On Binary Correlations of Multiplicative Functions

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Multiplicative functions

A function $f : \mathbb{N} \to \mathbb{C}$ is multiplicative if $f(mn) = f(m)f(n)$ whenever $m, n \in \mathbb{N}$ are coprime.
A function f is 1-bounded if it takes values in the unit disc $\mathbb{D} = \{z \in \mathbb{C} : \ |z| \leq 1\}$.
Let $\Omega(n)$ be the number of prime factors of n, with multiplicities.
Examples of multiplicative functions:

- $f(n) = \mu(n) = \begin{cases} (-1)^{\Omega(n)}, & n \text{ squarefree} \\ 0 & \text{otherwise} \end{cases}$ – the Möbius function
- $f(n) = \lambda(n) = (-1)^{\Omega(n)}$ – the Liouville function
- $f(n) = 1_{p|n \Rightarrow p \leq y}$ – indicator of y-smooth numbers
- $f(n) = n^{it}$, $t \in \mathbb{R}$ – the Archimedean characters
- $f(n) = \chi(n)$ – the Dirichlet characters
- $f(n) = d(n)$ – the divisor function (unbounded).

Pretentious multiplicative functions

- The characters n^{it} and $\chi(n)$ are very simple multiplicative functions (n^{it} is a smooth function and $\chi(n)$ is periodic).
- Yet these functions are troublesome, as they have rather different behavior compared to other multiplicative functions when it comes to mean values and correlations.
- To separate these troublesome multiplicative functions from the rest, we say that f pretends to be g if

$$\mathbb{D}(f,g;X) := \left(\sum_{p\leq X}\frac{1-\mathrm{Re}(f(p)\overline{g(p)})}{p}\right)^{\frac{1}{2}} = O(1).$$

- The case where a multiplicative function $g:\mathbb{N}\to\mathbb{D}$ pretends to be $\chi(n)n^{it}$ is the case we want to avoid.

Correlations of multiplicative functions

- It is a central question how additive and multiplicative structures interact (e.g. $g(n)$ and $g(n+1)$ for multiplicative g)
- The Archimedean and Dirichlet characters correlate with their shifts: e.g.

$$\frac{1}{x}\sum_{n\leq x} n^{it}(n+1)^{-it} = 1 + o(1), \quad t \in \mathbb{R}$$

$$\frac{1}{x}\sum_{n\leq x} \chi_3(n)\chi_3(n+1) = -\frac{1}{3} + o(1).$$

- But if $g_1, \ldots, g_k$ do not pretend to be characters, there is no reason to expect their shifts to correlate. This is known as Elliott's conjecture.

Elliott and Chowla conjectures

Conjecture (Elliott). Let $g_1, \ldots, g_k : \mathbb{N} \to \mathbb{D}$ be multiplicative and $h_1, \ldots, h_k \in \mathbb{Z}$ distinct. Then

$$\frac{1}{x}\sum_{n\leq x} g_1(n+h_1)\cdots g_k(n+h_k) = o(1) \quad \text{as} \quad x \to \infty,$$

provided that for some j the function g_j does not pretend to be any $\chi(n)n^{it}$ (in a suitable sense).

This contains as a special case Chowla's conjecture:

Conjecture (Chowla). Let $h_1, \ldots, h_k$ be distinct. Then

$$\frac{1}{x}\sum_{n\leq x} \lambda(n+h_1)\cdots \lambda(n+h_k) = o(1) \quad \text{as} \quad x \to \infty.$$

Progress towards Elliott's conjecture

- The case $k = 1$, $g(n) = \lambda(n)$ is the prime number theorem.
- The case $k = 1$ of Elliott's conjecture is a theorem of Halász.
- Little is known for $k \geq 2$ unless we introduce the logarithmic correlations

$$\frac{1}{\log x} \sum_{n \leq x} \frac{g_1(n+h_1) \cdots g_k(n+h_k)}{n}.$$

- The logarithmic correlation is easier, since if the non-logarithmic correlation is $o(1)$, so is the logarithmic one by partial summation.

Logarithmic Elliott conjecture

- Tao proved in 2015 the case $k = 2$ of the logarithmic Elliott conjecture.

Theorem (Tao). Let $g_1, g_2 : \mathbb{N} \to \mathbb{D}$ be multiplicative and $h \in \mathbb{Z} \setminus \{0\}$. Then

$$\frac{1}{\log x} \sum_{n \leq x} \frac{g_1(n) g_2(n+h)}{n} = o(1) \quad \text{as} \quad x \to \infty,$$

provided that g_1 or g_2 does not pretend to be any $\chi(n) n^{it}$.

- This was generalized to k-point correlations by Tao and T. under the additional assumption that $g_1 \cdots g_k$ is not weakly pretentious.
- We will consider asymptotics for the logarithmic binary correlations for a wider class of real-valued multiplicative functions.

Uniform distribution assumption

- If g_j is uniformly distributed in APs with mean value δ_j, we expect

$$\frac{1}{\log x}\sum_{n\leq x}\frac{g_1(n)g_2(n+h)}{n}=\delta_1\delta_2+o(1)\quad \text{as}\quad x\to\infty.$$

- We write $g\in\mathcal{U}(x,\varepsilon)$ if

$$\left|\frac{1}{x}\sum_{\substack{x\leq n\leq 2x\\ n\equiv b \pmod q}}g(n)-\frac{1}{qx}\sum_{x\leq n\leq 2x}g(n)\right|\leq\frac{\varepsilon}{q},\quad 1\leq b\leq q\leq\varepsilon^{-1}$$

  for all $x\geq x_0(\varepsilon)$. This is our uniformity assumption.

- The Liouville function $\lambda(n)$ and the indicator of smooth numbers $1_{p|n\Rightarrow p\leq x^a}$ satisfy the uniformity assumption.
- Dirichlet characters with fixed modulus do not satisfy the uniformity assumption.

Main Theorem

Theorem (T., 2017). Let $g_1, g_2 : \mathbb{N} \to [-1, 1]$ be multiplicative, and let $h \in \mathbb{Z} \setminus \{0\}$. Let $1 \le \omega(X) \le \log(3X)$ be any function with $\omega(X) \xrightarrow{X \to \infty} \infty$. Assume that $g_1 \in \mathcal{U}(x, \varepsilon)$. Then

$$\frac{1}{\log \omega} \sum_{\frac{x}{\omega(x)} \le n \le x} \frac{g_1(n) g_2(n+h)}{n} = \frac{1}{x^2} \left(\sum_{x \le n \le 2x} g_1(n) \right) \left(\sum_{x \le n \le 2x} g_2(n) \right) + o_{\varepsilon \to 0}(1)$$

with $o_{\varepsilon \to 0}(1)$ uniform in g_1, g_2.
When g_1, g_2 have nonzero mean values, this differs from Tao's result.

Remarks

- If we assume that g_j does not pretend to be any twisted Dirichlet character, from Halász's theorem we see that $g_j \in \mathcal{U}(x, \varepsilon)$, so our main theorem includes the real-valued case of Tao's result. This is not surprising, as we use the same proof method.
- Klurman showed in 2016 that one can get an asymptotic for the correlations

$$\frac{1}{x} \sum_{n \leq x} f_1(n + h_1) \cdots f_k(n + h_k)$$

when $f_1, \ldots, f_k$ are pretentious. The multiplicative functions we consider may be 1-pretentious, but we need uniformity in the multiplicative functions when considering e.g. smooth numbers or characters with large modulus, and this is not possible in Klurman's result.

Applications to smooth numbers and character sums

- Smooth numbers
- Consecutive smooth numbers
- Erdős and Turán's conjecture
- Independence of smoothness of n and $n+1$
- Erdős and Pomerance's conjecture
- Character sums
- Burgess bound for reducible quadratics

Smooth numbers

- To study smooth numbers, we introduce the largest prime factor function $P^+(n) = \max_{p|n} p$.
- We say that n is y-smooth if $P^+(n) \leq y$.
- The distribution of smooth numbers is well-understood. For example, it is known that there exists a function $\rho : [1, \infty) \to (0, 1]$ (Dickmann's function) such that

$$\frac{1}{x}|\{n \leq x : \ n \equiv b \pmod q, \ P^+(n) \leq x^a\}| = \frac{\rho\left(\frac{1}{a}\right)}{q} + o(1)$$

for $a \in (0, 1)$ and $q \leq (\log x)^A$. It is also known that one can get good error terms and allow larger values of q.

Consecutive smooth numbers

- The distribution of smooth numbers at consecutive integers is not well-understood and there are many conjectures about them.

In the 1930's, Erdős and Turán formulated

Conjecture (Erdős and Turán). The set

$$\{n \in \mathbb{N} : \ P^+(n) < P^+(n+1)\} \tag{1}$$

has an asymptotic density and it equals $\frac{1}{2}$.

- The asymptotic density (when it exists) is defined by

$$d(A) = \lim_{x \to \infty} \frac{1}{x} \sum_{\substack{n \leq x \\ n \in A}} 1.$$

- Erdős and Pomerance proved that (1) has positive lower density, and their lower bound was improved by la Bretèche, Pomerance and Tenenbaum, and further by Wang (to 0.1356).

Erdős and Turán's conjecture

- We can prove a logarithmic version of Erdős and Turán's conjecture.
- The logarithmic density (when it exists) is defined by

$$\delta(A) = \lim_{x\to\infty} \frac{1}{\log x} \sum_{\substack{n\le x \\ n\in A}} \frac{1}{n}.$$

Theorem (T., 2017). Erdős and Turan's conjecture holds when asymptotic density is replaced with logarithmic density. That is,

$$\delta(\{n \in \mathbb{N} : \;\; P^{+}(n) < P^{+}(n+1)\}) = \frac{1}{2}.$$

Independence of smoothness of n and $n+1$

Generalizing Erdős and Turán's conjecture, Erdős and Pomerance conjectured that the distributions of $P^+(n)$ and $P^+(n+1)$ are independent of each other.

Conjecture (Erdős-Pomerance.) Let $a, b \in (0,1)$ be real numbers. Then the set

$$\{n \in \mathbb{N}: \ P^+(n) \leq n^a, \ P^+(n+1) \leq n^b\}$$

has an asymptotic density, and it equals $\rho(\frac{1}{a})\rho(\frac{1}{b})$ (where $\rho(\cdot)$ is Dickmann's function).

Erdős and Pomerance's conjecture

Theorem (T., 2017). The conjecture of Erdős and Pomerance holds when asymptotic density is replaced with logarithmic density. That is,

$$\delta(\{n \in \mathbb{N} : \ P^{+}(n) \leq n^{a}, \ P^{+}(n+1) \leq n^{b}\}) = \rho\left(\frac{1}{a}\right)\rho\left(\frac{1}{b}\right).$$

We can also generalize a result of Hildebrand by proving that the asymptotic density of the set

$$\{n \in \mathbb{N} : \ n^{a} \leq P^{+}(n) \leq n^{b}, \ n^{c} \leq P^{+}(n+1) \leq n^{d}\}$$

is positive for $a < b, c < d$.

Character sums

- We can also use our main theorem to give some bounds for character sums over reducible quadratics.
- Note that if χ_Q is a real primitive Dirichlet character (mod Q), and Q grows neither too slowly nor too rapidly in terms of x, then $\chi_Q \in \mathcal{U}(x, \varepsilon)$.
- This can be seen for $\Omega(1) \le Q \le x^{4-\delta}$ and Q cube-free (i.e., $p^3 \nmid Q$ for all primes p) by using the famous Burgess bound

$$\frac{1}{x}\sum_{n \le x} \chi(n) = o(x)$$

for primitive characters $\chi \pmod{Q}$.

Burgess bound for reducible quadratics

Since the character χ_Q is uniformly distributed in APs in the range that we mentioned, we can prove a logarithmic variant of Burgess' bound for character sums over the quadratic $n(n+h)$.

Theorem (T., 2017). Let $h \neq 0$, $\delta > 0$, and let $Q = Q(x) \to \infty$ as $x \to \infty$, $Q(x) \leq x^{4-\delta}$ with Q cube-free. Then we have

$$\frac{1}{\log \log x} \sum_{\frac{x}{\log x} \leq n \leq x} \frac{\chi(n(n+h))}{n} = o(1).$$

From the Weil bound, we would obtain this in the smaller region $Q = o(\frac{x^2}{\log x})$.

Proof ideas

- Averaging over primes
- Entropy decrement argument
- Stability of mean values of multiplicative functions
- Circle method argument

Averaging over primes

- We follow the same strategy as in Tao's proof of the logarithmic Elliott conjecture for binary correlations. Two key ingredients are the entropy decrement argument and the breakthrough of Matomäki and Radziwiłł on multiplicative functions in short intervals.
- Let

$$f_h(x) := \frac{1}{\log x} \sum_{n \le x} \frac{g_1(n) g_2(n+h)}{n}.$$

Since $g_j(pn) = g_j(p)g(n)$ for primes $p \nmid n$, we can average over small primes p to get

$$\sum_{p \sim P} \frac{g_1(p) g_2(p)}{p} \cdot f_h(x) = \sum_{p \sim P} \frac{1}{\log x} \sum_{n \le x} \frac{g_1(n) g_2(n+ph)}{n} 1_{p|n} + o\left(\frac{1}{\log P}\right).$$

This increases the number of variables to two (n and p).

Entropy decrement argument

- Recall

$$\sum_{p\sim P}\frac{g_1(p)g_2(p)}{p}\cdot f_h(x)=\sum_{p\sim P}\frac{1}{\log x}\sum_{n\leq x}\frac{g_1(n)g_2(n+ph)}{n}1_{p|n}+o\left(\frac{1}{\log P}\right).$$

- The factor $1_{p|n}$ is problematic, but on average it is just $\frac{1}{p}$. The entropy decrement argument of Tao allows to indeed replace $1_{p|n}$ with $\frac{1}{p}$ above for suitable P (using some information theory inequalities).

Stability of mean values of multiplicative functions

- Now we have
$$\sum_{p\sim P}\frac{g_1(p)g_2(p)}{p}\cdot f_h(x)=\sum_{p\sim P}\frac{1}{p}\frac{1}{\log x}\sum_{n\le x}\frac{g_1(n)g_2(n+ph)}{n}+o\left(\frac{1}{\log P}\right).$$
- If δ_j is the mean value of g_j on $[x,2x]$, we wish to separate the term $\delta_1\delta_2$ from the right-hand side of the above expression.
- To do this, we need a slight generalization to arithmetic progressions of a lemma of Granville and Soundararajan, stating that g_j has mean value δ_j also on other intervals.

Lemma (Granville-Soundararajan). Let $g:\mathbb{N}\to[-1,1]$ be multiplicative and $1\le a\le q$. Then for all $1\le y\le \log^{10}x$ we have

$$\left|\frac{1}{x}\sum_{\substack{x\le n\le 2x\\ n\equiv a \pmod q}} g(n)-\frac{1}{x/y}\sum_{\substack{x/y\le n\le 2x/y\\ n\equiv a \pmod q}} g(n)\right|\ll_q (\log x)^{-c_0}.$$

Circle method argument

- After extracting the main term of $\delta_1\delta_2$ from the bilinear average of g_1 and g_2, it can be dealt with using the circle method, since we are essentially counting patterns $n, n+ph$ with some weights
- When applying the circle method, we get a main term $\delta_1\delta_2$ for the correlation $f_h(x)$, where δ_j is the mean value of g_j, so we need

$$\sup_{\theta\in\mathbb{R}} \frac{1}{X}\int_X^{2X} \left| \frac{1}{P}\sum_{x\leq n\leq x+P} g_1(n)e(n\theta) - \frac{1}{x}\sum_{x\leq n\leq 2x} g_1(n)e(n\theta)\right| dx = o_{\varepsilon\to 0}(1).$$

- This short exponential sum estimate was proved by Matomäki, Radziwiłł and Tao, based on the theorem of Matomäki and Radziwiłł on multiplicative functions in short intervals. Then we conclude that $f_h(x) = \delta_1\delta_2 + o_{\varepsilon\to 0}(1)$, which proves the theorem.

Thank you!