

Transitivity is a stronger property than sensitivity:

Proposition. Let $|S| \geq 2$. If CA $F : S^{\mathbb{Z}^d} \rightarrow S^{\mathbb{Z}^d}$ is transitive then F is sensitive.

Proof.

We prove

F not sensitive $\implies F$ not transitive.

If F is not sensitive then for every finite E there exists an E -blocking pattern. In particular, let $p \in S^D$ be E -blocking for $E = \{\vec{0}\}$.

We prove

$$F \text{ not sensitive} \implies F \text{ not transitive.}$$

If F is not sensitive then for every finite E there exists an E -blocking pattern. In particular, let $p \in S^D$ be E -blocking for $E = \{\vec{0}\}$.

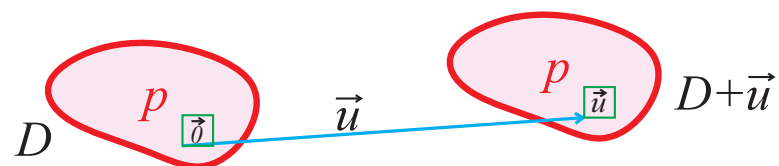
Take a non-zero translation vector $\vec{u} \in \mathbb{Z}^d$ such that

$$(D + \vec{u}) \cap D = \emptyset,$$

and consider the cylinder

$$U = [p] \cap \tau_{-\vec{u}}([p]),$$

which is determined by two copies of pattern p , in positions D and $D + \vec{u}$:



We prove

$$F \text{ not sensitive} \implies F \text{ not transitive.}$$

If F is not sensitive then for every finite E there exists an E -blocking pattern. In particular, let $p \in S^D$ be E -blocking for $E = \{\vec{0}\}$.

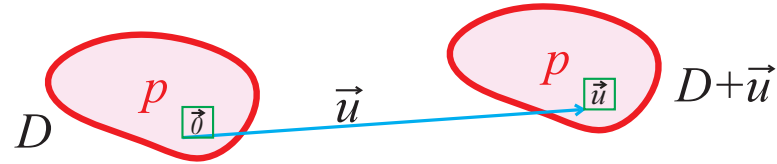
Take a non-zero translation vector $\vec{u} \in \mathbb{Z}^d$ such that

$$(D + \vec{u}) \cap D = \emptyset,$$

and consider the cylinder

$$U = [p] \cap \tau_{-\vec{u}}([p]),$$

which is determined by two copies of pattern p , in positions D and $D + \vec{u}$:



As p is $\{\vec{0}\}$ -blocking, for all $c \in U$ and all $t \in \mathbb{N}$ we have

$$F^t(c)_{\vec{0}} = F^t(\tau_{\vec{u}}(c))_{\vec{0}} = \tau_{\vec{u}}(F^t(c))_{\vec{0}} = F^t(c)_{\vec{u}}.$$

If V is a cylinder determined by a pattern with domain $\{\vec{0}, \vec{r}\}$ that assigns different states to cells $\vec{0}$ and \vec{r} then $F^t(U) \cap V = \emptyset$ for all $t \geq 0$. \square

Example. A simple (reversible) example showing the the converse implication is not true:

The cartesian product $\tau \times I$ of a non-zero translation τ and the identity function I , both over the binary state set $\{0, 1\}$.

(Cartesian product $G \times H$ means a CA with two independent layers. One layer applies G , the other layer H .)

This CA is sensitive because τ is sensitive, but it is not transitive because I is not transitive.

Example. A simple (reversible) example showing the the converse implication is not true:

The cartesian product $\tau \times I$ of a non-zero translation τ and the identity function I , both over the binary state set $\{0, 1\}$.

(Cartesian product $G \times H$ means a CA with two independent layers. One layer applies G , the other layer H .)

This CA is sensitive because τ is sensitive, but it is not transitive because I is not transitive.

Remark.

- If G is sensitive then $G \times H$ is sensitive,
- If G is not transitive then $G \times H$ is not transitive.

Point $x \in X$ is a **transitive point** of dynamical system $F : X \longrightarrow X$ if its orbit

$$\{x, F(x), F^2(x), \dots\}$$

is dense in X .

For a cellular automaton F then, a configuration c is a **transitive point** if for every pattern $p \in S^D$ there exists $t \geq 0$ such that $F^t(c) \in [p]$.

(In the orbit, every finite pattern shows up in every position.)

Point $x \in X$ is a **transitive point** of dynamical system $F : X \longrightarrow X$ if its orbit

$$\{x, F(x), F^2(x), \dots\}$$

is dense in X .

For a cellular automaton F then, a configuration c is a **transitive point** if for every pattern $p \in S^D$ there exists $t \geq 0$ such that $F^t(c) \in [p]$.

(In the orbit, every finite pattern shows up in every position.)

Let us denote by

$$\mathcal{T}_F = \mathcal{T} = \{c \in S^{\mathbb{Z}^d} \mid \text{the orbit } c, F(c), F^2(c), \dots \text{ is dense}\}$$

the set of the transitive points of F .

Proposition. The following three conditions are equivalent for a CA F :

- (i) F is transitive,
- (ii) \mathcal{T}_F is a residual set,
- (iii) $\mathcal{T}_F \neq \emptyset$.

Proof.

We prove **(i) \implies (ii)** and **(iii) \implies (i)**, The implication (ii) \implies (iii) is given by the **Baire theorem**.

F is transitive $\implies \mathcal{T}_F$ is a residual set

Let F be a transitive CA, and let

$$V_1, V_2, \dots$$

be an enumeration of all cylinders. (There are countably many finite patterns, so they can be enumerated.)

F is transitive $\implies \mathcal{T}_F$ is a residual set

Let F be a transitive CA, and let

$$V_1, V_2, \dots$$

be an enumeration of all cylinders. (There are countably many finite patterns, so they can be enumerated.)

For $k = 1, 2, \dots$ denote

$$X_k = \bigcup_{t=0}^{\infty} F^{-t}(V_k) = \{c \mid \exists t \geq 0 : F^t(c) \in V_k\},$$

the set of configurations whose orbit visits V_k . Clearly

$$\mathcal{T}_F = \bigcap_{k=1}^{\infty} X_k,$$

so it is enough to show that each X_k is open and dense.

F is transitive $\implies \mathcal{T}_F$ is a residual set

Let F be a transitive CA, and let

$$V_1, V_2, \dots$$

be an enumeration of all cylinders. (There are countably many finite patterns, so they can be enumerated.)

For $k = 1, 2, \dots$ denote

$$X_k = \bigcup_{t=0}^{\infty} F^{-t}(V_k) = \{c \mid \exists t \geq 0 : F^t(c) \in V_k\},$$

the set of configurations whose orbit visits V_k . Clearly

$$\mathcal{T}_F = \bigcap_{k=1}^{\infty} X_k,$$

so it is enough to show that each X_k is open and dense.

Indeed, the set X_k is

- **open** as a union of open sets $F^{-t}(V_k)$,
- **dense**: Because F is transitive, for any cylinder U there exists $x \in U$ and time t such that $F^t(x) \in V_k$. Thus $x \in U \cap X_k$.

$\mathcal{T}_F \neq \emptyset \implies F$ is transitive

Let c be a **transitive point** for F . Consider any two cylinders $[p]$ and $[q]$, and let us prove that there exists a configuration $x \in [p]$ and time $t \geq 0$ such that $F^t(x) \in [q]$.

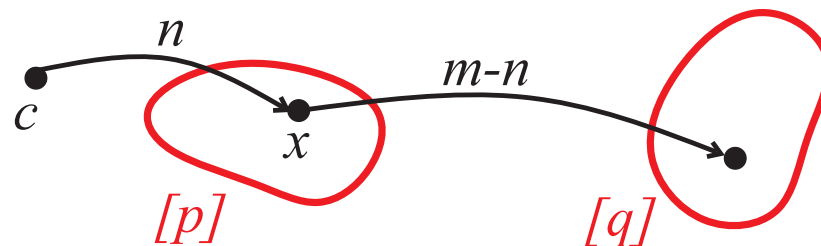
Transitivity of c means that there are times $n, m \geq 0$ such that $F^n(c) \in [p]$ and $F^m(c) \in [q]$.

$\mathcal{T}_F \neq \emptyset \implies F$ is transitive

Let c be a **transitive point** for F . Consider any two cylinders $[p]$ and $[q]$, and let us prove that there exists a configuration $x \in [p]$ and time $t \geq 0$ such that $F^t(x) \in [q]$.

Transitivity of c means that there are times $n, m \geq 0$ such that $F^n(c) \in [p]$ and $F^m(c) \in [q]$.

(1) If $n \leq m$ then we can let $x = F^n(c)$ and $t = m - n$. We have $x \in [p]$ and $F^t(x) = F^m(c) \in [q]$, as required:

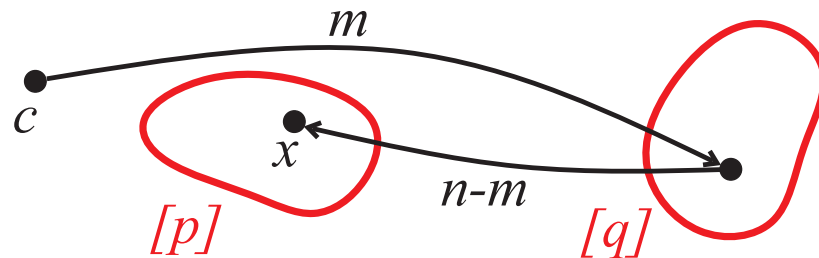


$\mathcal{T}_F \neq \emptyset \implies F$ is transitive

Let c be a **transitive point** for F . Consider any two cylinders $[p]$ and $[q]$, and let us prove that there exists a configuration $x \in [p]$ and time $t \geq 0$ such that $F^t(x) \in [q]$.

Transitivity of c means that there are times $n, m \geq 0$ such that $F^n(c) \in [p]$ and $F^m(c) \in [q]$.

(2) If $n > m$ then we let $U = [q] \setminus \{c, F(c), \dots, F^n(c)\}$. Set U is non-empty and open so there exists $m' \geq 0$ such that $F^{m'}(c) \in U$. Clearly $m' > n$ so we choose $x = F^n(c)$ and $t = m' - n$. These satisfy $x \in [p]$ and $F^t(x) = F^{m'}(c) \in [q]$, as required.

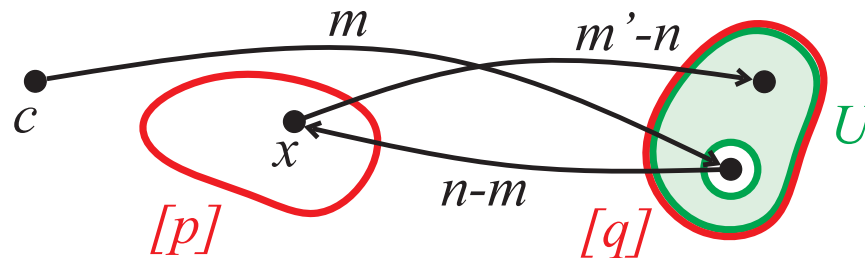


$\mathcal{T}_F \neq \emptyset \implies F$ is transitive

Let c be a **transitive point** for F . Consider any two cylinders $[p]$ and $[q]$, and let us prove that there exists a configuration $x \in [p]$ and time $t \geq 0$ such that $F^t(x) \in [q]$.

Transitivity of c means that there are times $n, m \geq 0$ such that $F^n(c) \in [p]$ and $F^m(c) \in [q]$.

(2) If $n > m$ then we let $U = [q] \setminus \{c, F(c), \dots, F^n(c)\}$. Set U is non-empty and open so there exists $m' \geq 0$ such that $F^{m'}(c) \in U$. Clearly $m' > n$ so we choose $x = F^n(c)$ and $t = m' - n$. These satisfy $x \in [p]$ and $F^t(x) = F^{m'}(c) \in [q]$, as required.



Proposition. The limit set $S^{\mathbb{Z}^d}$ is the only attractor of a transitive CA.

Proof.

Let G be a transitive CA, and let U be a non-empty clopen inward set (meaning $G(U) \subseteq U$).

We prove that $U = S^{\mathbb{Z}^d}$. If not, the complement $V = S^{\mathbb{Z}^d} \setminus U$ of U is non-empty and open. But for all $t \geq 0$

$$G^t(U) \cap V = \emptyset,$$

which contradicts transitivity. □

Chain transitivity.

There exist surjective cellular automata whose only attractor is the limit set $S^{\mathbb{Z}^d}$ but that are not transitive. So the converse implication of the previous proposition is not true. For that we need a weaker concept of chain-transitivity.

Chain transitivity.

There exist surjective cellular automata whose only attractor is the limit set $S^{\mathbb{Z}^d}$ but that are not transitive. So the converse implication of the previous proposition is not true. For that we need a weaker concept of chain-transitivity.

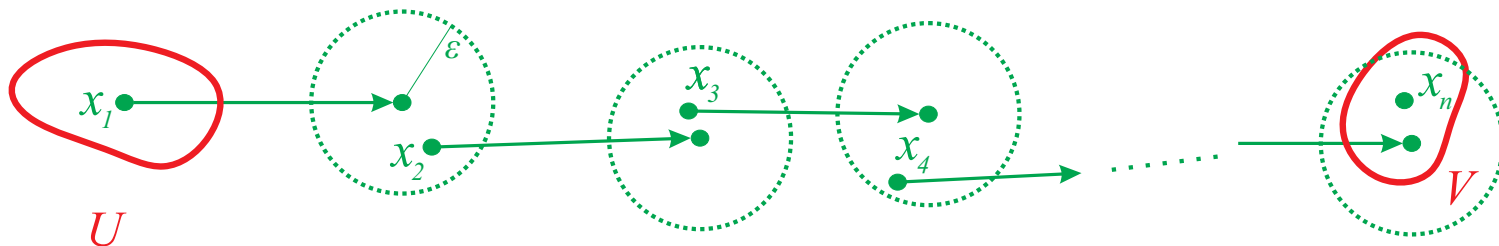
For dynamical systems: Let $F : X \rightarrow X$ be a dynamical system, and let $\varepsilon > 0$. An **ε -chain** is a sequence $x_1, x_2, \dots, x_n \in X$ such that

$$d(F(x_i), x_{i+1}) < \varepsilon \text{ for } 1 \leq i < n.$$

The system is **chain transitive** if

$\forall \varepsilon > 0$ and \forall non-empty open $U, V \subseteq X$

$\exists \varepsilon$ -chain $x_1, x_2, \dots, x_n : x_1 \in U, x_n \in V$



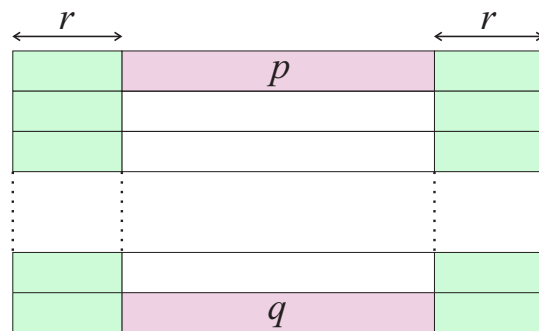
(One can get from U to V along an “almost orbit” where at each step the point may be changed a little.)

For CA in terms of cylinders: A CA G is **chain transitive** if for every every finite $D \subseteq \mathbb{Z}^d$ and for all patterns $p, q \in S^D$ there exists a sequence

$$c_0, c_1, \dots, c_n$$

of configurations such that

- $c_0|_D = p$, $c_n|_D = q$, and
- $G(c_i)|_D = c_{i+1}|_D$ for all $i \in \{0, 1, \dots, n-1\}$.



(The green states can be chosen freely to transform p into the target pattern q .)

For any finite $D \subseteq \mathbb{Z}^d$, consider the directed graph with

- vertex set S^D , all D -patterns, and
- an edge $p \longrightarrow q$ iff there exists a configuration c such that $c|_D = p$ and $G(c)|_D = q$.

The CA is chain transitive iff this graph is **strongly connected** for all D .

For any finite $D \subseteq \mathbb{Z}^d$, consider the directed graph with

- vertex set S^D , all D -patterns, and
- an edge $p \longrightarrow q$ iff there exists a configuration c such that $c|_D = p$ and $G(c)|_D = q$.

The CA is chain transitive iff this graph is **strongly connected** for all D .

Remark.

- Chain transitive CA are **surjective** as they can have no orphans.
- Every **transitive** CA is chain transitive.

Proposition. A CA G is chain transitive if and only if $S^{\mathbb{Z}^d}$ is the only attractor of G .

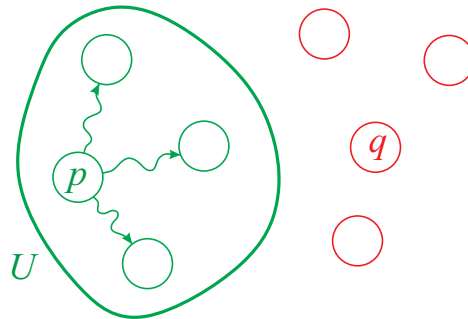
Proof.

Proposition. A CA G is chain transitive if and only if $S^{\mathbb{Z}^d}$ is the only attractor of G .

Proof.

\Leftarrow If G is not chain transitive then for some domain D the directed graph constructed for that domain is not strongly connected.

Let $p, q \in S^D$ be vertices such that there is no path in the graph from p to q . Let U be the union of all the cylinders determined by patterns that can be reached in the graph from p :



(No edge from a green vertex to a red one.)

Then U is clopen as a finite union of cylinders, $U \neq S^{\mathbb{Z}^d}$ because $U \cap [q] = \emptyset$, and $G(U) \subseteq U$. Hence U specifies an attractor different from $S^{\mathbb{Z}^d}$.

Proposition. A CA G is chain transitive if and only if $S^{\mathbb{Z}^d}$ is the only attractor of G .

Proof.

\implies Conversely, suppose that G is chain transitive. Then G is surjective so that $S^{\mathbb{Z}^d}$ is an attractor (=limit set).

Suppose, contrary to the claim, that there is also another attractor. It is determined by a clopen set $U \neq \emptyset, S^{\mathbb{Z}^d}$ such that $G(U) \subseteq U$.

Let $D \subseteq \mathbb{Z}^d$ be a finite domain such that U is a disjoint union of cylinders with domain D :

$$\begin{aligned} U &= [p_1] \cup \cdots \cup [p_n], \\ S^{\mathbb{Z}^d} \setminus U &= [q_1] \cup \cdots \cup [q_m], \end{aligned}$$

where $S^D = \{p_1, \dots, p_n, q_1, \dots, q_m\}$.

Because $G(U) \subseteq U$, in the graph corresponding to domain D there are no edges $p_i \longrightarrow q_j$ for any i, j . Thus the graph is not strongly connected, contradicting G being chain transitive.

Example. The following **Coven and Hedlund** CA is chain transitive but not transitive. So its only attractor is $\Omega = \{0, 1\}^{\mathbb{Z}}$.

The CA is $G = H \circ \sigma$, where H is the ECA number 180 and σ is the left shift: In G a bit is swapped if and only if it is immediately followed by the word 10.

Due to lack of time we skip the proofs of the following claims:

- (1) CA G is surjective because the local rule is left-permutive.
- (2) CA G is not sensitive: 000 is two blocking. It is then not transitive either.
- (3) CA G is chain transitive and hence $\{0, 1\}^{\mathbb{Z}}$ is the only attractor.

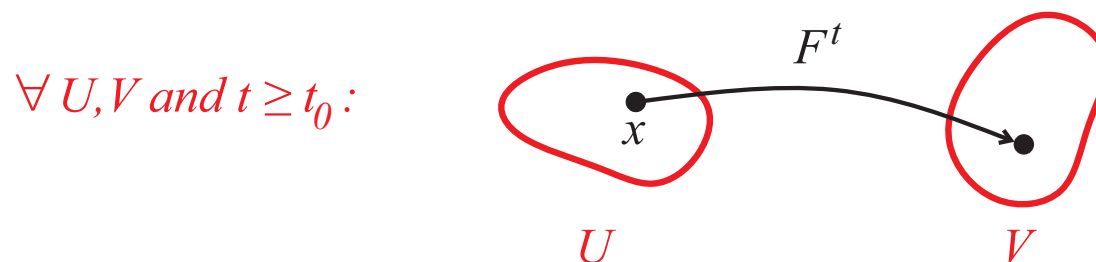
Mixingness

For dynamical systems: A dynamical system $F : X \longrightarrow X$ is **mixing** if

\forall non-empty open $U, V \subseteq X$

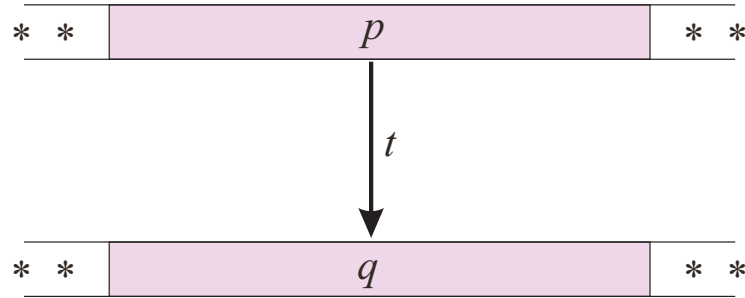
$$\exists t_0 : \forall t \geq t_0 : F^t(U) \cap V \neq \emptyset$$

(The condition is the same as for transitivity except that the last line “ $\exists t : F^t(U) \cap V \neq \emptyset$ ” got replaced by the more demanding “ $\exists t_0 : \forall t \geq t_0 : F^t(U) \cap V \neq \emptyset$ ”. So every mixing system is transitive.)



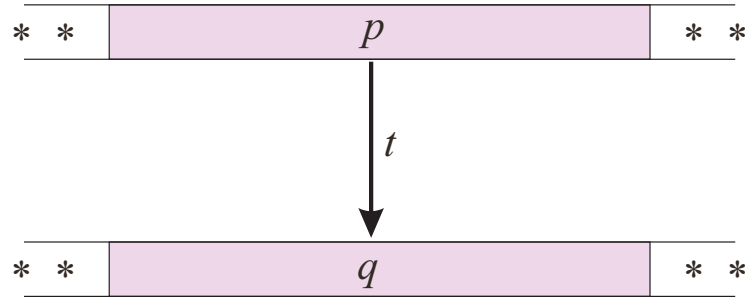
Mixingness means that one can get from any non-empty open set U into any non-empty open set V in any sufficiently large number of iteration steps.

For CA in terms of cylinders: A cellular automaton F is **mixing** if for all finite patterns p and q there exists t_0 such that for all $t \geq t_0$ there is $c \in [p]$ such that $F^t(c) \in [q]$:



Clearly a mixing CA is transitive. It is not known whether the converse implication also holds: are there any transitive, non-mixing cellular automata ?

For CA in terms of cylinders: A cellular automaton F is **mixing** if for all finite patterns p and q there exists t_0 such that for all $t \geq t_0$ there is $c \in [p]$ such that $F^t(c) \in [q]$:

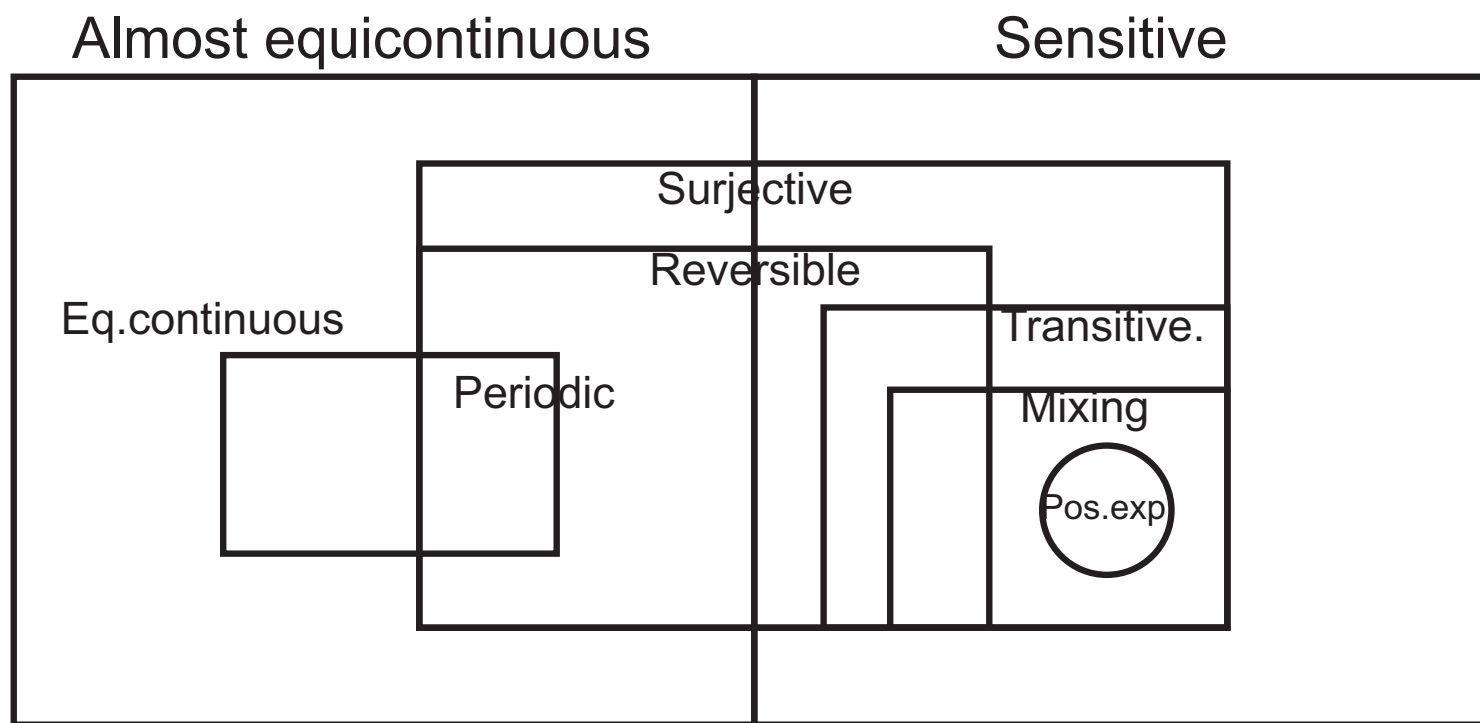


Clearly a mixing CA is transitive. It is not known whether the converse implication also holds: are there any transitive, non-mixing cellular automata ?

Proposition. A positively expansive CA is mixing.

Proof: skipped due to lack of time.

We can now add transitivity and mixing to the sensitivity classification diagram:



(I removed reversible expansive from the diagram as I do not know its relation to transitivity and mixingness. Also, I do not know whether transitivity and mixingness should be represented by the same box.)

Devaney chaos

In topological dynamics the term chaos has a number of different definitions. One of the most widely used is by Devaney: A dynamical system is **Devaney chaotic** if

- (1) it is **sensitive**,
- (2) it is **transitive**,
- (3) temporally **periodic points are dense**.

Since every transitive CA is sensitive, (2) implies (1).

It is famously open whether every surjective CA has dense temporally periodic points. If so, then Devaney chaos would be equivalent to transitivity.

Devaney chaos

In topological dynamics the term chaos has a number of different definitions. One of the most widely used is by Devaney: A dynamical system is **Devaney chaotic** if

- (1) it is **sensitive**,
- (2) it is **transitive**,
- (3) temporally **periodic points are dense**.

Since every transitive CA is sensitive, (2) implies (1).

It is famously open whether every surjective CA has dense temporally periodic points. If so, then Devaney chaos would be equivalent to transitivity.

Open Problem. Is it true that in every surjective CA every cylinder contains a temporally periodic configuration ?

Is this true at least for all transitive CA ?