

Measure invariance in surjective CA

We have time left only for a very short visit to measure theoretic tools.

To make notations shorter, in the following $X = S^{\mathbb{Z}^d}$.

A non-empty family Σ of subsets of X is called a **σ -algebra** if its is closed under complementation and countable unions:

$$\begin{aligned} A \in \Sigma &\implies X \setminus A \in \Sigma \\ A_1, A_2, \dots \in \Sigma &\implies A_1 \cup A_2 \cup \dots \in \Sigma. \end{aligned}$$

The **Borel σ -algebra** \mathcal{B} is the smallest σ -algebra that contains all open sets. (It is well defined as the intersection of all σ -algebras that contain the open sets.)

As open sets are countable unions of cylinders, \mathcal{B} is also the smallest σ -algebra that contains all cylinders.

In the following, \mathcal{B} always denotes the Borel σ -algebra. Its elements are called Borel sets, or measurable sets.

A **Borel probability measure** (or simply a **measure** in the following) is a function

$$\mu : \mathcal{B} \longrightarrow [0, 1]$$

that satisfies the following conditions:

- $\mu(\emptyset) = 0$ and $\mu(X) = 1$, and
- (σ -additivity) If A_1, A_2, \dots are pairwise disjoint elements of \mathcal{B} then

$$\mu(A_1 \cup A_2 \cup \dots) = \mu(A_1) + \mu(A_2) + \dots$$

Note that σ -additivity applies to **countable** unions of pairwise disjoint Borel sets.

Measure μ specifies a probability distribution on the configuration space, and $\mu(A)$ is the probability that a configuration drawn randomly according to μ is in the set A .

Example. Let $c \in X$ be fixed, and define

$$\mu(A) = \begin{cases} 1, & \text{if } c \in A, \\ 0, & \text{if } c \notin A. \end{cases}$$

This is the probability distribution that (deterministically) chooses c with probability 1.

It is called the **point measure**.

Another example. If $C \subseteq X$ is finite we can define a measure by

$$\mu(A) = \frac{|A \cap C|}{|C|}.$$

In this measure we choose each element $c \in C$ with probability $\frac{1}{|C|}$.

Let $[p]$ and $[q_1], \dots, [q_n]$ be cylinders such that

- $[p] = [q_1] \cup \dots \cup [q_n]$,
- $[q_i] \cap [q_j] = \emptyset$ for all $i \neq j$.

The union is disjoint: $\{[q_1], \dots, [q_n]\}$ partitions $[p]$.

Clearly for any measure μ then

$$\mu([p]) = \mu([q_1]) + \dots + \mu([q_n]).$$

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$$\mu([p]) = \mu([q_1]) + \dots + \mu([q_n]).$$

Motivated by this, let us call a function f that assigns to each cylinder $[p]$ a value in $[0, 1]$ a **consistent** probability assignment on cylinders if

- $f(X) = 1$ (X is the cylinder determined by the empty pattern!), and
- $f([p]) = f([q_1]) + \dots + f([q_n])$ whenever $\{[q_1], \dots, [q_n]\}$ partitions $[p]$.

A natural question arises: when can such f be extended to \mathcal{B} into a measure? And if it can be extended, is the extension unique?

It turns out (proof skipped) that every consistent probability assignment f can be extended into a unique measure.

Proposition. For every consistent assignment f of probabilities to cylinders there exists a unique Borel probability measure μ such that $\mu([p]) = f([p])$ for all finite patterns p .

This proposition is a special instance of the famous **Carathéodory's extension theorem** in measure theory.

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The uniqueness implies that any two measures that agree on cylinders, are equal:

Corollary. If μ_1 and μ_2 are Borel probability measures such that $\mu_1([p]) = \mu_2([p])$ for all finite patterns p , then $\mu_1 = \mu_2$.

It is easy to see that the consistency condition $f([p]) = f([q_1]) + \cdots + f([q_n])$ only needs to be checked for the cases where $p \in S^D$ and all q_i have the same domain $D \cup \{\vec{n}\}$ for some $\vec{n} \notin D$.

In other words: we only need that for each finite pattern $p \in S^D$ and every $\vec{n} \notin D$,

$$f([p]) = \sum_{a \in S} f([q_a])$$

where q_a is the pattern on the domain $D \cup \{\vec{n}\}$ such that $q_a|_D = p$ and $q_a(\vec{n}) = a$.

Example. Let $\pi : S \longrightarrow [0, 1]$ be a probability distribution on the state set. This means that

$$\sum_{a \in S} \pi(a) = 1.$$

It specifies a **Bernoulli** measure μ_π that corresponds to randomly choosing at each cell (independently of other cells) a state according to π . (=Biased coin flip at each cell.)

For each pattern $p \in S^D$, the measure μ_π gives a cylinder $[p]$ the value

$$\mu_\pi([p]) = \prod_{\vec{n} \in D} \pi(p(\vec{n})).$$

The consistency condition is easily seen to hold.

Example. A particular case of Bernoulli measures is the **uniform** measure that uses $\pi(a) = \frac{1}{|S|}$ for all $a \in S$. In this case

$$\mu_p([p]) = \left(\frac{1}{|S|} \right)^{|D|},$$

so each cylinder with the same domain is given the same probability. In this distribution all configurations are “equally likely”.

To relate Borel measures to cellular automata we need the following:

Lemma. For every continuous $G : X \longrightarrow X$ and every Borel set $A \in \mathcal{B}$, also the set $G^{-1}(A)$ is a Borel set.

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Justification:

$$\begin{aligned} A \in \Sigma &\implies G^{-1}(A) \in \mathcal{B} \\ &\implies G^{-1}(X \setminus A) = X \setminus G^{-1}(A) \in \mathcal{B} \\ &\implies X \setminus A \in \Sigma, \end{aligned}$$

$$\begin{aligned} A_1, A_2, \dots \in \Sigma &\implies G^{-1}(A_1), G^{-1}(A_2), \dots \in \mathcal{B} \\ &\implies G^{-1}(A_1 \cup A_2 \cup \dots) = G^{-1}(A_1) \cup G^{-1}(A_2) \cup \dots \in \mathcal{B} \\ &\implies A_1 \cup A_2 \cup \dots \in \Sigma. \end{aligned}$$

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Proof. Define the family

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of subsets of X . Family Σ is a σ -algebra.

For any open U the set $G^{-1}(U)$ is also open and therefore $G^{-1}(U) \in \mathcal{B}$. Thus Σ contains all open sets, Because \mathcal{B} is the smallest σ -algebra that contains open sets, $\mathcal{B} \subseteq \Sigma$.

Conclusion: $A \in \mathcal{B} \implies A \in \Sigma \implies G^{-1}(A) \in \mathcal{B}$. □

For any measure μ and any continuous $G : X \longrightarrow X$ we can now define a new measure $G(\mu)$ as follows:

$$G(\mu)(A) = \mu(G^{-1}(A)) \text{ for all } A \in \mathcal{B}.$$

The function $G(\mu)$ is defined on every Borel set A because $G^{-1}(A)$ is Borel. So defined $G(\mu)$ is a Borel probability measure:

Lemma. For any Borel probability measure μ on X and any continuous $G : X \longrightarrow X$ the function $G(\mu)$ is a Borel probability measure on X .

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Proof.

Follows directly from the facts that $G^{-1}(\emptyset) = \emptyset$, $G^{-1}(X) = X$, $G^{-1}(X \setminus A) = X \setminus G^{-1}(A)$ and

$$G^{-1}(A_1 \cup A_2 \cup \dots) = G^{-1}(A_1) \cup G^{-1}(A_2) \cup \dots$$

Moreover, we need to note that $G^{-1}(A_i)$ and $G^{-1}(A_j)$ are disjoint if A_i and A_j are disjoint.

Remark. The measure $G(\mu)$ is the probability distribution of the configurations after one application of CA G , when the initial configuration is drawn randomly according to μ .

Indeed, the probability of the event $G(c) \in A$ is the same as the probability of the event $c \in G^{-1}(A)$. This probability is $\mu(G^{-1}(A)) = G(\mu)(A)$.

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Another remark. So any CA G defines a function $\mu \mapsto G(\mu)$ on Borel probability measures. This can be iterated, and its n iterate gives $G^n(\mu)$, the distribution after n application of G when the initial configuration is drawn according to μ .

When simulating a CA on a μ -randomly chosen initial configuration, the configuration at time n is a configuration drawn randomly according to the distribution $G^n(\mu)$.

It turns out that the function $\mu \mapsto G(\mu)$ can be viewed as a dynamical system because the Borel probability measures can be given a compact metric such this application is continuous!

We say that measure μ is **invariant** under G if $G(\mu) = \mu$. **Ergodic theory** studies dynamics under invariant measures.

The following is a reformulation of the balance theorem:

Proposition. Let G be a surjective CA and let μ be the uniform Bernoulli measure. Then $G(\mu) = \mu$.

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Proof. It is enough to prove that

$$G(\mu)([p]) = \mu([p])$$

for all finite patterns $p \in S^D$. Let E be a finite domain that contains the neighborhood of D . By the balance property there is a collection of $n = |S|^{|E|-|D|}$ patterns $q_1, \dots, q_n \in S^E$ such that

$$G^{-1}([p]) = \cup_{i=1}^n [q_i].$$

We have

$$\mu(G^{-1}([p])) = \sum_{i=1}^n \mu([q_i]) = n \left(\frac{1}{|S|} \right)^{|E|} = \left(\frac{1}{|S|} \right)^{|D|} = \mu([p]).$$

The following result was proved in the homework as an application of the balance theorem. Using measure invariance the proof is shorter:

Proposition. Attractors of surjective CA G are exactly the clopen sets U that satisfy $G(U) = U$. The complement of an attractor is hence also an attractor (or the empty set).

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Proposition. Attractors of surjective CA G are exactly the clopen sets U that satisfy $G(U) = U$. The complement of an attractor is hence also an attractor (or the empty set).

Proof.

Let G be a surjective CA, and let U be any clopen set such that $G(U) \subseteq U$.

Clearly then $U \subseteq G^{-1}(U)$. Denote $V = G^{-1}(U) \setminus U$. By the invariance of the uniform measure μ ,

$$\mu(V) = \mu(G^{-1}(U)) - \mu(U) = 0.$$

But V is open, and the only open set of measure 0 is the empty set (all cylinders have positive measures). We conclude that $G^{-1}(U) \subseteq U$.

By surjectivity, $G(U) = U$ and $G(S^{\mathbb{Z}^d} \setminus U) = S^{\mathbb{Z}^d} \setminus U$.

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The previous proposition implies that all attractors of a surjective CA are clopen. It is then enough to show that the only translation invariant clopen sets are $S^{\mathbb{Z}^d}$ and \emptyset .

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The previous proposition implies that all attractors of a surjective CA are clopen. It is then enough to show that the only translation invariant clopen sets are $S^{\mathbb{Z}^d}$ and \emptyset .

Let A be a translation invariant clopen set, and let B be its complement. For any non-empty open sets U and V there exists a translation τ such that $\tau(U) \cap V \neq \emptyset$. So if $A, B \neq \emptyset$ then for some τ

$$\emptyset \neq \tau(A) \cap B = A \cap B = \emptyset,$$

a contradiction. So necessarily $A = \emptyset$ or $B = \emptyset$. □

Poincaré recurrence theorem

One of the basic theorems of Ergodic theory is the **Poincaré recurrence theorem**. It deals with measure invariant transformations so it directly applies to surjective CA. We first prove a variant which states that among the points inside any measurable set A , the set of points that do not return back to A infinitely many times has measure zero:

Proposition. Let G be a surjective CA and let μ denote the uniform measure. For every Borel set A the set

$$B = \{c \in A \mid \exists k : \forall n > k : G^n(c) \notin A\}$$

of points of A that only return finitely many times to A has measure $\mu(B) = 0$.

Proof.

$$\mathbf{B} = \{\mathbf{c} \in \mathbf{A} \mid \exists \mathbf{k} : \forall \mathbf{n} > \mathbf{k} : \mathbf{G}^{\mathbf{n}}(\mathbf{c}) \notin \mathbf{A}\}$$

For every $k \geq 0$ denote by

$$A_k = G^{-k}(A) \cup G^{-k-1}(A) \cup G^{-k-2}(A) \cup \dots$$

the set of points of X that visit A at time k or later. Then

- $A_0 \supseteq A_1 \supseteq A_2 \supseteq \dots$
- Each A_k **is a Borel set** as a countable union of Borel sets $G^{-i}(A)$.
- $G^{-1}(A_k) = A_{k+1}$ for all k , because $c \in A_{k+1} \iff G(c) \in A_k$.

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Because the uniform measure is invariant the last point gives $\mu(A_k) = \mu(A_{k+1})$ for all k , and therefore

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Because $A \subseteq A_0$, we have that $A \setminus A_k \subseteq A_0 \setminus A_k$ for every k , and consequently

$$\mu(A \setminus A_k) \leq \mu(A_0 \setminus A_k) = \mu(A_0) - \mu(A_k) = 0.$$

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$$\mu(A \setminus A_k) \leq \mu(A_0 \setminus A_k) = \mu(A_0) - \mu(A_k) = 0.$$

Set B consists of those points of A that do not belong to some A_k , that is,

$$B = (A \setminus A_0) \cup (A \setminus A_1) \cup (A \setminus A_2) \cup \dots$$

is a countable union of measure 0 sets $A \setminus A_k$. Thus $\mu(B) = 0$. □

Remark. In particular, the set of points of A that never return to A has measure zero. (This is the set $A \setminus A_1$ in the previous proof.)