e)|_E = F^t(c)|_E.$$

The property remains if we add any additional cells in D, so we may assume that D is a contiguous segment and hence $u = c|_D$ is a word.



The word u is r-blocking.

This complete the proof of our proposition (stated here again):

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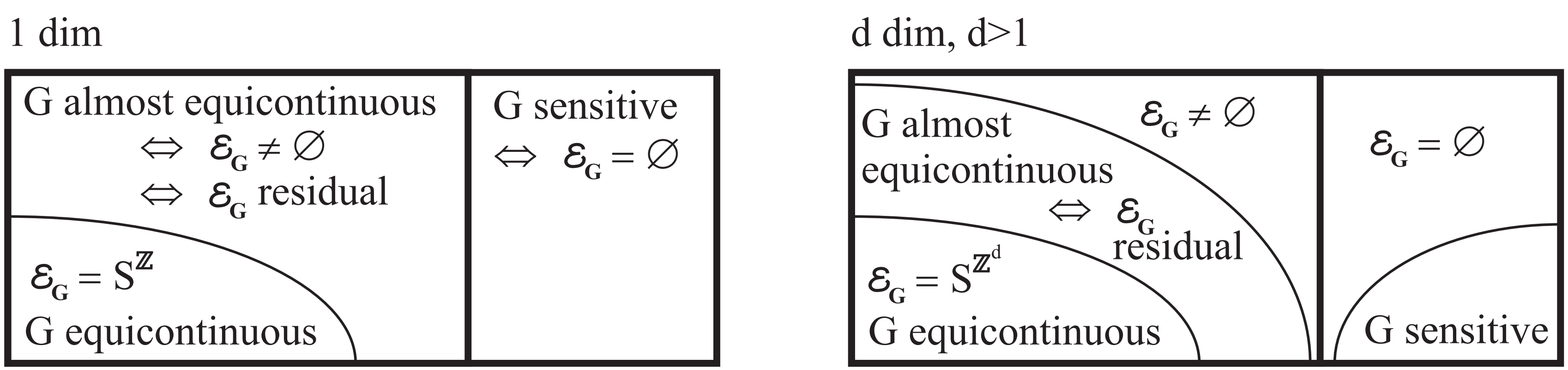
Remark on higher dimensions: In the two-dimensional case:

- there are two-dimensional CA G such that $\mathcal{E}_G \neq \emptyset$ but $\mathcal{E}_G$ is not dense,
- there are two-dimensional non-sensitive CA G such that $\mathcal{E}_G = \emptyset$.

The following picture illustrates the difference in one- and higher dimensional cellular automata:



If $\mathcal{E}_G$ is residual then the CA G is called **almost equicontinuous**. A one-dimensional CA is either sensitive or almost equicontinuous, but not both.

Remark on residual sets: Residual sets are bigger than just dense. They are "fat" in a similar sense as full measure sets are in measure theory.

For example, (countable) intersections of residual sets are residual (direct from the definition), and therefore still dense. In measure theory, countable intersections of full measure sets have full measure. In contrast, an intersection of just two dense sets may be empty (think of $\mathbb{Q}$ and $\mathbb{R} \setminus \mathbb{Q}$ inside $\mathbb{R}$.)

Expansivity

Expansivity is a very **strong form of instability**, in which all changes to the initial configuration will eventually propagate to an observation window.

A dynamical system $F : X \longrightarrow X$ is **(positively) expansive** if there exists $\varepsilon > 0$, an **expansivity constant**, such that

$$x \neq y \qquad \Longrightarrow \qquad (\exists t \geq 0)\ d(F^t(x), F^t(y)) > \varepsilon.$$

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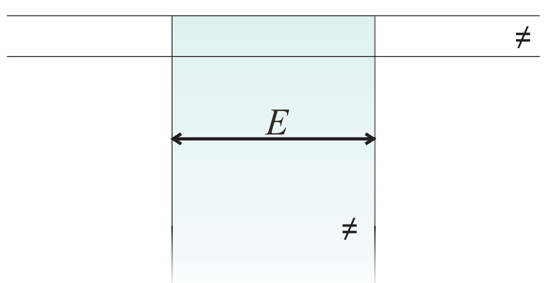
For CA in terms of cylinders: A CA F is **positively expansive** if there is a finite observation window $E \subseteq \mathbb{Z}^d$ such that for any $c \neq e$ there exists $t \geq 0$ such that

$$F^t(c)|_E \neq F^t(e)|_E.$$

We call E an **expansivity constant** of F.



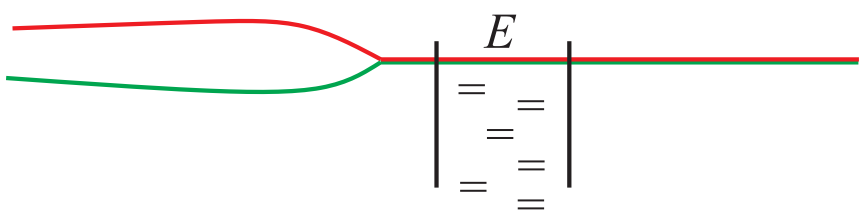
The condition is clearly stronger than sensitivity: Every positively expansive CA is sensitive (with the same window $E \subseteq \mathbb{Z}^d$) because there are no isolated points in $S^{\mathbb{Z}^d}$.

Indeed, for every configuration c and finite $D \subseteq \mathbb{Z}^d$ there exists $e \neq c$ such that $e|_D = c|_D$. By positive expansivity, there is $t \geq 0$ such that

$$F^t(e)|_E \neq F^t(c)|_E.$$

Example. XOR is not positively expansive.

In the XOR CA differences propagate only to the left. For any finite $E \subseteq \mathbb{Z}$ we can thus consider two configurations c, e that only differ in cells to the left of E.



For all $t \geq 0$ then $F^t(c)|_E = F^t(e)|_E$. Thus XOR is not positively expansive.

Example. Rule 150 is positively expansive.

In contrast, the three element XOR (elementary CA 150) with the local rule

$$f(a, b, c) = a + b + c \pmod 2$$

is positively expansive. We can take $E = \{0, 1\}$ as the expansivity constant.

Let $c \neq e$ be two different configurations. If $c|_E \neq e|_E$, we are done.

Assume then that $c|_E = e|_E$. The first difference to the left of E propagates one cell to the right at each step, and the first difference to the right of E propagates one cell to the left at each step. So a difference will be eventually seen inside E.

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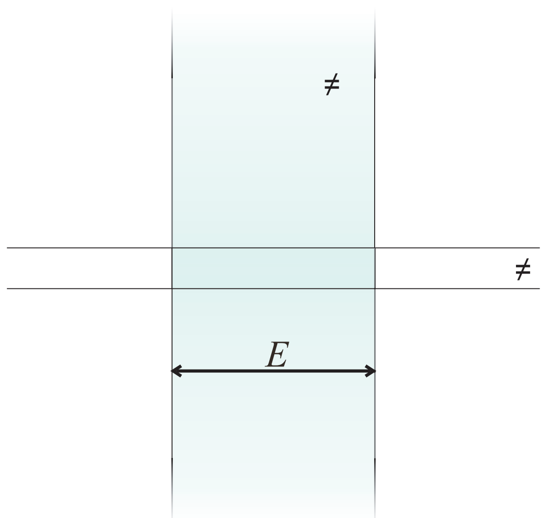
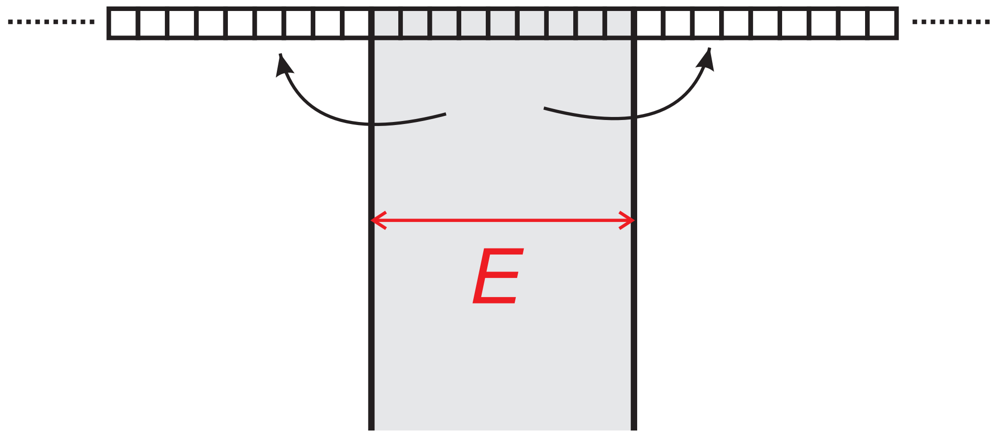
Remark. A size one observation window $E = \{0\}$ does not work. (Think of the two configurations $\ldots 00000 \ldots$ and $\ldots 011\ 011\ 011 \ldots$ that both have state 0 in E.)

If we allow in the definition of positive expansivity the time t take also **negative values** we obtain the definition of expansivity. This concept is only defined for reversible CA.

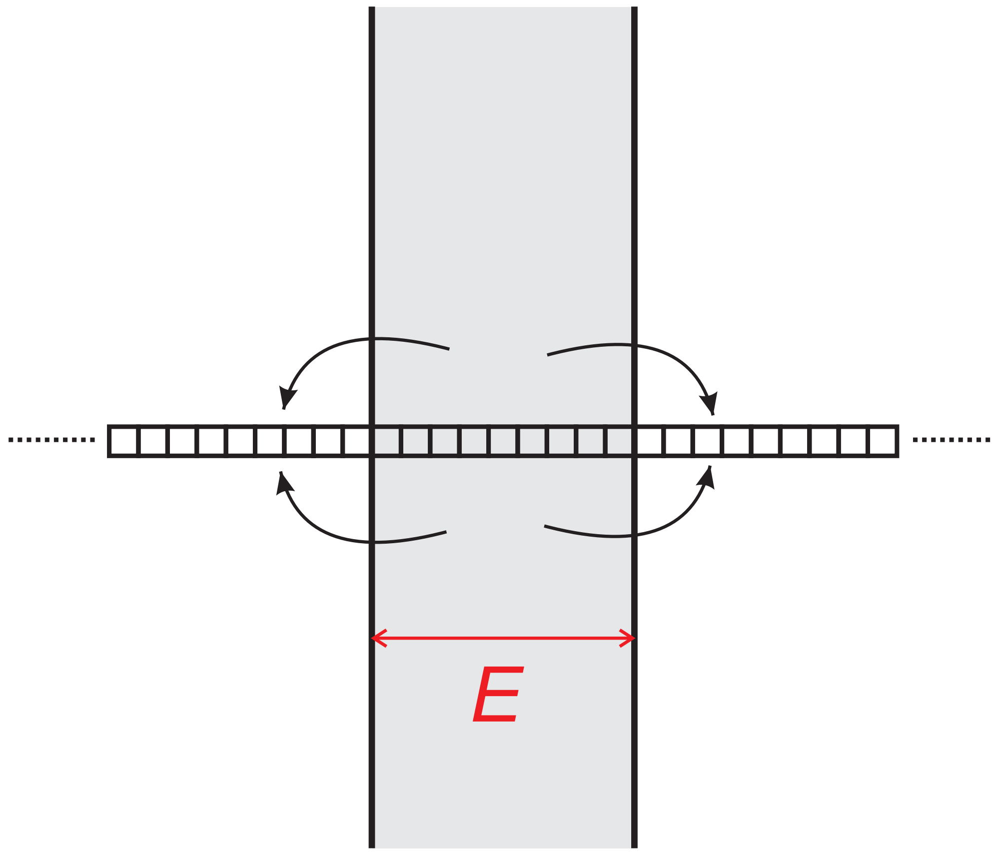
A reversible CA is called **expansive** if there exists a finite $E \subseteq \mathbb{Z}^d$ such that for all $c \neq e$ there exists an integer $t \in \mathbb{Z}$ (which may be also negative) such that $G^t(e)|_E \neq G^t(c)|_E$.



Again: In positively expansive CA a positive half column of finite width in the space-time diagram uniquely determines the initial configuration:



In expansive CA a two-sided column of finite width in the space-time diagram uniquely determines the initial configuration:



Example. In the one-dimensional case, any **non-zero translation** is expansive but not positively expansive.

But not in the higher dimensional cases: No higher dimensional translation is expansive since for any finite observation window the difference of two configurations may shift past the window, missing the observer.

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But not in the higher dimensional cases: No higher dimensional translation is expansive since for any finite observation window the difference of two configurations may shift past the window, missing the observer.

In fact (as proved soon):

- There are no expansive or positively expansive cellular automata in dimensions > 1.
- No reversible CA is positively expansive (so no CA is both expansive and positively expansive).

Proposition. Expansive and positively expansive CA are sensitive.

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Proof. Let G be (positively) expansive CA with constant $E \subseteq \mathbb{Z}^d$.

To prove sensitivity, let $D \subseteq \mathbb{Z}^d$ and $c \in S^{\mathbb{Z}^d}$ be arbitrary. Let $e \neq c$ be such that $e|_D = c|_D$. Such an e exists because c is not isolated.

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(1) If G is **positively expansive** then there exists $t \geq 0$ such that

$$G^t(e)|_E \neq G^t(c)|_E,$$

confirming sensitivity with constant E. (This was already explained on a previous slide.)

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(2) If G is **reversible and expansive** then there exists $t \in \mathbb{Z}$ such that $G^t(e)|_E \neq G^t(c)|_E$. If $t \geq 0$ we are done, so **suppose** $t < 0$.

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In reversible CA periodic points are dense, so there is a periodic configuration c' satisfying $c'|_D = c|_D$ and $G^t(c')|_E = G^t(c)|_E$.

Indeed: the intersection

$$[c|_D] \cap G^{-t}([G^t(c)|_E])$$

is open and non-empty (contains c)

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To prove sensitivity, let $D \subseteq \mathbb{Z}^d$ and $c \in S^{\mathbb{Z}^d}$ be arbitrary. Let $e \neq c$ be such that $e|_D = c|_D$. Such an e exists because c is not isolated.

(2) If G is **reversible and expansive** then there exists $t \in \mathbb{Z}$ such that $G^t(e)|_E \neq G^t(c)|_E$. If $t \geq 0$ we are done, so **suppose** $t < 0$.

In reversible CA periodic points are dense, so there is a periodic configuration c' satisfying $c'|_D = c|_D$ and $G^t(c')|_E = G^t(c)|_E$.

Similarly, there is a periodic e' with $e'|_D = e|_D = c|_D$ and $G^t(e')|_E = G^t(e)|_E$.

Proposition. Expansive and positively expansive CA are sensitive.

Proof. Let G be (positively) expansive CA with constant $E \subseteq \mathbb{Z}^d$.

To prove sensitivity, let $D \subseteq \mathbb{Z}^d$ and $c \in S^{\mathbb{Z}^d}$ be arbitrary. Let $e \neq c$ be such that $e|_D = c|_D$. Such an e exists because c is not isolated.

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In reversible CA periodic points are dense, so there is a periodic configuration c' satisfying $c'|_D = c|_D$ and $G^t(c')|_E = G^t(c)|_E$.

Similarly, there is a periodic e' with $e'|_D = e|_D = c|_D$ and $G^t(e')|_E = G^t(e)|_E$.

Now c' and e' have a **common temporal period** $p > 0$. For a suitably large $k \in \mathbb{Z}$ we have $t + kp > 0$. Then

$$G^{t+kp}(e')|_E = G^t(e')|_E = G^t(e)|_E \neq G^t(c)|_E = G^t(c')|_E = G^{t+kp}(c')|_E.$$

This means that either $G^{t+kp}(e')|_E \neq G^{t+kp}(c)|_E$ or $G^{t+kp}(c')|_E \neq G^{t+kp}(c)|_E$. So either c' or e' confirms sensitivity. (Recall that $e'|_D = c'|_D = c|_D$.)