A configuration is called **recurrent** if it returns to each of its open neighborhoods. Denote by

$$\mathcal{R}_G = \mathcal{R} = \{c \in S^{\mathbb{Z}^d} \mid \forall \text{ open } U\colon \;\; c \in U \Longrightarrow \exists n \geq 1 : G^n(c) \in U\}.$$

the set of recurrent points.

Clearly temporally periodic points are recurrent: periodicity is a very strong form of recurrence.

In fact, if c is recurrent then it returns infinitely many times to all its open neighborhoods U: Either c is periodic, or for every n the set $U \setminus \{G(c), G^2(c), \ldots, G^n(c)\}$ is an open neighborhood of c, which then must be visited by c.

Another variant of the Poincaré recurrence theorem tells us that in surjective CA recurrent points are residual and have full uniform measure.

Proposition. If G is surjective then $\mathcal{R}$ is a countable intersection of open sets of uniform measure 1. In particular, $\mathcal{R}$ is residual and $\mu(\mathcal{R}) = 1$.

Proof.

Let $U_1, U_2, \ldots$ be an enumeration of all cylinders. For each $k = 1, 2, \ldots$ define

$$X_k = \{c \in U_k \mid \forall n \geq 1 : G^n(c) \notin U_k\},$$

the set of points of U_k that never return to U_k. Note that c **is not recurrent if and only if it belongs to some** X_k.

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the set of points of U_k that never return to U_k. Note that c **is not recurrent if and only if it belongs to some** X_k.

- $\mu(X_k) = 0$ for all k, by the Poincaré recurrence theorem.
- Each X_k **is closed** because

$$X_k = U_k \setminus \bigcup_{n=1}^{\infty} G^{-n}(U_k)$$

  where U_k is clopen (both open and closed).

We see that the complement $Y_k = S^{\mathbb{Z}^d} \setminus X_k$ is a full measure open set. Such a set is dense (because otherwise its complement would contain a cylinder and all cylinders have positive measure).

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From $S^{\mathbb{Z}^d} \setminus \mathcal{R} = X_1 \cup X_2 \cup \ldots$ we obtain, by complementing and using de Morgan's laws, that

$$\mathcal{R} = Y_1 \cap Y_2 \cap \ldots$$

Thus $\mathcal{R}$ is residual and $\mu(\mathcal{R}) = 1$. $\square$

The Poincaré recurrence theorem is a useful tool. Let's see some applications.

Proposition. If a one-dimensional CA G is surjective and not sensitive then the periodic configurations are dense in $S^{\mathbb{Z}}$.

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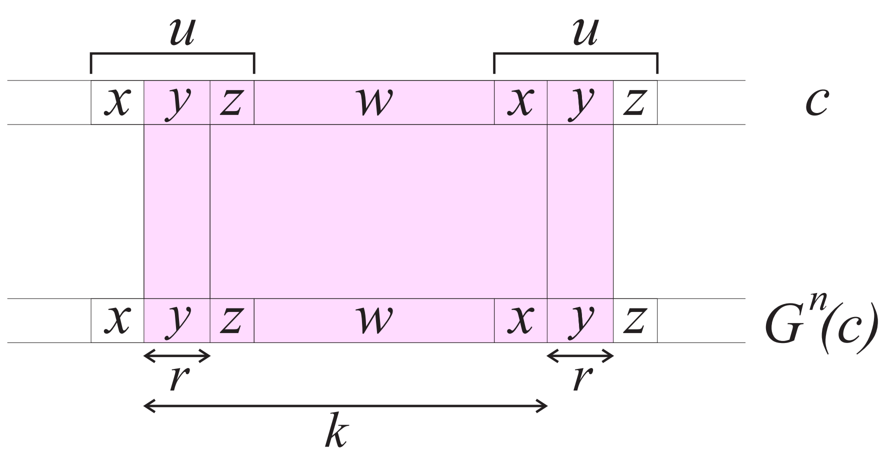
Proof.

Because G is not sensitive, there is an r-blocking word u. Write $u = xyz$ where y of length r is in the blocked position.

Let $w \in S^*$ be an arbitrary word. It is enough to show that the cylinder $[uwu]$ contains a periodic configuration.

By the recurrence theorem there is at least a recurrent configuration $c \in [uwu]$. Recurrence implies that $G^n(c) \in [uwu]$ for some $n \geq 1$.



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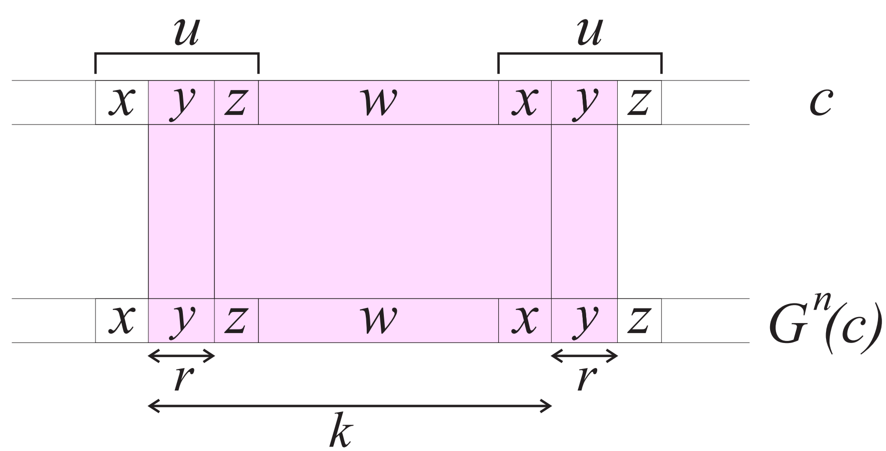

Note that changing c outside the uwu-region will not affect the pink region.

Let $e = \ldots wuwuwu \ldots$ be a spatially periodic configuration in $[uwu]$ with a spatial period $k = |wu|$. As the pink region in the figure is forced, $G^n(e)$ has the word $yzwx$ of length k in the same position with e.

Because e and $G^n(e)$ have a spatial period k and they agree in a segment of length k, we have $G^n(e) = e$. So e is temporally periodic. $\square$

Ergodicity

A measure theoretic concept that corresponds to transitivity in topological dynamics is ergodicity. We say that a surjective CA G is **ergodic** with respect to the uniform measure μ if for all Borel sets A hold

$$G^{-1}(A) = A \quad \Longrightarrow \quad \mu(A) = 0 \text{ or } \mu(A) = 1.$$

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Next we see that ergodicity implies transitivity.

Recall: $\mathcal{T}$ is the set of transitive points (=points with dense orbits). If a CA is transitive then $\mathcal{T}$ is a residual set; otherwise $\mathcal{T}$ is empty.

In ergodic CA the set $\mathcal{T}$ of transitive points has measure one:

Proposition. If a surjective CA G ergodic (with respect to the uniform measure μ) then $\mu(\mathcal{T}) = 1$.

Proof.

Let $U_1, U_2, \ldots$ be an enumeration of all cylinders. For every k, let

$$X_k = \bigcap_{n=1}^{\infty} \left(\bigcup_{i=n}^{\infty} G^{-i}(U_k) \right) = \{c \in X \mid \forall n : \ \exists i \geq n : G^i(c) \in U_k\}$$

be the set of points that visit U_k infinitely many times. Sets X_k are Borel sets and satisfy $G^{-1}(X_k) = X_k$. Due to ergodicity, either $\mu(X_k) = 0$ or $\mu(X_k) = 1$.

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The result now follows from the fact that

$$\mathcal{T} = X_1 \cap X_2 \cap \ldots.$$

$\square$

Since a full measure set is not empty, an ergodic cellular automaton has transitive points and so it is transitive:

Corollary. If a surjective CA is ergodic then it is transitive. □