On the Logarithmic Chowla and Elliott Conjectures

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joint work with Terence Tao

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Multiplicative functions

A function $g : \mathbb{N} \to \mathbb{C}$ is multiplicative if $g(mn) = g(m)g(n)$ whenever $m, n \in \mathbb{N}$ are coprime.
A function g is 1-bounded if it takes values in the unit disc $\mathbb{D} = \{z \in \mathbb{C} : \ |z| \leq 1\}$.
Let $\Omega(n)$ be the number of prime factors of n, with multiplicities.
Examples of multiplicative functions:

- $g(n) = \mu(n) = \begin{cases} (-1)^{\Omega(n)}, & n \text{ squarefree} \\ 0 & \text{otherwise} \end{cases}$ – the Möbius function
- $g(n) = \lambda(n) = (-1)^{\Omega(n)}$ – the Liouville function
- $g(n) = e^{2\pi i \alpha \Omega(n)}$ – generalized Liouville function
- $g(n) = n^{it}$, $t \in \mathbb{R}$ – the Archimedean characters
- $g(n) = \chi(n)$ – the Dirichlet characters
- $g(n) = d(n)$ – the divisor function (unbounded).

Mean values

- Mean values for multiplicative functions
$$\frac{1}{x}\sum_{n\leq x} g(n) \tag{1}$$
have a long history, and are related to many important topics (PNT, sieve methods,...)
- They are quite well-understood by a theorem of Halász, which tells that (1) is $o(1)$, unless
$$\inf_{t\in\mathbb{R}} \mathbb{D}(g, n\mapsto n^{it}, X) = O(1),$$
where $\mathbb{D}(\cdot)$ is the pretentious distance
$$\mathbb{D}(f,g;X) = \left(\sum_{p\leq x}\frac{1-\mathrm{Re}(f(p)\overline{g(p)})}{p}\right)^{1/2}.$$
- We want to avoid functions g with $\mathbb{D}(g, n\mapsto n^{it}, X)$ or $\mathbb{D}(g,\chi,X)$ bounded (pretentious functions).

Elliott's conjecture

- Correlations of multiplicative functions also have found various applications (sign patterns, rigidity theorems for multiplicative functions, Erdös discrepancy problem(!),...), but have proved to be much more difficult than mean values.
- For any multiplicative functions $g_1, \dots, g_k : \mathbb{N} \to \mathbb{D}$, there is no reason why their shifts would correlate, unless the g_j are pretentious

Elliott's conjecture

Let $g_1, \dots, g_k : \mathbb{N} \to \mathbb{D}$ be 1-bounded multiplicative functions and $h_1, \dots, h_k \in \mathbb{Z}$ distinct. Then we have the bound

$$\frac{1}{x}\sum_{n \leq x} g_1(n+h_1)\cdots g_k(n+h_k) = o(1),$$

provided that $\exists j$: $\inf_\chi \inf_{|t|\leq X} \mathbb{D}(g_j, \chi(n)n^{it}, X) \to \infty$.

Progress towards Elliott's conjecture

- The case $k = 1$ of Elliott's conjecture is the theorem of Halász (and includes PNT in arithmetic progressions).
- Elliott proved that, under certain additional assumptions, the correleation is bounded by $1 - \delta_k$ (as opposed to $o(1)$).
- Matomäki-Radziwiłł-Tao (2015): Elliott's conjecture holds for almost all shifts $h_1, \ldots, h_k \in [-H, H]^k$, where $H = H(x) \to \infty$ arbitrarily slowly.
- Tao (2015): The two-point case $k = 2$ holds with logarithmic averaging:

$$\frac{1}{\log x} \sum_{n \le x} \frac{g_1(n + h_1) g_2(n + h_2)}{n} = o(1),$$

provided again that g_1 or g_2 does not pretend to be any twisted character $\chi(n) n^{it}$.
- The logarithmic averages are easier, since if ordinary averages are $o(1)$, so are the logarithmic ones.

Connections to other conjectures

The Elliott conjecture, and its special case, the Chowla conjecture, are connected to many other important questions.

Chowla's conjecture

For any $k \geq 1$ and any distinct shifts $h_1, \ldots, h_k \in \mathbb{Z}$,

$$\frac{1}{x}\sum_{n \leq x} \lambda(n+h_1)\cdots\lambda(n+h_k) = o(1).$$

- Sarnak's conjecture in turn states that the Möbius function does not correlate with any determinisitic sequence:

$$\frac{1}{x}\sum_{n \leq x} \mu(n)F(T^n x) = o(1), \text{ if } (X,T) \text{ has 0 topol. entropy}, \; F \in C(X).$$

- Tao (2015): The Chowla and Sarnak cojectures are equivalent, if both are in their logarithmic forms.

Connections to other conjectures

- Sarnak's conjecture has been verified in the case of nilsequences, horocycle flows, automatic sequences,...
- Frantzikinakis-Host (2017): The logarithmic Sarnak conjecture holds in the case of uniquely ergodic systems. They also proved that shifted products of the Liouville function do not correlate with ergodic sequences, when weighted logarithmically.
- We make no progress on Sarnak's conjecture here.

Connections to other conjectures

- Correlations of the Liouville function have been shown by Tao to have some relation to the difficult problem of local Gowers uniformity of λ.
- This conjecture states that $\|\lambda\|_{U^d[x,x+H]} = o(1)$ for almost all x, with $H = H(x) \to \infty$ slowly, and U^d is the Gowers norm. The case $d = 1$ is already the Matomäki-Radziwiłł theorem.
- In particular, the case of $d = 2$ is related to the sup norm problem for λ:

$$\frac{1}{X}\int_0^X \sup_{\alpha\in\mathbb{R}} \left| \frac{1}{H} \sum_{x\leq n\leq x+H} \lambda(n)e(\alpha n)\right|^2 dx = o(1).$$

- We manage to go around these problems, and make no progress on them. This is possible, since we only consider odd order Chowlas.

Results and applications

- Main result
- Applications
- Structural theorem
- A new dichotomy

Main result

Weak form of logarithmic Elliott conjecture (Tao, T., 2017)

Let $g_1, \dots, g_k : \mathbb{N} \to \mathbb{D}$ be 1-bounded multiplicative functions and $h_1, \dots, h_k \in \mathbb{Z}$ distinct shifts. Then

$$\frac{1}{\log x} \sum_{n \leq x} \frac{g_1(n+h_1) \cdots g_k(n+h_k)}{n} = o(1),$$

provided that the product $g_1 \cdots g_k$ does not weakly pretend to be any Dirichlet character χ.

- We say that $f : \mathbb{N} \to \mathbb{D}$ pretends to be $g : \mathbb{N} \to \mathbb{D}$ weakly if

$$\sum_{p \leq x} \frac{1 - \mathrm{Re}(f(p)\overline{g(p)})}{p} = o(\log \log x).$$

Applications

If k is odd, then $\lambda^k = \lambda$ does not weakly pretend to be χ. On the other hand, if k is even, then $\lambda^k = 1$ is pretentious. Hence, we can prove the following.

Odd order cases of logarithmic Chowla conjecture

Let k be odd, and let $h_1, \ldots, h_k \in \mathbb{Z}$ be distinct integers. Then

$$\frac{1}{\log x} \sum_{n \le x} \frac{\lambda(n+h_1)\cdots\lambda(n+h_k)}{n} = o(1).$$

- Previously the $k = 2$ case was proved by Tao. The even order cases $k \geq 4$ remain open.
- Using Kátai's orthogonality criterion, one can show that even order Chowlas imply odd order Chowlas. Hence it is natural that we could prove only the odd order cases.

Applications

The odd order cases of log-Chowla allow us to prove some new results about sign patterns of the Liouville and Möbius functions.

Sign patterns of Liouville and Möbius functions

The Liouville function λ attains all 8 sign patterns of length 3 with expected log-density and all 16 sign patterns of length 4 with positive density.
The Möbius function μ attains all 65 possible sign patterns of length 4 with expected log-density.

- For the Möbus function, there cannot be 4 consecutive nonzero values; hence only $3^4 - 2^4 = 65$ possible patterns.
- Earlier, Matomäki, Radziwiłł and Tao showed that sign patterns of length 3 occur for λ with positive density, and same for sign patterns of length 2 for μ. Tao later proved the correct log-density of sign patterns of length 2 for λ and μ.

Structural theorem

Structural theorem for correlations (Tao, T., 2017)

Let $g_1, \dots, g_k : \mathbb{N} \to \mathbb{D}$ be arbitrary 1-bounded multiplicative functions and $h_1, \dots, h_k \in \mathbb{Z}$, and let $\widetilde{\lim}$ be a generalized limit. Then the correlation sequence

$$f(a) := \widetilde{\lim_{x \to \infty}} \frac{1}{\log x} \sum_{n \le x} \frac{g_1(n + ah_1) \cdots g_k(n + ah_k)}{n}$$

is almost periodic, meaning that f can be approximated uniformly with periodic functions f_i.

- By $\widetilde{\lim}$, we mean a linear functional extending the usual lim, making all bounded sequences convergent.
- Generalized limits are a way to talk about asymptotics of arbitrary sequences. Their construction uses Hahn-Banach.

A new dichotomy

- In Elliott's original conjecture there was the dichotomy

  g_j pretentious vs. g_j non-pretentious.

- In our result, however, there is a different dichotomy:

  $g_1 \cdots g_k$ invariant vs. $g_1 \cdots g_k$ non-invariant.

  We say that a function $G(n)$ is invariant if $G(pn) \approx \chi(p)G(n)$ for most primes p (more precisely, G weakly pretends to be χ).

- The two distinctions are somewhat different:
  If one takes $2k$ copies of Liouville's function, they are non-pretentious, but still invariant ($\lambda^{2k} = 1$).

- If one takes the functions $n^{it_1}, \cdots, n^{it_k}$ with $t_1 + \cdots + t_k \neq 0$, they are non-invariant, but still pretentious.

Proof ideas

- Proof ingredients
- Nilsequences
- Averaging over small primes
- Application of ergodic theory
- Combining the two approaches
- Weak logarithmic Elliott conjecture

Proof ingredients

- The proof starts with the averaging over small primes and entropy decrement ideas introduced by Tao for the case of two-point correlations.
- The proof continues by appealing to some tools from ergodic theory. Previously ergodic theory has successfully been applied to problems related to the Chowla and Elliott conjectures (e.g. by Frantzikinakis).
- Also nilsequences have a major role in the proof. Nilsequences are the characters of higher order Fourier analysis and include, but are more general than, polynomial phases $n \mapsto e(\alpha n^k)$.

Nilsequences

It is natural that nilsequences appear in work on higher order correlations, since they have long been known to be crucial for bounding certain "bilinear correlations" for multiplicative functions, such as

$$\frac{1}{x^2}\sum_{h\leq x}\sum_{n\leq x}\lambda(n)\lambda(n+h)\cdots\lambda(n+(k-1)h)=o(1).$$

The result above was established by Green-Tao and Green-Tao-Ziegler, and is closely related to linear equations in primes, which pioneered the use of nilsequences in number theory.

Averaging over small primes

- Let

$$f(a) := \widetilde{\lim_{x\to\infty}} \frac{1}{\log x} \sum_{n\le x} \frac{g_1(n+ah_1)\cdots g_k(n+ah_k)}{n}.$$

For any prime p, by multiplicativity we have $g_j(p(n+ah_j)) = g_j(n+ah_j)g_j(p)$ whenever $p \nmid n$. Therefore, averaging over the primes $p \sim P$, we get

$$f(a) = \frac{\log P}{P} \sum_{p\sim P} \frac{\overline{g_1(p)}\cdots\overline{g_k(p)}}{\log x} \sum_{n\le x} \frac{g_1(n+aph_1)\cdots g_k(n+aph_k)}{n} 1_{p|n} + o(1).$$

- Here we used the fact that the average is a logarithmic one.
- We have now two variables p and n to work with, whereas earlier we only had n.
- The factor $1_{p|n}$ is troublesome, but using the entropy decrement argument, we can replace it with its expected value $\frac{1}{p}$ for most P.

Averaging over small primes

- We iterate the argument of averaging over primes. Then we end up with an average over semiprimes

$$f(a) = \frac{\log P \log Q}{PQ} \sum_{p \sim P} \sum_{q \sim Q} \frac{\overline{G(p)G(q)}}{pq \log x} \sum_{n \le x} \frac{g_1(n + apqh_1) \cdots g_k(n + apqh_k)}{n} + o(1),$$

  for most scales P and Q, where the numbers $G(p) := g_1(p) \cdots g_k(p)$ could be called structural constants (they express whether the product $g_1 \cdots g_k$ is invariant or not).

- This can be rewritten as an approximate functional equation for f:

$$f(a) = \frac{\log P \log Q}{PQ} \sum_{p \sim P} \frac{1}{p} \sum_{q \sim Q} \frac{1}{q} \overline{G(p)}\,\overline{G(q)} f(apq) + o(1).$$

  for most P, Q.

Application of ergodic theory

- Another way to approach the correlation sequence $f(a)$ is via ergodic theory.
- Applying the Furstenberg correspondence principle, we can find some abstract measure space (X, μ), a measure-preserving transformation $T : X \to X$, and functions $G_1, \dots, G_k : X \to \mathbb{D}$ such that

$$f(a) = \int_X G_1(T^{ah_1}x) \cdots G_k(T^{ah_k}x) d\mu.$$

- There are strong tools for analyzing such multiple correlation sequences, and a deep theorem of Leibman tells that

$$\int_X G_1(T^{ah_1}x) \cdots G_k(T^{ah_k}x) d\mu = f_1(a) + f_2(a),$$

where f_1 is a nilsequence (generalization of a polynomial phase $n \mapsto e(\alpha n^k)$) and f_2 converges to zero in uniform density: $\lim_{N\to\infty} \sup_M \frac{1}{M} \sum_{N \le n \le N+M} |f(n)| = 0$.

Combining the two approaches

- Now the correlations satisfy an approximate functional equation
$$f(a)=\frac{\log P\log Q}{PQ}\sum_{p\sim P}\frac{1}{p}\sum_{q\sim Q}\frac{1}{q}\overline{G(p)}\,\overline{G(q)}f(apq)+o(1),$$
for most P,Q, and on the other hand $f(a)=f_1(a)+f_2(a)$, where f_1 is a nilsequence and f_2 converges to zero in uniform density.
- Using ergodic theory arguments, similar to those of Le, f_2 has negligible contribution.
- Then for some nilsequence f_1 we have
$$f(a)=\frac{\log P\log Q}{PQ}\sum_{p\sim P}\frac{1}{p}\sum_{q\sim Q}\frac{1}{q}\overline{G(p)}\,\overline{G(q)}f_1(apq)+o(1). \qquad (2)$$
- The right-hand side is a bilinear sum, and nilsequences have negligible bilinear sums, unless they are periodic (cf. $n\mapsto e(\alpha n)$, which has small bilinear sums, unless α is a rational number).
- Thus f_1 is periodic, so by (2) f is a uniform limit of periodic sequences, proving our structural theorem.

Weak logarithmic Elliott conjecture

- We then turn to the weak logarithmic Elliott conjecture.
- By the structure theorem and the approximate functional equation,

$$f_i(an) = \frac{\log P}{P} \sum_{p \sim P} \frac{\overline{G(p)}}{p} f_i(apn) + o(1),$$

for most P, where $f_i(n + D_i) = f_i(n)$ and the f_i converge uniformly to f.

- Multiplying both sides with a Dirichlet character $\chi(n)$ and taking averages over $a \leq D_i$, we get

$$\frac{1}{D_i} \sum_{a \leq D_i} f_i(an)\chi(n) = \frac{\log P}{P} \sum_{p \sim P} \frac{\overline{G(p)\,\chi(p)}}{p} \frac{1}{D_i} \sum_{a \leq D_i} f_i(apn)\chi(pn).$$

Weak logarithmic Elliott conjecture

- Crucially, by periodicity, the functions $n \mapsto f_i(an)\chi(n)$ and $n \mapsto f_i(apn)\chi(pn)$ have the same mean value for large p. Also, since $G(n) = g_1(n) \cdots g_k(n)$ was assumed strongly non-pretentious, we have

$$\frac{\log P}{P} \sum_{p \sim P} \frac{\overline{G(p)}\,\overline{\chi(p)}}{p} \gg 1$$

for many P. This can only happen if

$$\frac{1}{D_i} \sum_{a \leq D_i} f_i(an)\chi(n) = 0.$$

- Now the sequences $f_i(an)$ are orthogonal to all Dirichlet characters, but are periodic, so they must be identically zero. Thus the correlation sequence f, which is the limit of f_i, is identically zero, proving our main theorem.

Thank you!