Proposition. A positively expansive CA F is surjective.

Proof.

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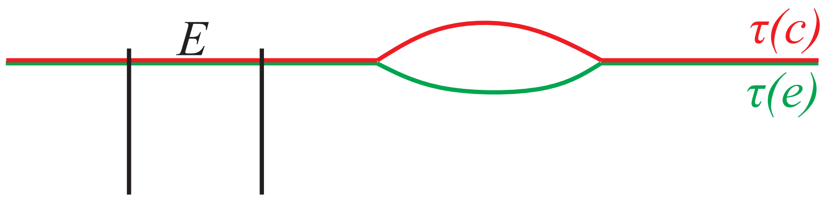
Proof. If F is not surjective then it is not pre-injective, so there are asymptotic $c \neq e$ such that $F(c) = F(e)$.

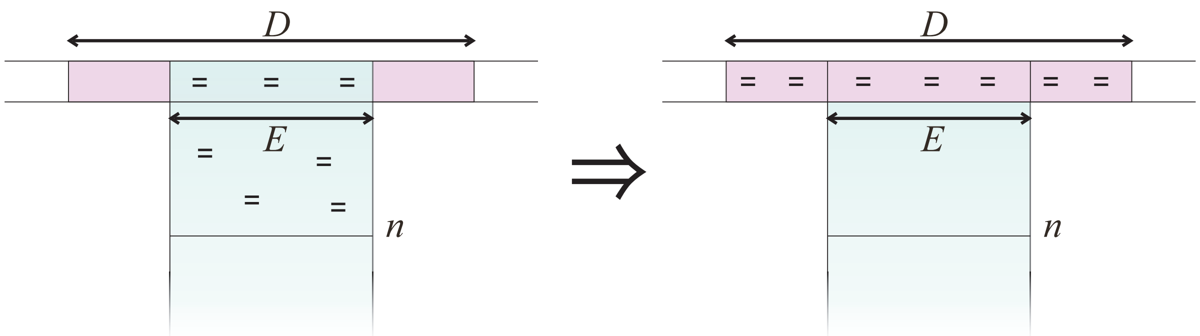
Consider any finite observation window E. For some translation τ we have that $\tau(c)|_E = \tau(e)|_E$. After the first time step configurations $\tau(c)$ and $\tau(e)$ become identical so the difference in them is never seen inside window E. Thus F is not positively expansive.



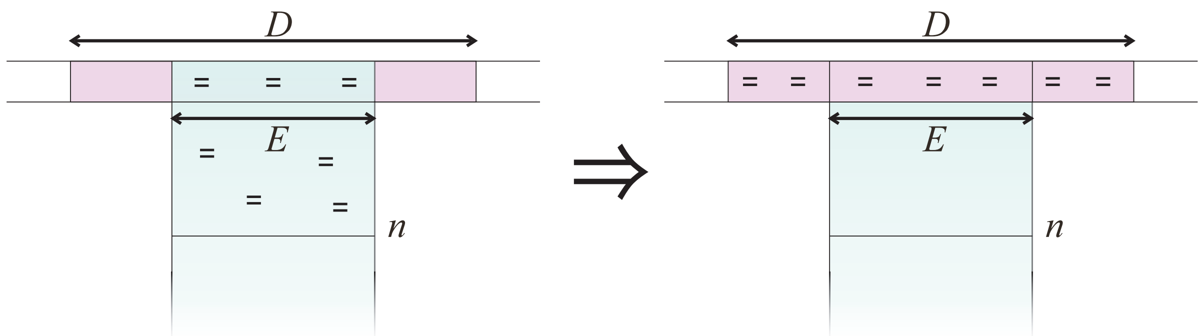
Compactness gives the following:

Lemma. Let F be a positively expansive CA with an expansivity constant $E \subseteq \mathbb{Z}^d$. For every finite $D \subseteq \mathbb{Z}^d$ there exists $n \geq 0$ such that for any configurations c, e the following holds:

$$(\forall t \leq n) F^t(c)|_E = F^t(e)|_E \quad \Longrightarrow \quad c|_D = e|_D.$$



Proof.



Proof. Suppose, on the contrary to the claim that for every n there are c_n and e_n such that for all $t \leq n$ holds $F^t(c_n)|_E = F^t(e_n)|_E$ but $c_n|_D \neq e_n|_D$.

By compactness there is a sequence of indices $i_1 < i_2 < \ldots$ such that

$$\begin{aligned} c &= \lim_{k\to\infty} c_{i_k}, \text{ and} \\ e &= \lim_{k\to\infty} e_{i_k} \end{aligned}$$

exist. Now clearly $c|_D \neq e|_D$ but $F^t(c)|_E = F^t(e)|_E$ for all $t \geq 0$, contradicting positive expansiveness.

Proposition. There do not exist two-or higher dimensional positively expansive CA.

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Proof. Let $D_i \subseteq \mathbb{Z}^d$ denote the hypercube of size

$$(2i+1) \times (2i+1) \times \cdots \times (2i+1)$$

centered at the origin.

Let F be a d-dimensional positively expansive CA, and let $E \subseteq \mathbb{Z}^d$ be its expansivity constant. We may assume that $E = D_k$ for some k. Let $D = D_{k+1}$, and let n be the number from the lemma so that we have the implication

$$(\forall t \leq n)\ F^t(c)|_{D_k} = F^t(e)|_{D_k} \quad \Longrightarrow \quad c|_{D_{k+1}} = e|_{D_{k+1}}.$$

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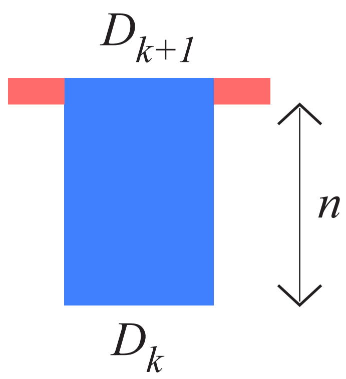
A "birthday cake" construction gives (by induction on j) that for every $j \geq 0$

$$(\forall t \leq jn)\ F^t(c)|_{D_k} = F^t(e)|_{D_k} \quad \Longrightarrow \quad c|_{D_{k+j}} = e|_{D_{k+j}}.$$

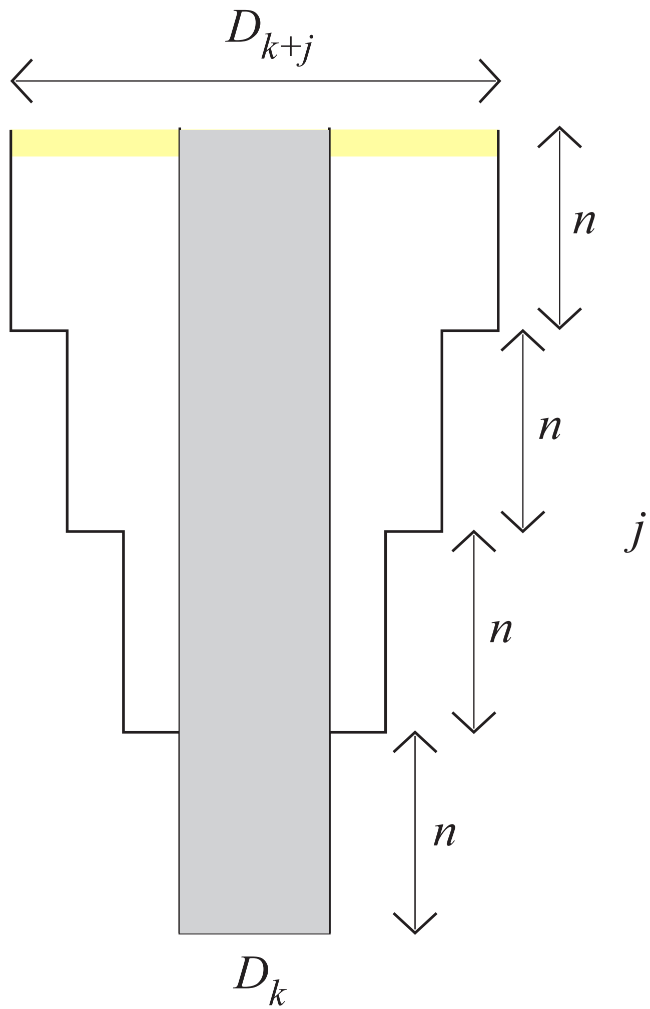
(Knowing the temporal sequence in window D_k for another n steps grows by one the radius of a uniquely identified region of the initial configuration.)

The **birthday cake** more precisely:

In space-time diagrams, the blue states uniquely identify the red states:



And we want to show that the grey states uniquely identify the yellow ones:



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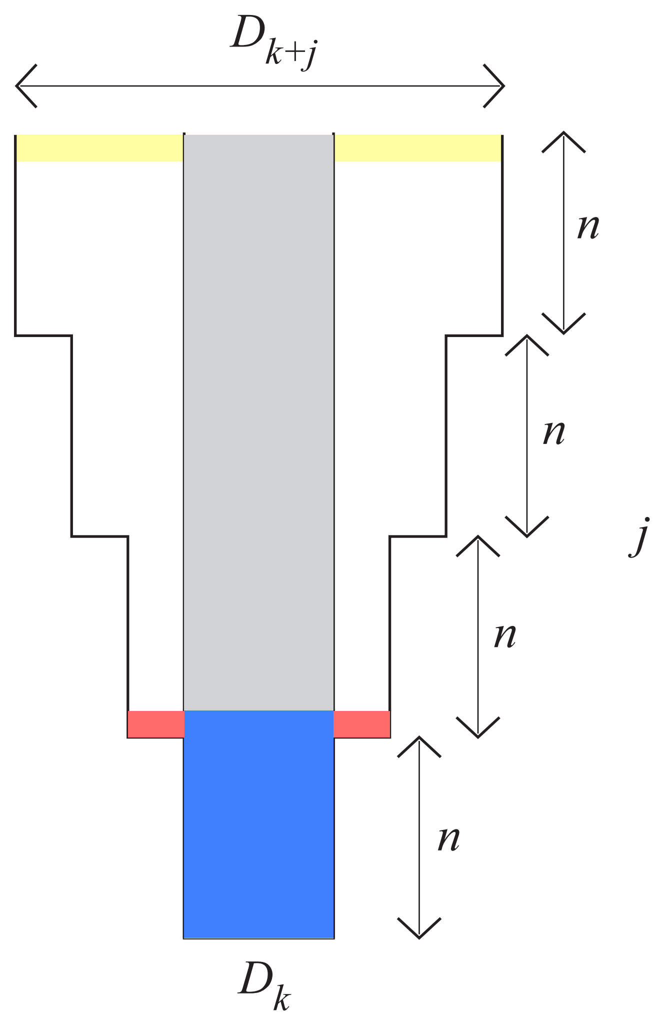

Knowing the states in D_k for times $t \leq jn$ uniquely determine the states in D_{k+1} for $t \leq (j-1)n$.

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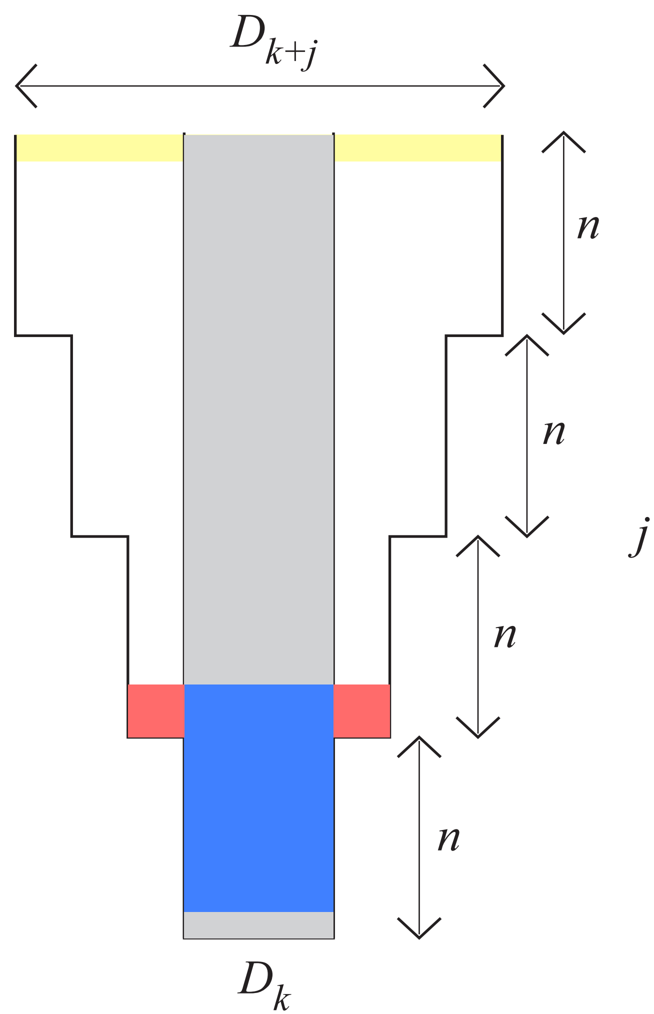

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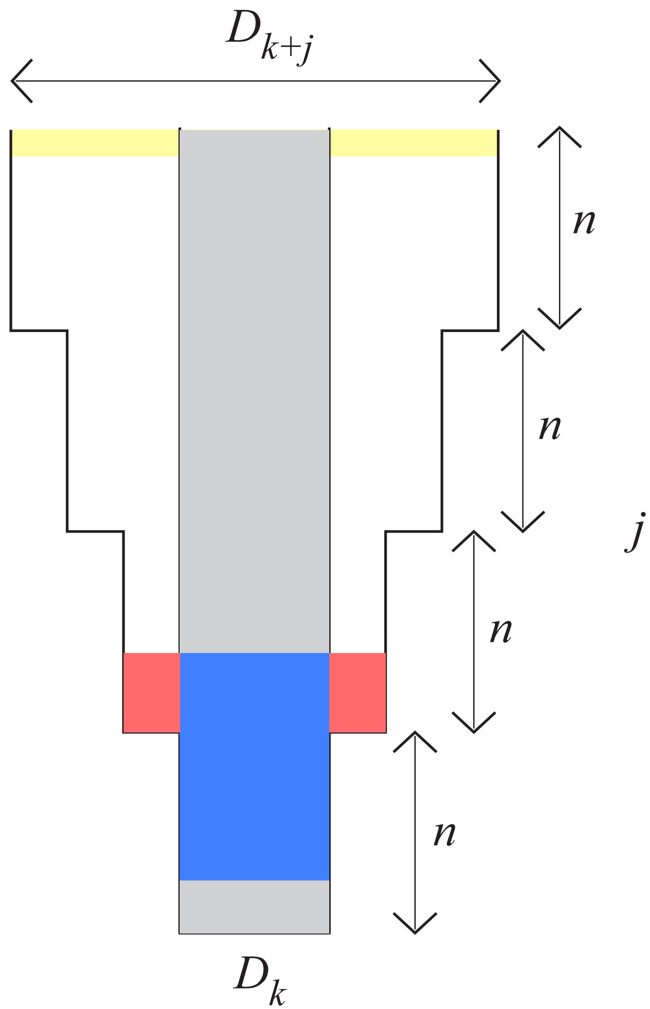

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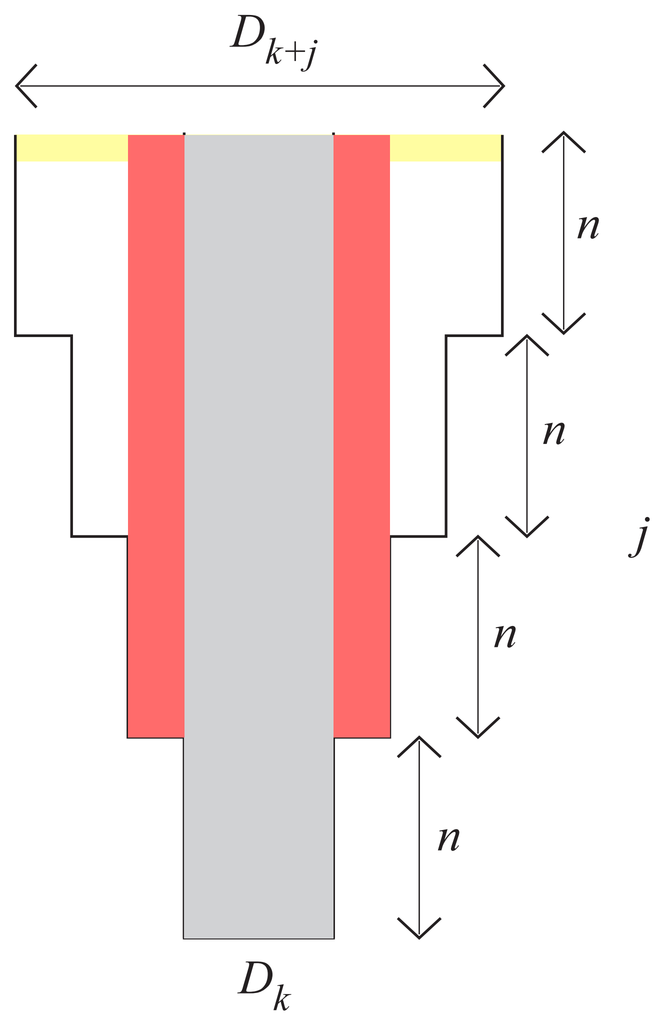
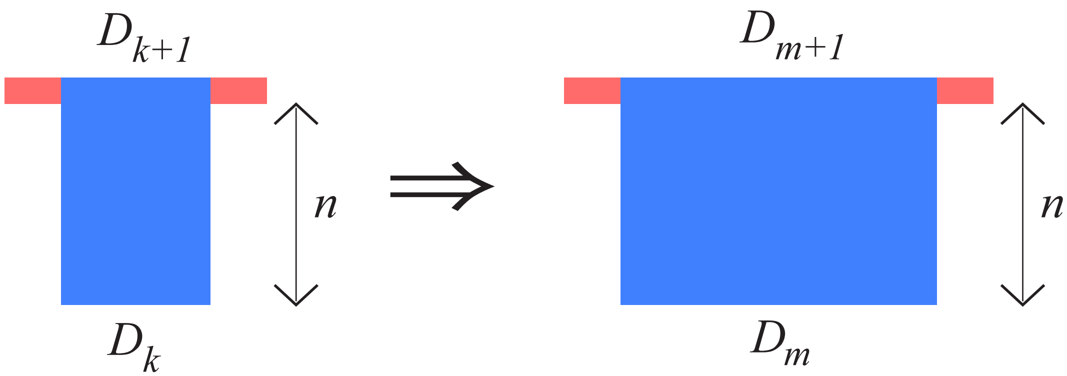


Knowing the states in D_k for times $t \leq jn$ uniquely determine the states in D_{k+1} for $t \leq (j-1)n$.

Next we use the fact that for all $m \geq k$

$$(\forall t \leq n)\ F^t(c)|_{D_m} = F^t(e)|_{D_m} \quad \Longrightarrow \quad c|_{D_{m+1}} = e|_{D_{m+1}}.$$

We know this for $m = k$, and this gives it for all $m \geq k$:

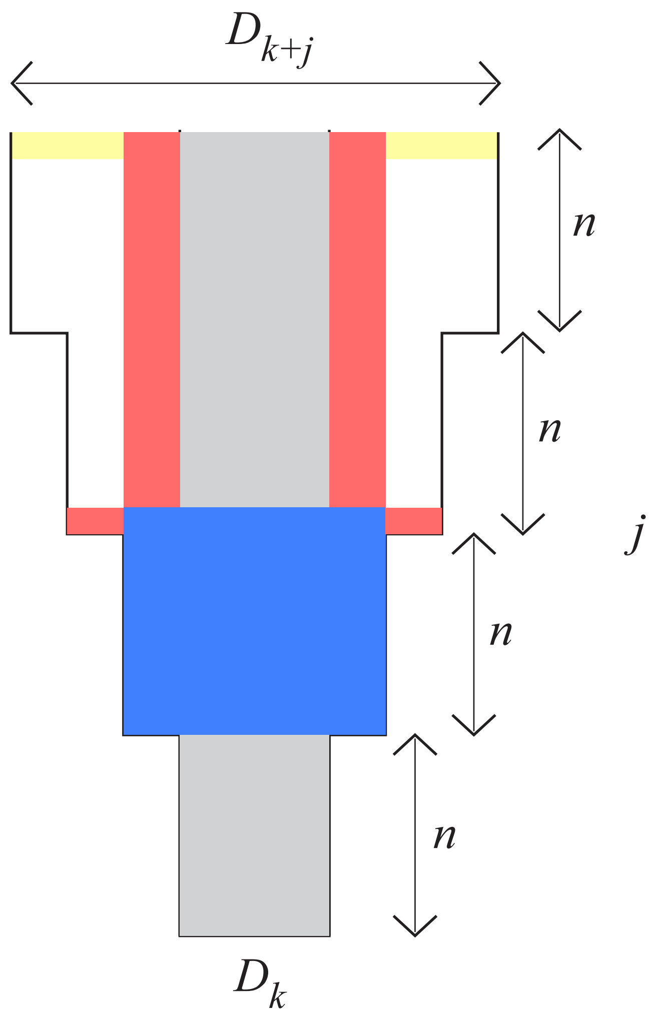


Indeed, every $\vec{n} \in D_{m+1}$ is covered by a translate $\vec{t} + D_{k+1}$ of D_{k+1} that is completely inside D_{m+1}:

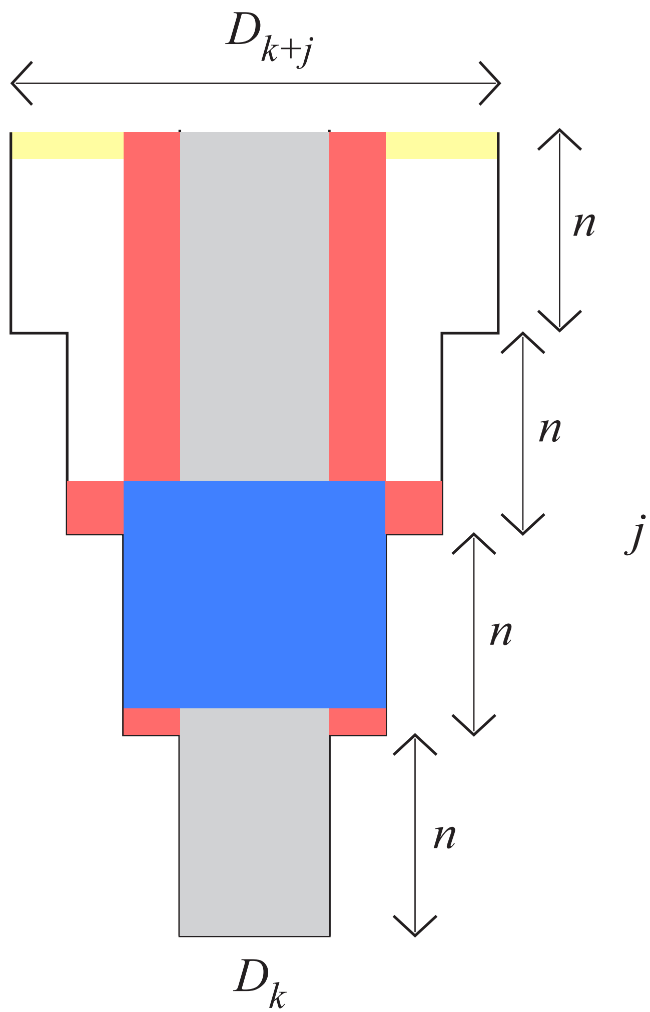
$$\vec{t} + D_{k+1} \subseteq D_{m+1}.$$

Then

$$\vec{t} + D_k \subseteq D_m.$$



So knowing the states in D_{k+1} for times $t \le (j-1)n$ uniquely determine the states in D_{k+2} for $t \le (j-2)n$.



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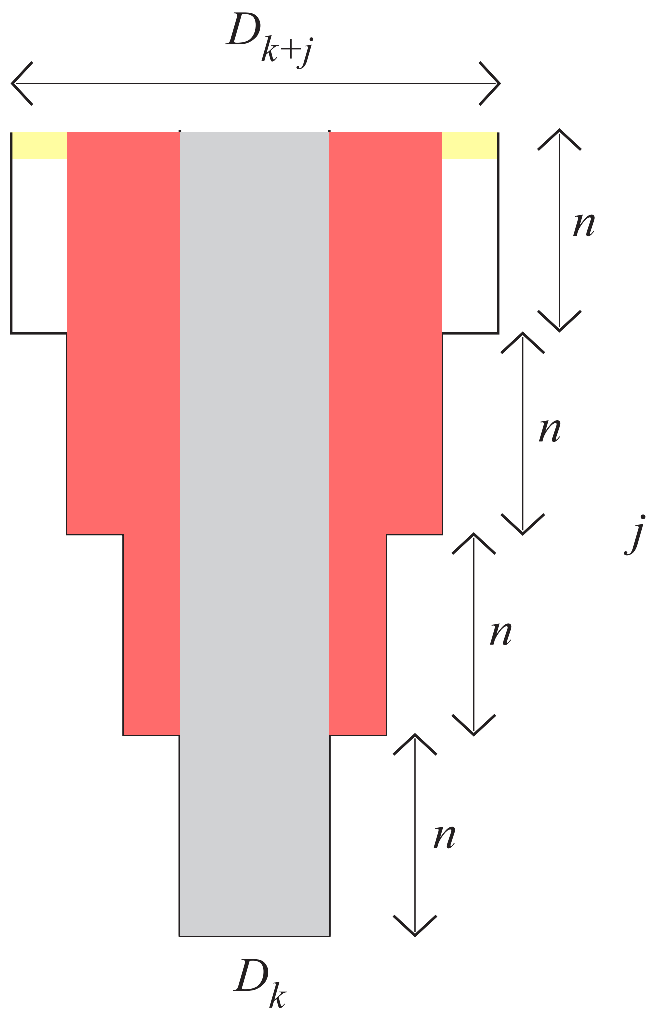

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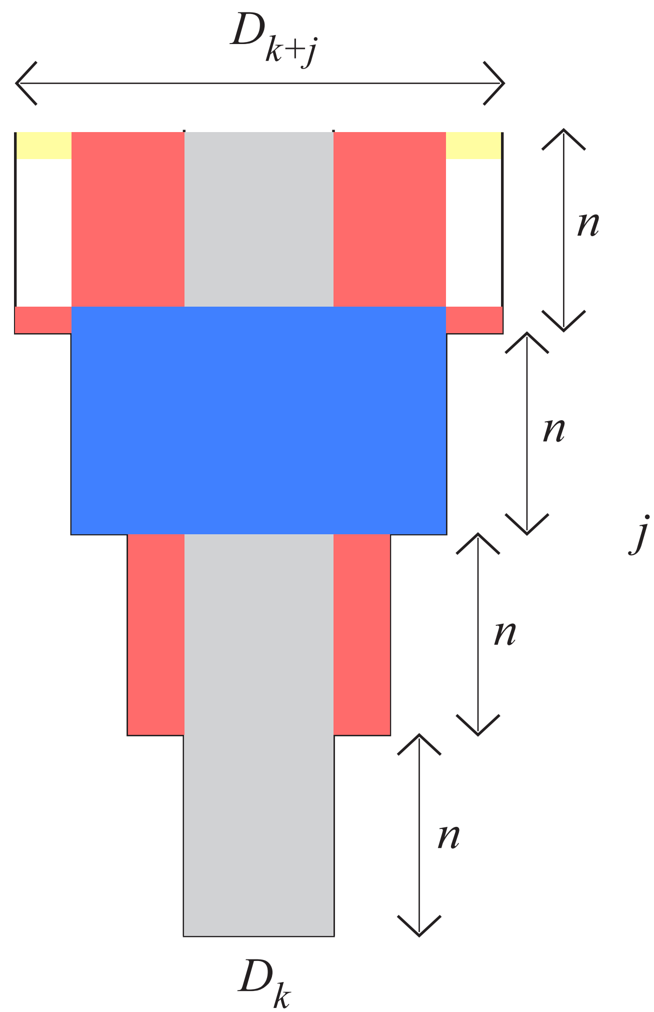
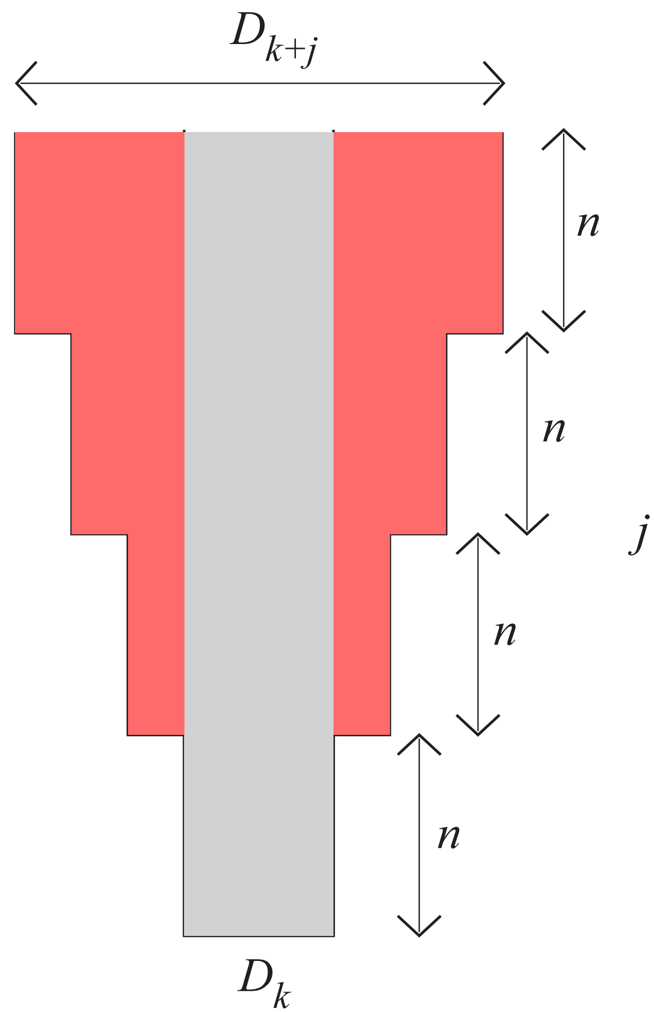
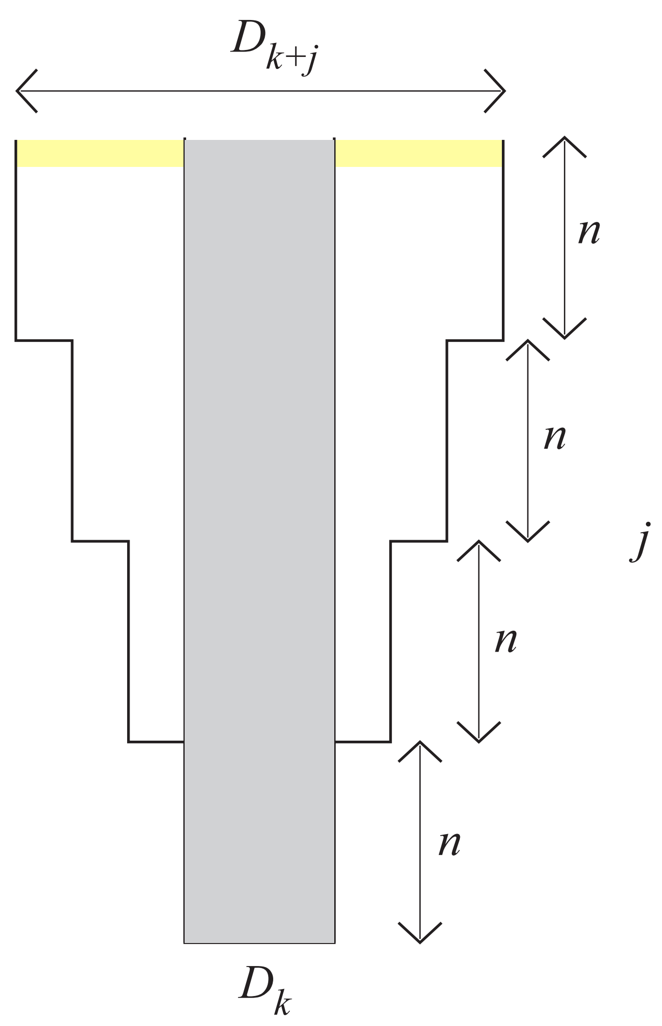


Next knowing the states in D_{k+2} for times $t \leq (j-2)n$ uniquely determine the states in D_{k+3} for $t \leq (j-3)n$.



And so on. Continuing likewise the states in D_{k+j} are uniquely identified up to time $t \leq (j-j)n$, *i.e.*, in the initial configuration at time $t = 0$.

Now the cake is ready: the grey states uniquely identify the yellow ones:



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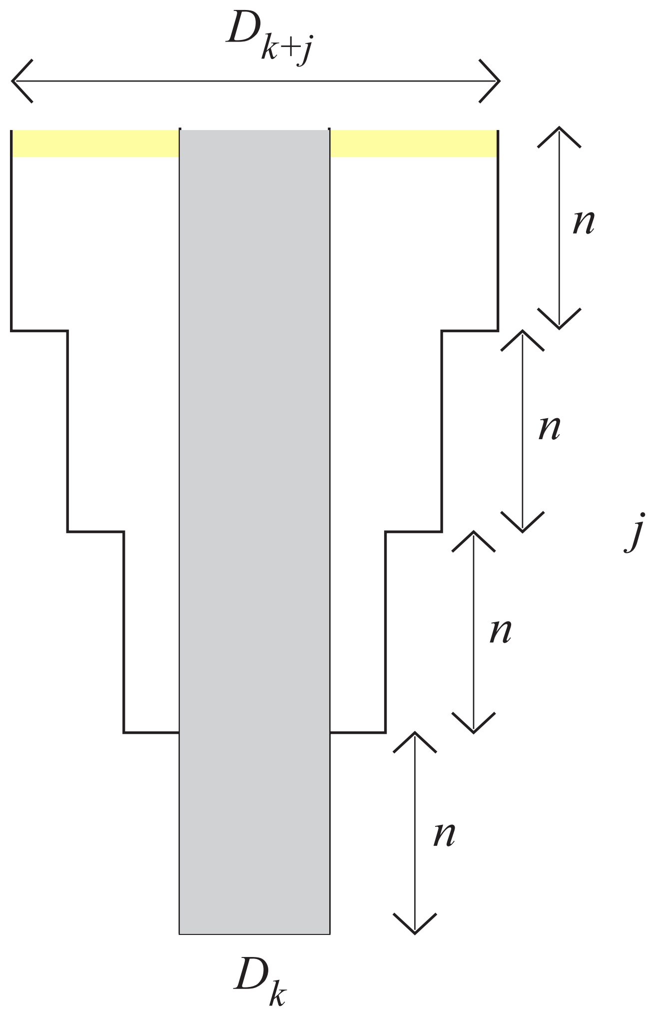


Note that if the dimension $d > 1$, the size of the yellow area grows faster than the size of the grey area when j grows:

$$\begin{aligned} |D_{k+j}| &= (2k + 2j + 1)^d, \\ (jn + 1)|D_k| &= (jn + 1)(2k + 1)^d. \end{aligned}$$

Let j be such that

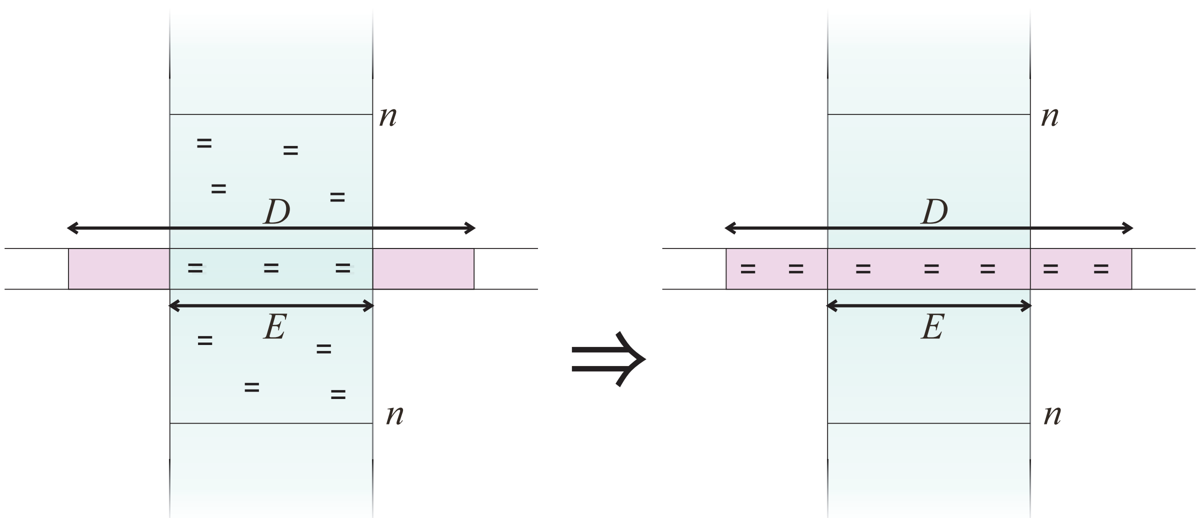
$$|D_{k+j}| > (jn + 1)|D_k|.$$

By the pigeon hole principle, some configurations with a difference in D_{k+j} have identical states in the grey area, which is a contradiction.

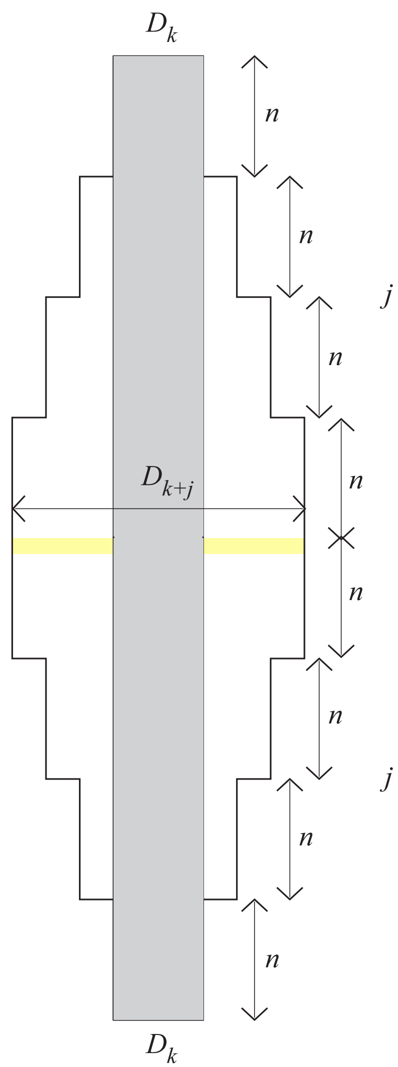
The same arguments work with **reversible expansive** cases:

Lemma. Let F be a **reversible expansive** CA with an expansivity constant $E \subseteq \mathbb{Z}^d$. For every finite $D \subseteq \mathbb{Z}^d$ there exists $n \geq 0$ such that for any configurations c, e the following holds:

$$(\forall t, -n \leq t \leq n) \; F^t(c)|_E = F^t(e)|_E \quad \Longrightarrow \quad c|_D = e|_D.$$



Proposition. There do not exist two-or higher dimensional **reversible expansive** CA.



Proposition. A reversible CA is not positively expansive.

Proof.

Proposition. A reversible CA is not positively expansive.

Proof. Enough to focus on one-dimensional CA.

Let G be a one-dimensional reversible CA that is positively expansive. We want to reach a contradiction.

Let $E = \{-m, -m+1, \ldots, m\}$ be an observation window confirming positive expansivity. Let r be the neighborhood radius of the inverse CA G^{-1}, and let $D = \{-m-r, \ldots, m+r\}$.

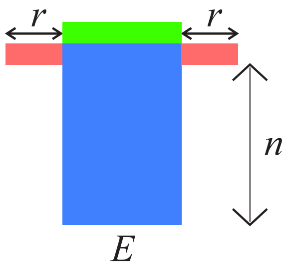
By our lemma there is n such that

$$(\forall t \leq n) F^t(c)|_E = F^t(e)|_E \quad \Longrightarrow \quad c|_D = e|_D.$$

Because r is the neighborhood radius of G^{-1}, in turn $c|_D = e|_D$ implies that

$$G^{-1}(c)|_E = G^{-1}(e)|_E.$$

So in the space-time diagram the blue states uniquely determine (by the Lemma) the red states, which in turn determine (by the inverse local rule of radius r) the green states:



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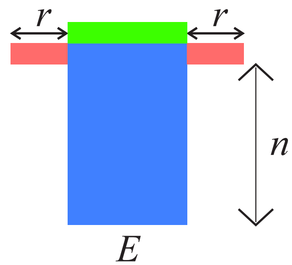
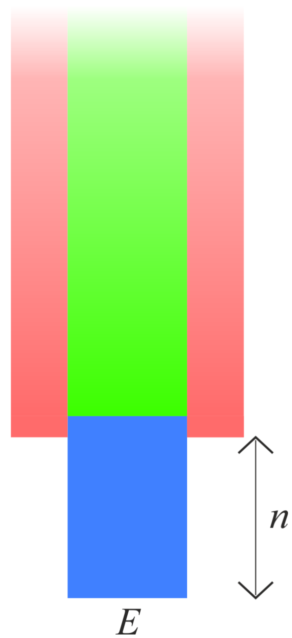
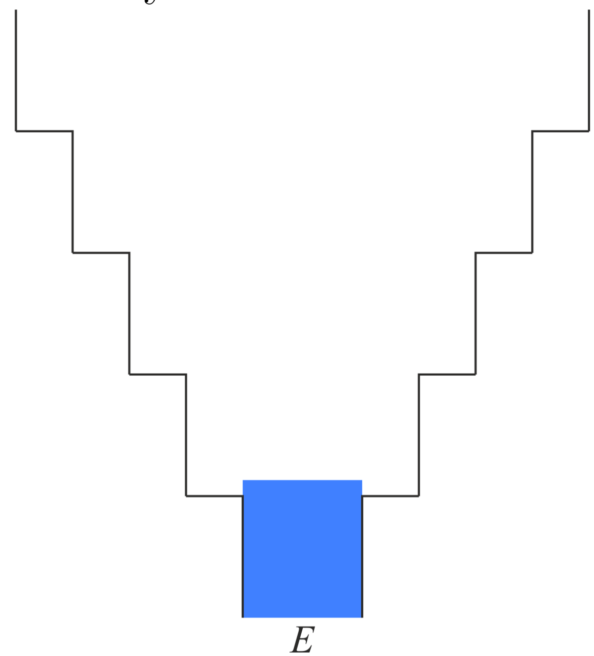


Considering $F^{-1}(c)$ instead of c, also the states above the green and red ones are determined. Repeating likewise, we have infinite vertical green and red strips determined by the blue states:



Further states are uniquely determined by considering the strips translated by r to the right and to the left. Overall, states inside an infinite cone (or upside cake) are uniquely determined by this method:



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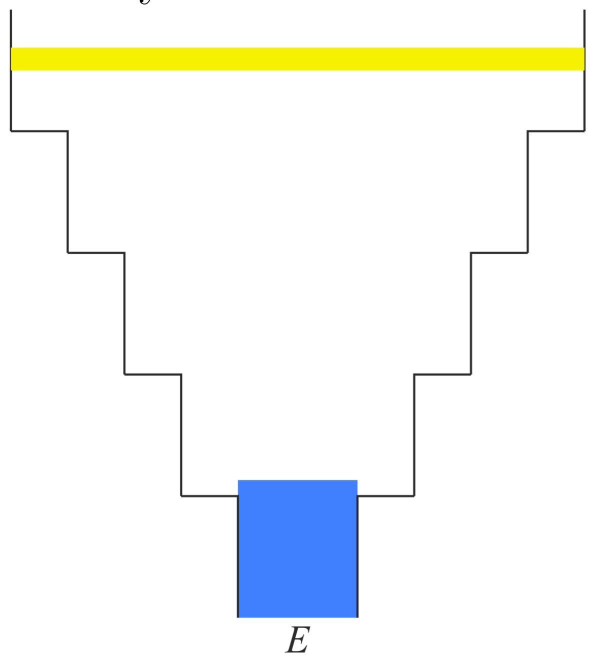
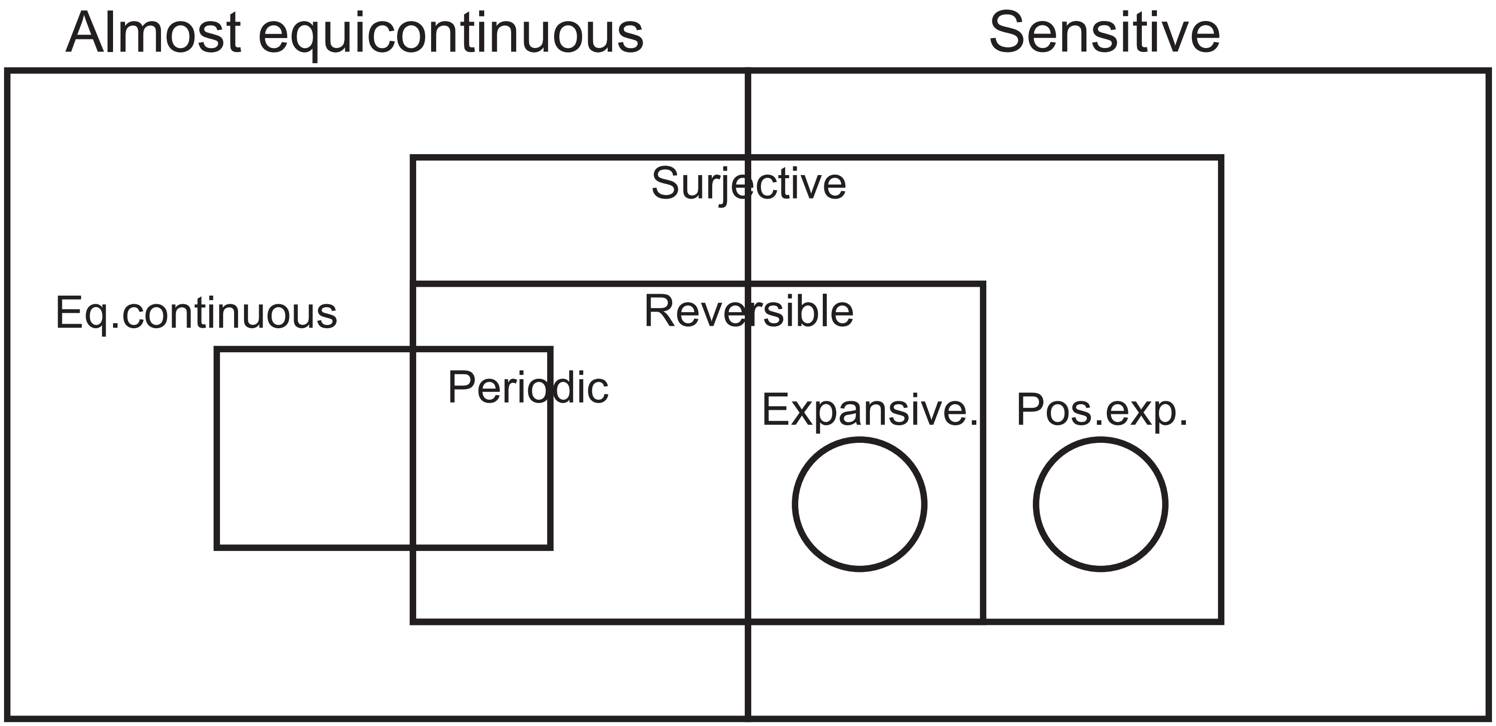


However, sufficiently far up, the cone has a row that is longer than the size of the blue area.

By a pigeon hole principle, there are two configurations with different yellow states but the same blue states, contradicting the fact that the blues states uniquely determine the yellow ones.

Summary (one-dimensional CA):



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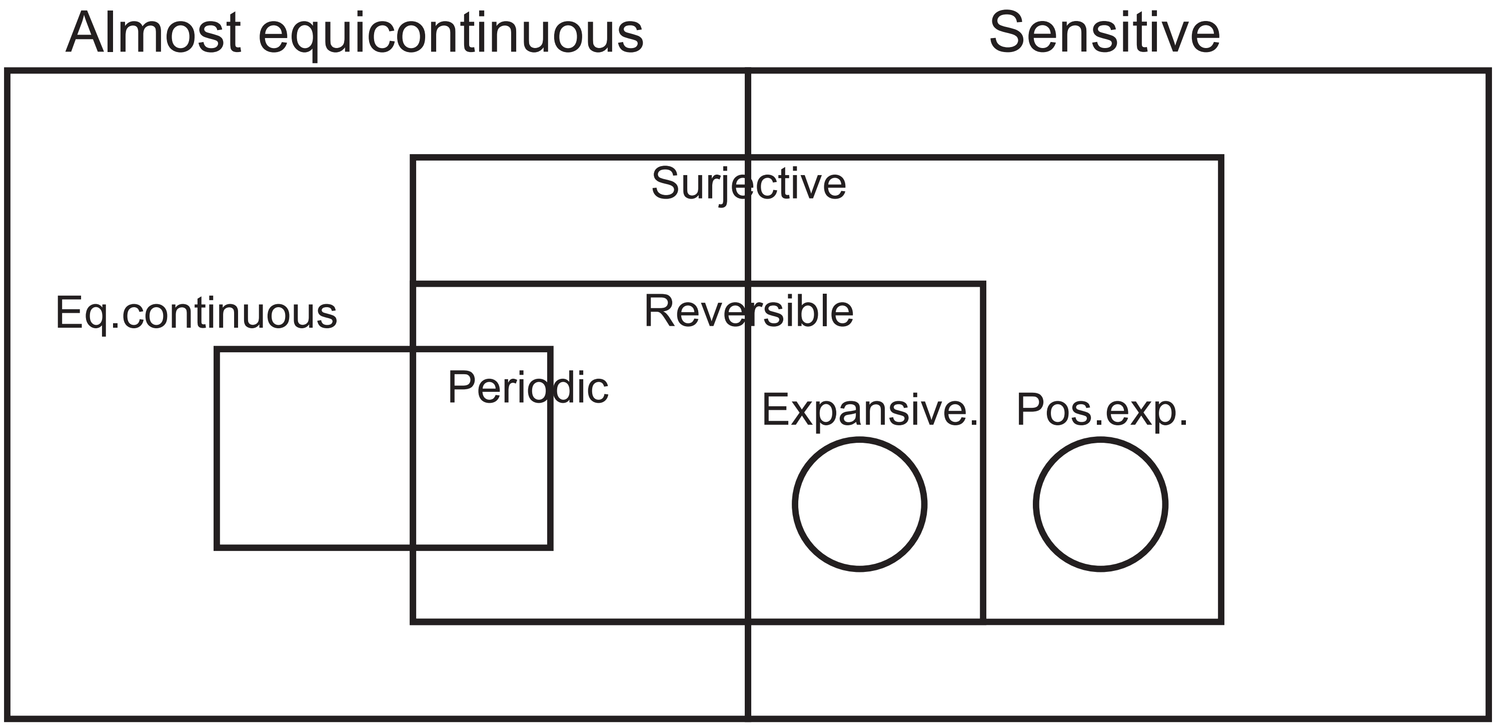

In higher dimensions expansive and positively expansive CA are missing, and the complement of sensitive CA is not the set of almost equicontinuous CA. Other aspects of the diagram are valid.

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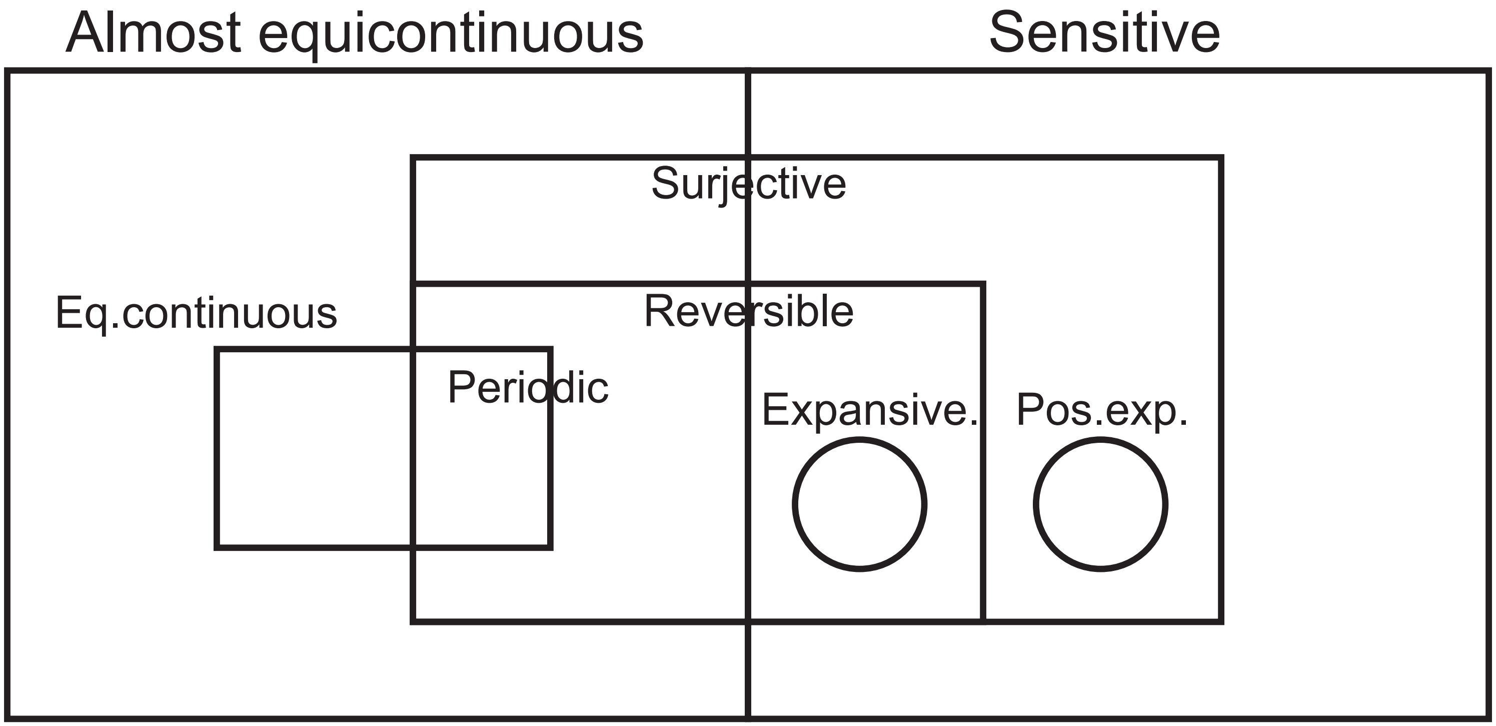


In higher dimensions expansive and positively expansive CA are missing, and the complement of sensitive CA is not the set of almost equicontinuous CA. Other aspects of the diagram are valid.

Open problem. It is not known whether expansivity or positive expansivity is decidable for a given 1D CA.

Mixing properties

By mixing properties of a dynamical system $F : X \longrightarrow X$ we mean its tendency to mix different parts of its phase space X. As sensitivity, also mixingness comes in different variants of various strengths.

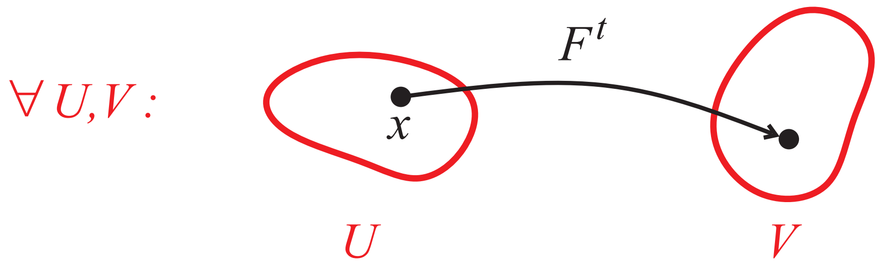
Transitivity

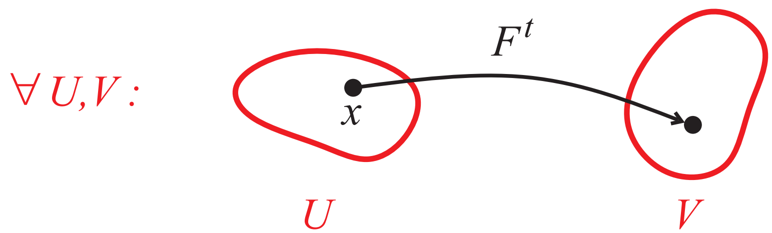
Our weakest mixing property is transitivity, stating that there are orbits between all cylinders.

For dynamical systems: A system $F : X \longrightarrow X$ is **transitive** if

$\forall$ non-empty open $U, V \subseteq X$

$$\exists x \in U \;\; : \;\; \exists t \geq 0 \;\; : \;\; F^t(x) \in V$$



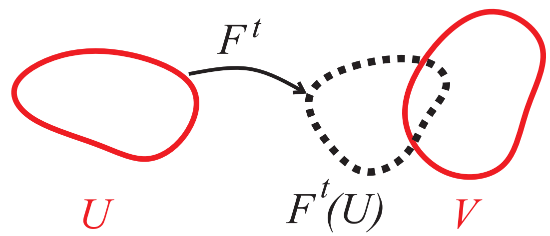


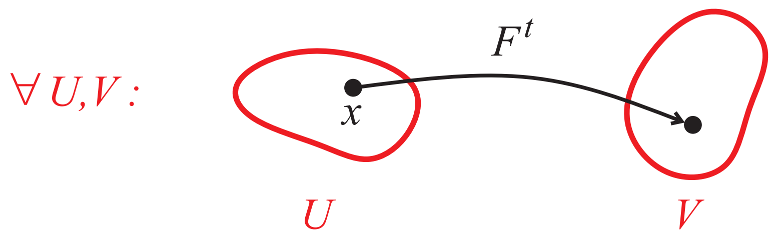
Property

$$\exists x \in U \ : \ \exists t \geq 0 \ : \ F^t(x) \in V$$

can be equivalently written as

$$\exists t \geq 0 \ : \ F^t(U) \cap V \neq \emptyset$$



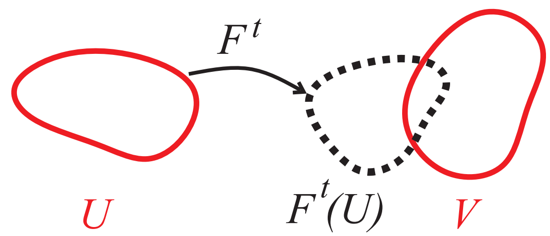


Property

$$\exists x \in U \;\;:\;\; \exists t \geq 0 \;\;:\;\; F^t(x) \in V$$

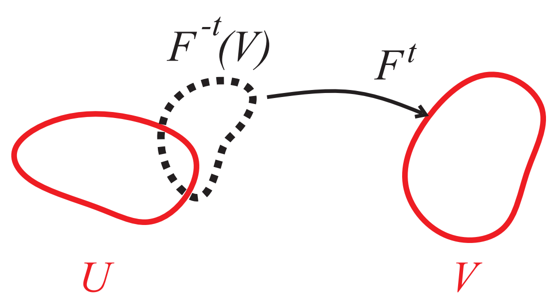
can be equivalently written as

$$\exists t \geq 0 \;\;:\;\; g^t(U) \cap V \neq \emptyset$$

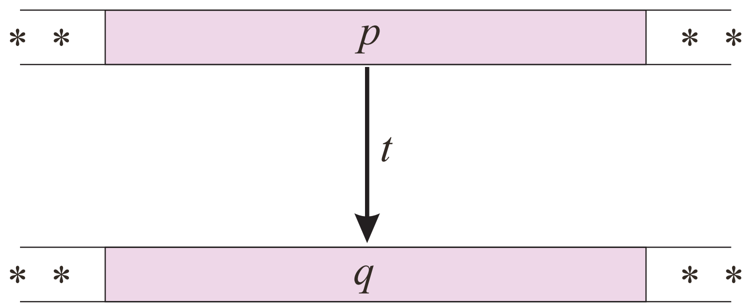


or as

$$\exists t \geq 0 \;\;:\;\; F^{-t}(V) \cap U \neq \emptyset,$$



For CA in terms of cylinders: A cellular automaton F is **transitive** if for all finite patterns p and q there exists $c \in [p]$ such that $F^t(c) \in [q]$ for some $t \geq 0$:



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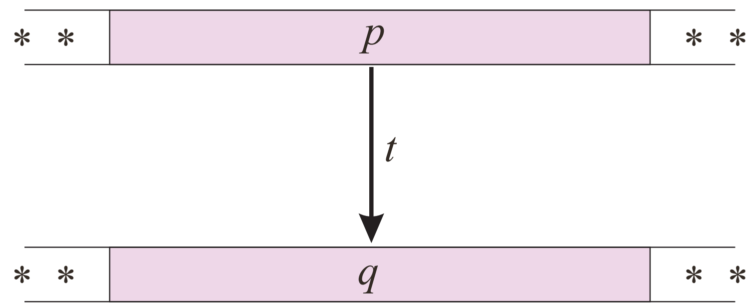

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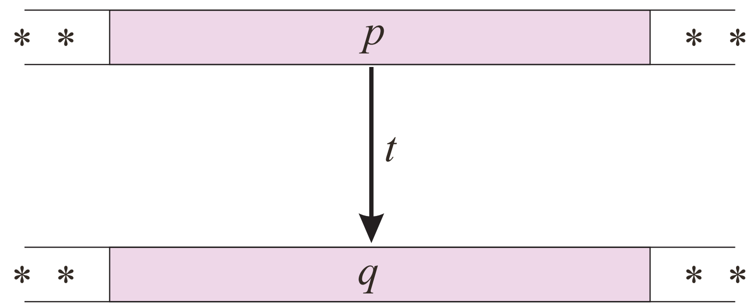

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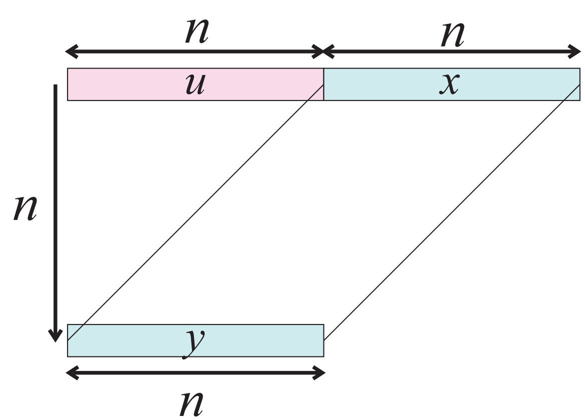
Remark. It is enough to verify transitivity for patterns $p, q \in S^D$ having the same shape D. Indeed, for any patterns $p \in S^A$ and $q \in S^B$ there are patterns p', q' having the same shape $D = A \cup B$ such that $[p'] \subseteq [p]$ and $[q'] \subseteq [q]$.

Also, it is enough to consider shapes D such that every finite set of cells is a subset of one of the considered sets D.

Example. The XOR CA F is transitive.

To show transitivity of F it is enough to consider two cylinders determined by two words u, v of equal length n at identical positions of $\mathbb{Z}$.

Append another word x of length n at the end of u and consider the word y of length n that the local rule gives n time steps later from ux:



Because of right-permutivity of the local rule, any change to x causes a change in y. Thus the mapping

$$x \mapsto y$$

between length n words is injective (for a fixed u). Thus the mapping is also surjective so that some choice of x produces $y = v$.