The Goldbach Problem for Primes of the Form $x^2 + y^2 + 1$

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The Goldbach Conjectures

In 1742, in a letter to L. Euler, C. Goldbach formulated two conjectures that became later known as the Goldbach conjectures.

Goldbach's weak (ternary) conjecture

Every odd integer $n \geq 7$ is the sum of three primes.

Goldbach's strong (binary) conjecture

Every even integer $n \geq 4$ is the sum of two primes.

The Goldbach Conjectures

- In 1937, using the Hardy–Littlewood circle method, Vinogradov succeded in proving that every large enough odd integer is the sum of three primes.
- A year later, Chudakov, Van der Corput and Estermann proved (independently) that almost all even n are the sum of two primes, meaning that the number of exceptional even numbers $n \leq N$ is $o(N)$.
- Montgomery and Vaughan improved in 1975 the number of exceptional n in the strong Goldbach conjecture to $\ll N^{1-c}$ for some $c > 0$. This estimate has been improved several times, the latest exponent being Jia's $c = 0.121$.
- The weak conjecture was solved for all $n \geq 7$ by Helfgott in 2013, but the strong conjecture remains open.

Goldbach-type Problems

Strengthening the weak Goldbach conjecture, we may ask the following.

Goldbach problem

Given an interesting subset $\mathcal{P}$ of the primes, is every large enough odd n of the form $n = p + q + r$ with $p, q, r \in \mathcal{P}$?

- The answer to Goldbach problem depends on how $\mathcal{P}$ is distributed in arithmetic progressions and Bohr sets of the form $\{n : \ \|\alpha_i n + \beta_i\| < \eta, \ \forall i \leq M\}$ with $\alpha_i, \eta > 0, \ \beta_i \in \mathbb{R}$
- Shao (2014): If $\mathcal{P}$ has relative density greater than $\frac{5}{8}$ in the primes, then the Goldbach problem is true.
- Matomäki and Shao (2015): If $\mathcal{P}$ is the set of Chen primes (primes p such that $p + 2$ has at most two prime factors), then the Goldbach problem holds. Also if $\mathcal{P}$ is the set of primes such that $[p, p + C]$ contains two primes, then the Goldbach problem holds.

Primes of the Form $x^2 + y^2 + 1$

- Fermat (1640) asserted: A prime is of the form $p = x^2 + y^2$ if and only if $p = 2$ or $p \equiv 1 \pmod 4$. This was proved by Euler (1755).
- However, the infinitude of primes of the form $p = x^2 + y^2 + 1$ was only proved in 1960 by Linnik.
- Iwaniec (1972) gave a sieve-theoretic proof of the infinitude of such primes, based on the semilinear sieve developed by him. His proof gives the lower bound of $\gg \frac{N}{(\log N)^{\frac{3}{2}}}$ for such primes up to N, which is the correct order of magnitude by Selberg's sieve.
- Subsequently, Huxley&Iwaniec, Wu and Matomäki considered the problem of finding primes of the form $x^2 + y^2 + 1$ in short intervals.

Results

Let $\mathscr{P} = \{p \in \mathbb{P} : \ p = x^2 + y^2 + 1\}$.

Theorem 1 (T., 2016)

Almost all even integers $n \not\equiv 5, 8 \pmod 9$ are the sum of two primes from $\mathscr{P}$.

Theorem 2 (T., 2016)

All odd integers $n \geq N_0$ are the sum of the three primes from $\mathscr{P}$.

As a byproduct of the proof method, we can solve two more additive problems for the set $\mathscr{P}$.

Theorem 3 (T., 2016)

The set $\mathscr{P}$ contains infinitely many three term arithmetic progressions.

Theorem 4 (T., 2016)

Given $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ and $\beta \in \mathbb{R}$, there are infinitely many $p \in \mathscr{P}$ such that $\|\alpha p + \beta\| < p^{-0.01}$.

Remarks

- The local conditions $n \not\equiv 5, 8 \pmod 9$ in Theorem 1 are necessary.
- Matomäki (2008): Almost all even n satisfying some local conditions are of the form $n = p + q$ with $p \in \mathscr{P}$ and q a generic prime.
- Several subsets of the primes have been shown to contain infinitely many three term progressions. Green (2003): any positive density subset of the primes contains 3-term APs. Green–Tao (2004): The Chen primes contain infinitely many 3-term APs.
- The distribution of $\alpha p \pmod 1$ has attracted a lot of attention, both for generic primes and primes of a special form. Matomäki (2008) showed that, given $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ and $\beta \in \mathbb{R}$, there are infinitely many Chen primes p such that $\|\alpha p + \beta\| < p^{-\theta}$ for some $\theta > 0$.
- We do not get an asymptotic formula for the number of representations in Theorems 1–2, nor do we get a bound of the form $\ll \frac{N}{(\log N)^A}$ for the exceptional set in Theorem 1 (but only $o(N)$).

Harmonic Analysis Side of the Problem

- The Hardy–Littlewood Circle Method
- Transference Principle
- Transference Type Result for Goldbach Problems
- Applying the Transference Type Result for Goldbach Problems

The Hardy–Littelwood Circle Method

- Consider, as Vinogradov did, the number of solutions to $N = p_1 + p_2 + p_3$ in primes. If $f_N(\alpha) = \sum_{p \leq N} e(\alpha p)$ is the exponential sum corresponding to primes, the number of solutions equals
$$I = \int_0^1 f_N(\alpha)^3 e(-N\alpha)\, d\alpha.$$
- The integrand of I is large when $\alpha \approx \frac{a}{q}$ with q small (that is, α lies on a major arc). The contribution of these can be calculated asymptotically, and is approximately $N^2(\log N)^{-3}$ for odd N.
- If α is not close to any $\frac{a}{q}$ with q small (the minor arc case), one can show that $f_N(\alpha) \ll \frac{N}{(\log N)^A}$, so that by Parseval's formula the minor arcs contribute to I
$$\ll \frac{N}{(\log N)^A} \int_0^1 |f_N(\alpha)|^2\, d\alpha \ll \frac{N^2}{(\log N)^A},$$
which is small enough.

Transference Principle

Green and Tao developed an alternative strategy for solving translation-invariant additive problems in the primes. This approach (in a much more elaborate form) led to the Green–Tao theorem on primes in arithmetic progressions.

Suppose that, for some $\delta > 0$, $A \subset [1, N]$ satisfies

(i) Pseudorandomness: There exists a function $\nu(n) \geq \delta^{-1} 1_A(n)$ such that ν is "pseudorandom";

(ii) Fourier uniformity: $\sum_{\xi \in \mathbb{Z}_N} |\delta^{-1} \widehat{1_A}(\xi)|^{\frac{5}{2}} \ll 1$;

(iii) Density: $|A| \geq \delta N$. Then A contains $\gg \delta^3 N^2$ 3-term APs.

- In applications we have $\delta \leq \frac{1}{\log N}$.
- The name transference principle comes from the fact that the problem is transferred to Roth's theorem.

Transference Type Result for Goldbach Problems

Matomäki and Shao (2015) formulated a version of the transference principle that works for Goldbach-type problems:

Suppose that, for some $\delta > 0$, $A \subset (\frac{N}{3}, \frac{2N}{3})$ satisfies
(i) Well-distribution in Bohr sets: For every Bohr set B "of finite complexity", $|A \cap B| \geq \delta |B|$;

(ii) Fourier uniformity: $\sum_{\xi \in \mathbb{Z}_N} |\delta^{-1} \widehat{1_A}(\xi)|^{\frac{5}{2}} \ll 1$;

(iii) Density: $|A| \geq \delta N$. Then there exist $\gg \delta^3 N^2$ triples

$a_1, a_2, a_3 \in A$ such that $N = a_1 + a_2 + a_3$.

- We require a version of this principle that works for binary problems (this is an easy modification).
- The traditional circle method demands proving equidistribution in Bohr sets, while the above needs only that every Bohr set contains a fair amount of the elements of A.

Applying the Transference Type Result for Goldbach Problems

- For the set $\mathscr{P}$ of primes of the form $x^2 + y^2 + 1$, the density condition (iii) follows from Iwaniec's work.
- The Fourier uniformity condition (ii) follows from the work of Ramaré and Rusza, together with the Green–Tao restriction theory for the primes.
- We are left with the Bohr set condition (i), which is the most central part of the proof. This is proved using sieve theory.

Sieve Theory Side of the Problem

- Applying the Sieves
- Bombieri-Vinogradov Estimates for Major and Minor Arcs
- The Level of Distribution

Applying the Sieves

- We want to estimate $\Sigma = \sum_{\substack{p\sim N \\ p\in\mathscr{P}}} B(p)$, where $B(\cdot)$ is (a smoothed version of) the characteristic function of a Bohr set (and $\mathscr{P}$ is restricted to the progression $3 \pmod 8$ for simplicity).
- Denoting $P(z) = \prod_{p<z} p$, $Q(z) = \prod_{\substack{2<p<z \\ p\equiv -1 \pmod 4}} p$, Buchstab's identity gives, for any z,

$$\Sigma = \sum_{\substack{p\sim N \\ (p-1,Q(N^{\frac{1}{2}}))=1}} B(p) = \sum_{\substack{p\sim N \\ (p-1,Q(z))=1}} B(p) - \sum_{\substack{p\sim N \\ z\le p_2<N^{1/2} \\ p-1=2p_2k \\ (k,Q(p_2))=1}} B(p) := \Sigma_1 - \Sigma_2.$$

- The semilinear sieve gives a lower bound of the form $\Sigma_1 \geq (A+o(1))N(\log N)^{-\frac{3}{2}}$.
- We need an upper bound of the form $\Sigma_2 \leq (B+o(1))N(\log N)^{-\frac{3}{2}}$, $B<A$.

Applying the Sieves

- Since every integer that is $\equiv 1 \pmod 4$ has an even number of prime factors that are $-1 \pmod 4$ (with multiplicities), we have

$$\Sigma_2 = \sum_{\substack{p\sim N \\ p-1=2p_1p_2m \\ z<p_2\le p_1 \\ q|m \Rightarrow q\equiv 1\,(4)}} B(p).$$

- Writing $\ell = p_1 m$, we obtain, for any w,

$$\Sigma_2 \le \sum_{\ell\in\mathcal{L}} \sum_{\substack{n\sim N \\ n=\ell p_2+1 \\ p_2\equiv -1\,(4) \\ (n,P(w))=1}} B(n) := \sum_{\ell\in\mathcal{L}} \Sigma_2(\ell),$$

  where $\mathcal{L}$ is some appropriate set.

- Estimating $\Sigma_2(\ell)$ can be dealt with the linear sieve, so we can give a good upper bound for it.

Bombieri-Vinogradov Estimates for Major and Minor Arcs

- We have obtained a lower bound for Σ_1 and an upper bound for Σ_2. Whether we get $\Sigma = \Sigma_1 - \Sigma_2 \gg N(\log N)^{-\frac{3}{2}}$ or not depends on how large $\rho_+, \rho_- \in (0, \frac{1}{2})$ we can take in the Bombieri-Vinogradov estimates

$$\sum_{\substack{d \leq N^{\rho_\pm} \\ (d,c)=1}} \lambda_d^{\pm} \Bigg(\sum_{\substack{p \sim N \\ p \equiv c \pmod d}} e(\alpha p) - \frac{1}{\varphi(d)} \sum_{p \sim N} e(\alpha p) \Bigg) \ll \frac{N}{(\log N)^{100}},$$

since $B(n)$ is a linear combination of terms of the form $e(\alpha n)$. Here λ_d^+ are the upper bound linear sieve weights and λ_d^- are the lower bound semilinear sieve weights (with suitable sieving limits).

Bombieri-Vinogradov Estimates for Major and Minor Arcs

- One considers the Bombieri-Vinogradov estimate separately for major and minor arcs.
- For major arcs, $\alpha \approx \frac{a}{q}$ with q small, so we can essentially reduce to the classical Bombieri-Vinogradov theorem and achieve $\rho_+ = \rho_- = \frac{1}{2} - \varepsilon$.
- For minor arcs, we use Vaughan's identity to decompose the sums over primes into type I and II averages

$$\sum_{\substack{d \leq N^{\rho_\pm} \\ (d,c)=1}} |\lambda_d^\pm| \Bigg| \sum_{\substack{mn \sim N \\ mn \equiv c \pmod{d} \\ m \sim M}} \alpha_m \beta_n e(\alpha mn) \Bigg|,$$

with $|\alpha_m|, |\beta_m| \leq 1$ (and the average is of type I if $\alpha_m \equiv 1$).

- How good estimates we get for these type I and II averages depends on the combinatorial properties of $\lambda_d^\pm$.

The Level of Distribution

- The linear sieve weights λ_d^+ are "well-factorable", so the results of earlier authors tell that $\rho_+ = \frac{1}{2} - \varepsilon$ is admissible (and the best one could hope for).
- The lower bound semilinear sieve weights λ_d^- are not well-factorable, so existing results would only permit $\rho_- = \frac{1}{3} - \varepsilon$, while we need $\rho_- \geq 0.385$ to succeed.
- However, one can study the combinatorial properties of λ_d^- to show that it is well-factorable in some specific range, giving us the exponent $\rho_- = \frac{3}{7} - \varepsilon = 0.428\ldots - \varepsilon$, which is good enough to prove the transference condition (i), and hence proves Theorems 1–2 via the transference result.
- Applying the same argument with $B(n)$ the characteristic function of $\{n : \ \|\alpha n + \beta\| < N^{-0.01}\}$ gives us Theorem 4.