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Compton scattering of polarized light: scattering matrix for isotropic electron gas

D.I. Nagirner[1] **and J. Poutanen**[2]

[1] Astronomical Observatory, St.Petersburg University, Petergof, St.Petersburg 198904, Russia
[2] Observatory and Astrophysics Laboratory, P.O.Box 14, SF-00014 University of Helsinki, Finland

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Abstract. The scattering matrix for the scattering of radiation by isotropic monoenergetic relativistic electron gas is found. The expression for the absorption coefficient by such gas is also given. The relativistic kinetic equation is obtained which describes the Compton scattering of radiation with arbitrary energy and polarization for random electron directions. The Compton scattering matrix (CSM) contains five redistribution functions (RFs) describing the redistribution of radiation on frequency and depending also on the scattering angle. The expressions for these functions are reduced to the forms free of cancellations.

Key words: polarization – scattering – radiative transfer

1. Introduction

Although a large number of papers has been devoted to the problem of interaction between the radiation and relativistic electrons (see e.g. Blumenthal & Gould 1970; Zel'dovich 1975; Illarionov et al. 1979; Lightman & Rybicki 1980; Sunyaev & Titarchuk 1980; Nishimura et al. 1986; Kershaw et al. 1986; Nagirner et al. 1991), Compton scattering of polarized light is studied not enough. In the limiting case of small frequencies and low temperatures of electron gas (in comparison with the rest electron energy $m_e c^2$) Sunyaev & Titarchuk (1985) reduced the problem of calculation of the spectral polarization to the calculation of Rayleigh scattering of different orders. That is possible because in the process of the multiple scattering, the photon frequency corresponds to the definite mean number of scattering and the scattering matrix is close to the Rayleigh matrix. The scattering of the high energy photon on the cool electron gas was considered by Williams (1984).

The attempts to take into account the effects of the induced scattering were made in some works. The results of Pomraning (1974) were criticized and corrected for linear polarization by Stark (1981). The Fokker-Planck equation for the two-dimensional radiative transfer equation (two Stokes parameters) was formulated there for the case of non-relativistic temperature of the Maxwellian electron gas. This equation is the generalization of the equations, where the Compton scattering was interpreted without including the polarization effects, namely the Kompaneets equation (Kompaneets 1956), where the radiation field is supposed to be isotropic and homogeneous, and the equation by Babuel-Peyrissac & Rouvillois (1969), where the dependence of intensity on coordinates and directions is admitted.

The scattering matrix for the case of the ultrarelativistic isotropic electron gas with a power energy spectrum was obtained by Bonometto et al. (1970).

In several works the scattering of polarized radiation on the electrons in magnetic field was considered (see, for example, Bussard et al. 1986). Naturally, it is necessary to investigate in some detail the simpler case of the Compton scattering of polarized light without magnetic field. In the present paper the relativistic kinetic equation which describes such a scattering is concretized for the isotropic electron distribution.

We suppose that the non-degenerate electron gas occupies some volume. We connect with the gas the co-moving frame of reference E and assume that in this frame the electron distribution is isotropic. Although, the kinetic equation we consider is relativistically covariant, all calculations are performed in the frame E. For simplicity we assume that macroscopic motions of the gas in the frame E are absent.

So far as we consider the photon scattering only by electrons, we use in calculations in the frame E the relativistic quantum system of units. In such a system the main units are the electron mass $m_e = m$, the speed of light and the Planck constant $m = c = \hbar = 1$. Other values are measured in this system in the following units: length in the Compton wavelength $\hbar/mc$, energy in mc^2, frequency in $mc^2/\hbar$. The electron charge is $e = \sqrt{e^2/\hbar c} = 1/\sqrt{137.036}$, the classical electron radius $r_0 = e^2/mc^2 = e^2/\hbar c = e^2 = 1/137.036$ is equal to the fine structure constant.

In Sect. 2 and 3 the description of photon and electron gases and the act of Compton scattering are given. Then, in Sect. 4 we

Send offprint requests to: J. Poutanen formulate the relativistic kinetic equation for Compton scattering. We average the scattering matrix over directions of electron momentum in the frame E in Sect. 5 and 6. The formulae for five redistribution functions are found in Sect. 7. The expressions for these functions were published in the paper by authors (Nagirner & Poutanen 1991). We consider limiting cases of the scattering matrix in Sect. 8.

The definitions and the notations of the vectors in the Minkowski space, the operations with them, the polarization bases and parameters describing the polarization of radiation are given in the Appendices A, B and C. The technical details of calculations and auxiliary quantities are given in the Appendices D and E.

2. Description of photons and electrons

We choose the arbitrary inertial system and ascribe to any point of the space-time the four-vector $\underline{r} = \{ct, \boldsymbol{r}\}$. In the same system the photon is described by the four-momentum $\underline{k} = \{k, \boldsymbol{k}\}$. In the quasi-classical approximation the photon has definite values of the coordinates $\boldsymbol{r}$ and momentum $\boldsymbol{k}$.

The polarization state of radiation is defined relative to two vectors of some polarization basis either by the polarization density matrix $\check{n}$ or by the Stokes parameters connected with each other by:

$$\check{n} = \begin{pmatrix} n_{(11)} & n_{(12)} \\ n_{(21)} & n_{(22)} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} n_{\rm I} + n_{\rm Q} & n_{\rm U} - i n_{\rm V} \\ n_{\rm U} + i n_{\rm V} & n_{\rm I} - n_{\rm Q} \end{pmatrix}. \tag{1}$$

Here $\tilde{n}, \tilde{n}^{\rm T} = (n_{\rm I}, n_{\rm Q}, n_{\rm U}, n_{\rm V})$ is the Stokes vector with the dimensionless and relativistically invariant components — Stokes parameters, which are related to the usual parameters — spectral intensities — by multiplying them by $c\,k^3/4\pi^2\hbar^2$. In the main part of our paper we use so defined Stokes parameters.

Let us choose in the frame E an orthogonal basis $\boldsymbol{l}_1, \boldsymbol{l}_2$, $\boldsymbol{l}_3 = \boldsymbol{l}$ and set $\boldsymbol{k} = x\,\boldsymbol{\omega}$. The direction of the photon propagation $\boldsymbol{\omega}$ is defined by the cosine $\eta = (\boldsymbol{\omega}\,\boldsymbol{l})$ and azimuth φ:

$$\boldsymbol{\omega} = \sqrt{1-\eta^2}(\cos\varphi\,\boldsymbol{l}_1 + \sin\varphi\,\boldsymbol{l}_2) + \eta\,\boldsymbol{l}. \tag{2}$$

For this photon we connect with the unit vector $\boldsymbol{l}$ the *external* polarization basis consisting of the four four-vectors $\underline{e}^{\rm ex}_{\mu}(\boldsymbol{k})$, which relativistic expressions are given by Eqs. (B6) in the Appendix B. The vectors of this basis $\underline{e}^{\rm ex}_{(1)}(\boldsymbol{k})$ and $\underline{e}^{\rm ex}_{(2)}(\boldsymbol{k})$ in the frame E have only the space-components — usual unit polarization vectors:

$$\boldsymbol{e}^{\rm ex}_1(\boldsymbol{\omega}) = (\boldsymbol{l} - \eta\boldsymbol{\omega})/\sqrt{1-\eta^2}, \quad \boldsymbol{e}^{\rm ex}_2(\boldsymbol{\omega}) = \boldsymbol{\omega} \times \boldsymbol{l}/\sqrt{1-\eta^2}. \tag{3}$$

The electron is characterized by the four-momentum $\underline{p} = \{p_0, \boldsymbol{p}\}$. In the frame E we set $\underline{p} = \{\gamma, z\boldsymbol{\Omega}\}$, $\gamma = \sqrt{1+z^2}$. Let the cosine of the angle between the $\boldsymbol{\omega}$ and $\boldsymbol{\Omega}$ be $\zeta = (\boldsymbol{\omega}\,\boldsymbol{\Omega})$.

The distribution function of the electron gas $f_{\rm e}(\boldsymbol{p})$ in momenta $\boldsymbol{p}$ is assumed to be relativistically invariant. We suppose that the distribution is isotropic in E, i.e. $f_{\rm e}(\boldsymbol{p}) = n_{\rm e}\,f_{\rm e}(\gamma)/m^3c^3$, where $n_{\rm e}$ is the electron density in E. The functions $f_{\rm e}(\boldsymbol{p})$ and $f_{\rm e}(\gamma)$ are normalized as

$$\int f_{\rm e}(\boldsymbol{p})\,\underline{p}\,\frac{{\rm d}^3p}{p_0} = n_{\rm e}\,\frac{\{1,\boldsymbol{\beta}\}}{\sqrt{1-\beta^2}},$$

$$4\pi\int_0^\infty z^2{\rm d}z\,f_{\rm e}(\gamma) = 1, \tag{4}$$

where $\boldsymbol{\beta}$ is the dimensionless (in units of the speed of light) velocity of an arbitrary frame of reference relative to E.

The polarization matrix and the vector of the Stokes parameters, as well as the distribution function of electrons, can depend on the space-coordinates and on time. But we will not point them out as arguments of these functions because their values in both sides of the kinetic equation are the same.

3. Description of the scattering act

We consider a scattering of a photon of four-momentum $\underline{k}$ by an electron of four-momentum $\underline{p}$. Let $\underline{k}_1$ and $\underline{p}_1$ be the four-momenta of the scattered particles. The conservation laws

$$\underline{p} + \underline{k} = \underline{p}_1 + \underline{k}_1 \tag{5}$$

give a possibility to eliminate the three-dimensional momentum of the scattered electron $\boldsymbol{p}_1 = \boldsymbol{p} + \boldsymbol{k} - \boldsymbol{k}_1$ from all expressions. From (5) we obtain

$$\underline{k}\,\underline{p} = \underline{k}_1\,\underline{p}_1, \quad \underline{k}_1\,\underline{p} = \underline{k}\,\underline{p}_1, \quad \underline{k}_1\,(\underline{p}+\underline{k}) = \underline{k}\,\underline{p}. \tag{6}$$

We can find the frequency of the scattered photon k_1 (Compton formula, see below) from the last equation in (6).

Let us introduce the designations for the scalar products of the electron momentum $\underline{p}$ and the photon momenta $\underline{k}$ and $\underline{k}_1$:

$$\xi = \frac{\underline{p}\,\underline{k}}{m^2c^2}, \quad \xi_1 = \frac{\underline{p}\,\underline{k}_1}{m^2c^2}, \quad q = \frac{\underline{k}\,\underline{k}_1}{m^2c^2} = \xi - \xi_1. \tag{7}$$

In the last relation we take into account Eqs. (6).

For the considered photons and electrons we introduce the *internal* basis $\underline{e}^{\rm in}_{(\lambda)}(\boldsymbol{k})$ (see Appendix B, Eq. (B8)). In this basis the tensor of rank four $\overset{\diamond}{\mathsf{T}}$ transforming the elements of the matrix $\check{n}$ due to the scattering of photons on the unpolarized electrons (in accordance with the rule in Eq. (C7)) has the simplest form (Berestetskii et al. 1971). Its non-zero elements are

$$\begin{aligned} T_{1111} &= [B+2+4A(A+2)]/2, \\ T_{1212} &= T_{2121} = (B-2)/2, \\ T_{2222} &= (B+2)/2, \\ T_{1122} &= T_{2211} = (B+2)(A+1)/2, \\ T_{1221} &= T_{2112} = -(B-2)(A+1)/2. \end{aligned} \tag{8}$$

Here we use the notations

$$A = 1/\xi - 1/\xi_1, \quad B = \xi/\xi_1 + \xi_1/\xi. \tag{9}$$

According to the formulae (C8) the elements of the matrix $\hat{F}$, which describe the transformation of the Stokes parameters due to the scattering in the same *internal* basis, are the following

$$\hat{F} = \begin{pmatrix} F & A(A+2) & 0 & 0 \\ A(A+2) & A(A+2)+2 & 0 & 0 \\ 0 & 0 & 2(A+1) & 0 \\ 0 & 0 & 0 & B(A+1) \end{pmatrix}, \quad (10)$$

where $F = A(A+2) + B$.

The photon with momentum $\boldsymbol{k}_1$ in system E is characterized by the same quantities as $\boldsymbol{k}$, but with a subscript 1: $\boldsymbol{\omega}_1, \eta_1, \varphi_1$ etc. Let $\mu = \boldsymbol{\omega}\, \boldsymbol{\omega}_1$ and $\zeta_1 = \boldsymbol{\Omega}\, \boldsymbol{\omega}_1$ be the cosines of the angles between the momenta of the scattered photon and the initial photon and the electron. The cosine of the photon scattering angle is

$$\mu = \eta\eta_1 + \sqrt{1-\eta^2}\sqrt{1-\eta_1^2}\cos(\varphi - \varphi_1). \quad (11)$$

In the system E the quantities ξ, ξ_1, q are given by

$$\xi = x(\gamma - z\zeta), \quad \xi_1 = x_1(\gamma - z\zeta_1), \quad q = xx_1(1-\mu). \quad (12)$$

The Compton formula for the frequency of the scattered photon can be easily obtained from (6) and (12):

$$x_1 = x\frac{\gamma - z\zeta}{\gamma - z\zeta_1 + x(1-\mu)}. \quad (13)$$

The vectors of the *internal* basis $\underline{e}^{\rm in}_{(1)}(\boldsymbol{k})$ and $\underline{e}^{\rm in}_{(2)}(\boldsymbol{k})$ in E, if the scattering is undergoing by the electron at rest, are reduced to the three-dimensional vectors (see Appendix B) which we denote as

$$\begin{aligned} \boldsymbol{e}_1(\boldsymbol{\omega}) &= (\boldsymbol{\omega}_1 - \mu\boldsymbol{\omega})/\sqrt{1-\mu^2}, \\ \boldsymbol{e}_2(\boldsymbol{\omega}) &= \boldsymbol{\omega} \times \boldsymbol{\omega}_1/\sqrt{1-\mu^2}. \end{aligned} \quad (14)$$

These two vectors together with $\boldsymbol{\omega}$ form the three-dimensional basis, which we name as the main basis. In this basis the direction of electron propagation is defined by cosine ζ and azimuth Φ:

$$\boldsymbol{\Omega} = \sqrt{1-\zeta^2}\,[\cos\Phi\, \boldsymbol{e}_1(\boldsymbol{\omega}) + \sin\Phi\, \boldsymbol{e}_2(\boldsymbol{\omega})] + \zeta\boldsymbol{\omega}. \quad (15)$$

The relation connecting various functions of angles between the vectors $\boldsymbol{\omega}, \boldsymbol{\omega}_1, \boldsymbol{\Omega}, \boldsymbol{l}$ and others, are given in the Appendix D. The formulae of transformations from one basis to another are presented in the Appendix E.

4. Kinetic equation

The relativistic kinetic equation describing the scattering of the polarized radiation by the electron gas in terms of the polarization density matrices in the linear approximation in the arbitrary frame of reference may be written as:

$$\begin{aligned} \underline{k}\,\underline{\nabla}\,\check{n} &= \frac{r_0^2}{2} m^2 c^2 \int \frac{{\rm d}^3p}{p_0}\frac{{\rm d}^3p_1}{p_{10}}\frac{{\rm d}^3k_1}{k_1}\delta(\underline{p} + \underline{k} - \underline{p}_1 - \underline{k}_1) \\ &\times \left[f_{\rm e}(\boldsymbol{p}_1)\check{n}_1^{\rm s} - f_{\rm e}(\boldsymbol{p})F\check{n}\right]. \end{aligned} \quad (16)$$

Here

$$\check{n}_1^{\rm s} = \check{f}(-\chi)\, \overset{\diamond}{\mathsf{T}}\, (\check{f}(\chi_1)\,\check{n}_1\,\check{f}(-\chi_1))\,\check{f}(\chi), \quad \check{n}_1 = \check{n}(\boldsymbol{k}_1), \quad (17)$$

the matrix $\check{f}$ is defined in (C11) and the angles χ and χ_1 in Eqs. (E4) and analogous for the photon $\underline{k}_1$. The four-dimensional gradient operator is the following $\underline{\nabla} = \left\{\frac{1}{c}\frac{\partial}{\partial t}, -\boldsymbol{\nabla}\right\}$. This equation can be derived from the quantum Liouville equation by the Bogolubov method in the manner by Silin (1971) for another problem. The same equation was obtained by other authors (e.g. Acquista & Anderson 1974).

Further, from (16) using (1), (C11) and (C12) we obtain the equation for the Stokes vector also in the linear approximation

$$\begin{aligned} \underline{k}\,\underline{\nabla}\,\tilde{n} &= \frac{r_0^2}{2} m^2 c^2 \int \frac{{\rm d}^3p}{p_0}\frac{{\rm d}^3p_1}{p_{10}}\frac{{\rm d}^3k_1}{k_1}\delta(\underline{p} + \underline{k} - \underline{p}_1 - \underline{k}_1) \\ &\times \left[f_{\rm e}(\boldsymbol{p}_1)\tilde{n}_1^{\rm s} - f_{\rm e}(\boldsymbol{p})F\tilde{n}\right], \end{aligned} \quad (18)$$

where $\tilde{n} = \tilde{n}(\boldsymbol{k})$, $\tilde{n}_1 = \tilde{n}(\boldsymbol{k}_1)$, $\hat{L}$ is given by Eq. (C12) and

$$\tilde{n}_1^{\rm s} = \hat{L}(-\chi)\,\hat{F}\,\hat{L}(\chi_1)\,\tilde{n}_1. \quad (19)$$

The δ-function in the integrand reflects the conservation laws. The matrices $\check{f}$ and $\hat{L}$ transform the polarization matrices and the Stokes parameters from the *external* polarization basis to the *internal* one and backward. In the last term in the square brackets of both Eqs. (16) and (18), the scalar quantity F is the Klein-Nishina cross-section which is not a matrix. This fact is the consequence of the isotropy of the medium.

By means of the designation

$$\begin{aligned} \hat{R}(\boldsymbol{k}_1 \to \boldsymbol{k}) = \frac{3}{16\pi}\frac{m^3c^3}{n_{\rm e}}\int \frac{{\rm d}^3p}{p_0}\frac{{\rm d}^3p_1}{p_{10}} \\ \delta(\underline{p}_1 + \underline{k}_1 - \underline{p} - \underline{k}) f_{\rm e}(\boldsymbol{p}_1)\hat{L}(-\chi)\hat{F}\hat{L}(\chi_1), \end{aligned} \quad (20)$$

we rewrite Eq. (18) in the following form

$$\underline{k}\,\underline{\nabla}\,\tilde{n} =$$

$$\frac{\sigma_0 n_{\rm e}}{mc}\int \frac{{\rm d}^3k_1}{k_1}\hat{R}(\boldsymbol{k}_1 \to \boldsymbol{k})\tilde{n}_1 - \tilde{n}\int \sigma(\xi)\,\xi\, f_{\rm e}(\boldsymbol{p})\frac{{\rm d}^3p}{p_0}, \quad (21)$$

where the total Compton cross-section is (cf. Berestetskii et al. 1971)

$$\begin{aligned} \sigma(\xi) &= \frac{r_0^2}{2}\frac{m^2c^2}{\underline{pk}}\int \frac{{\rm d}^3p_1}{p_{10}}\frac{{\rm d}^3k_1}{k_1}\delta(\underline{p} + \underline{k} - \underline{p}_1 - \underline{k}_1)\, F \\ &= \sigma_0\frac{3}{8\xi^2}\left[4 + \left(\xi - 2 - \frac{2}{\xi}\right)\ln(1+2\xi) + 2\xi^2\frac{1+\xi}{(1+2\xi)^2}\right] \end{aligned} \quad (22)$$

and $\sigma_0 = 8\pi r_0^2/3$ is the Thomson cross-section.

The three-dimensional integral in (20) over p disappears due to the δ-function. In the other quantities one can substitute $\boldsymbol{p} = \boldsymbol{p}_1 + \boldsymbol{k}_1 - \boldsymbol{k}$. Using (5) and the identity

$$\delta(p_{10} + k_1 - p_0 - k) = p_0\delta\left(\underline{k}_1(\underline{p} + \underline{k}) - \underline{k}\,\underline{p}\right), \quad (23)$$

we reduce (20) to the following

$$\hat{R}(\boldsymbol{k}_1 \to \boldsymbol{k}) = \frac{3}{16\pi}\frac{m^3c^3}{n_e}\int \frac{\mathrm{d}^3p_1}{p_{10}}\delta\left(\underline{k}(\underline{p}_1+\underline{k}_1)-\underline{k}_1\underline{p}_1\right) \times f_e(\boldsymbol{p}_1)\hat{L}(-\chi)\hat{F}\hat{L}(\chi_1). \quad (24)$$

Now, let us pass to the frame E and take into account the isotropy of the electron distribution. Then we can write Eq. (21) in the usual form of the radiative transfer equation:

$$(\frac{\partial}{\partial t}+\boldsymbol{\omega}\boldsymbol{\nabla})\tilde{n} = -n_e\,\overline{\sigma}(x)\tilde{n} + \frac{\sigma_0 n_e}{x}\int_0^\infty x_1\mathrm{d}x_1\int \mathrm{d}^2\omega_1\hat{R}(x_1,\boldsymbol{\omega}_1\to x,\boldsymbol{\omega})\tilde{n}_1, \quad (25)$$

where the scattering cross-section is (Nagirner et al. 1991)

$$\begin{aligned}\overline{\sigma}(x) &= \frac{1}{x}\int \sigma(\xi)\,\xi\, f_e(\gamma)\frac{\mathrm{d}^3z}{\gamma}\\ &= \sigma_0\frac{3\pi}{4x^2}\int_1^\infty f_e(\gamma)\mathrm{d}\gamma\left[\left(x\gamma+\frac{9}{2}+\frac{2}{x}\gamma\right)\ln\frac{1+2x(\gamma+z)}{1+2x(\gamma-z)}\right.\\ &- 2xz + z\left(x-\frac{2}{x}\right)\ln(1+4x\gamma+4x^2)\\ &+ \left.4x^2z\frac{\gamma+x}{1+4x\gamma+4x^2}-2\int_{x(\gamma-z)}^{x(\gamma+z)}\ln(1+2\xi)\frac{\mathrm{d}\xi}{\xi}\right]. \quad (26)\end{aligned}$$

The scattering matrix (24) in the frame E takes the form:

$$\hat{R}(x_1,\boldsymbol{\omega}_1\to x,\boldsymbol{\omega}) = \frac{3}{16\pi}\int\frac{\mathrm{d}^3z_1}{\gamma_1}f_e(\gamma_1)\hat{L}(-\chi)\hat{F}\hat{L}(\chi_1)$$

$$\delta\left(x\left(\gamma_1+x_1-z_1\,\boldsymbol{\omega}\boldsymbol{\Omega}_1-x_1\,\boldsymbol{\omega}_1\,\boldsymbol{\omega}\right)-x_1\left(\gamma_1-z_1\,\boldsymbol{\omega}_1\boldsymbol{\Omega}_1\right)\right). \quad (27)$$

Next we want to reduce this triple integral to a single one through the calculation of the integrals over the directions of unit vector $\boldsymbol{\Omega}_1$, i.e. on azimuth Φ_1 and cosine ζ_1.

5. Calculation of the integral over the azimuth

In Sect. 3 we considered the scattering act $\boldsymbol{k},\boldsymbol{p}\to\boldsymbol{k}_1,\boldsymbol{p}_1$. All bases and angles were introduced for this process. The scattering matrix (27) as the whole *incoming* term (the first term in the right hand side of the kinetic equation (21)) corresponds to the process $\boldsymbol{k}_1,\boldsymbol{p}_1\to\boldsymbol{k},\boldsymbol{p}$. But this fact does not demand large changing of our formulae. We can not change the vectors of the *external* basis because the angles χ and χ_1 enter through the transformation matrices in a definite order. At the same time, we can change the order of the electron momenta by making the change $\underline{p}\leftrightarrow\underline{p}_1=\underline{p}+\underline{k}_1-\underline{k}$. If we make this change without changing the photon momenta, the quantities ξ and ξ_1 will be replaced by $\xi-q$ and ξ_1+q, respectively, i.e. ξ and ξ_1 will exchange their places. The last equation in (7) can be rewritten in the form $\xi_1=\xi+q$. The corresponding changes in the matrix (10) will be limited by changing the sign of the quantity A. But the vectors of the *internal* basis (as can be seen from the Eqs. (B8) of the Appendix B) stay invariable.

Keeping in mind all what was mentioned above, we rewrite the three-dimensional integral in (27) in new variables and substitute instead of scalar products in the argument of δ-function their explicit expressions through the introduced angles. We thus arrive at the following expression:

$$\hat{R}(x_1,\boldsymbol{\omega}_1\to x,\boldsymbol{\omega}) = \frac{3}{16\pi}\int_0^\infty\frac{z^2\mathrm{d}z}{\gamma}f_e(\gamma)$$

$$\int_{-1}^1\mathrm{d}\zeta\int_0^{2\pi}\mathrm{d}\Phi\hat{L}(-\chi)\hat{F}\hat{L}(\chi_1)\delta(\Gamma). \quad (28)$$

Here we have introduced the quantity

$$\Gamma = \gamma(x_1-x)-q-z\,(x_1\zeta_1-x\zeta), \quad (29)$$

where

$$\zeta_1 = \boldsymbol{\Omega}\,\boldsymbol{\omega}_1 = \zeta\mu+\sqrt{1-\zeta^2}\sqrt{1-\mu^2}\cos\Phi.$$

Our next task is to calculate the integrals over ζ and Φ. We shall make this in two steps.

At first we calculate the integral over the azimuth Φ of the direction of the electron momentum. The matrix in the integrand is

$$\hat{L}(-\chi)\hat{F}\hat{L}(\chi_1) = \begin{pmatrix} F & F_c' & F_s' & 0\\ F_c & F_{c+} & F_{s-} & 0\\ F_s & F_{s+} & F_{c-} & 0\\ 0 & 0 & 0 & B(A+1)\end{pmatrix}, \quad (30)$$

where

$$\begin{aligned}F_c &= A(A+2)\cos 2\chi,\\ F_s &= A(A+2)\sin 2\chi,\\ F_c' &= A(A+2)\cos 2\chi_1,\\ F_s' &= A(A+2)\sin 2\chi_1, \quad (31)\\ F_{c\pm} &= \pm\frac{A^2}{2}\cos 2(\chi+\chi_1)+\frac{(A+2)^2}{2}\cos 2(\chi-\chi_1),\\ F_{s\pm} &= \frac{A^2}{2}\sin 2(\chi+\chi_1)\pm\frac{(A+2)^2}{2}\sin 2(\chi-\chi_1).\end{aligned}$$

Here the cosines and sines depend on Φ through the quantity Ψ: $\chi=\chi^0-\Psi$, $\chi_1=\chi_1^0+\Psi$ and $\chi-\chi_1=\chi^0-\chi_1^0-2\Psi$. This quantity is defined in the Appendix E by the formulae (E8).

To calculate the integral over Φ, we note the following. The argument of the δ-function becomes equal to zero if $\cos\Phi=\cos\Phi_0$, where

$$\begin{aligned}\cos\Phi_0 &= \frac{x_1(\gamma-z\zeta\mu)-x(\gamma-z\zeta)-q}{x_1z\sqrt{1-\zeta^2}\sqrt{1-\mu^2}}\\ &= \frac{(x_1\mu-x)\xi+q(\gamma-x)}{qz\sqrt{1-\zeta^2}\sqrt{r}} \quad (32)\end{aligned}$$

and $r=(1+\mu)/(1-\mu)$. The integral (28) is not equal to zero if the absolute value of the $\cos\Phi_0$ defined by (32) is not larger than 1, or

$$\begin{aligned}D^2 &= z^2(1-\zeta^2)r\\ &- \frac{1}{x_1^2(1-\mu)^2}[x_1(\gamma-z\zeta\mu)-x(\gamma-z\zeta)-q]^2\geq 0. \quad (33)\end{aligned}$$

This inequality demands the value ζ to be between the roots of D^2, i.e. $\zeta_- \le \zeta \le \zeta_+$, where

$$\begin{aligned}\zeta_\pm &= \frac{[\gamma(x_1 - x) - q](x_1\mu - x) \pm x_1(1-\mu)b}{zQ^2},\\ b &= \sqrt{r}\sqrt{z^2Q^2 - [\gamma(x_1-x) - q]^2},\\ Q^2 &= (x - x_1)^2 + 2q.\end{aligned} \tag{34}$$

For the existence of the real roots $\zeta_\pm$ of D^2, the satisfaction of the condition

$$z^2Q^2 - [\gamma(x_1 - x) - q]^2 \ge 0 \tag{35}$$

is necessary. If this inequality satisfies, it is easy to prove that $-1 \le \zeta_- \le \zeta_+ \le 1$.

Taking into account these conditions, one can find two values of the argument Φ in the interval $[0, 2\pi]$ where Γ vanishes: $\Phi = \Phi_0 = \arccos\cos\Phi_0$ and $\Phi = 2\pi - \Phi_0$. The values of $\sin\Phi$ in these points differs in sign $\sin\Phi = \pm\sin\Phi_0$, and

$$\sin\Phi_0 = \frac{D}{z\sqrt{1-\zeta^2}\sqrt{r}} \ge 0. \tag{36}$$

Owing to these facts, the integrals over Φ from all quantities proportional to $\cos\Phi$ must be taken into account twice, but those proportional to $\sin\Phi$ enter with opposite signs and annihilate. It is easy to calculate the integral over Φ of the δ-function:

$$\int_0^{2\pi} \delta(\Gamma)\,\mathrm{d}\Phi = \frac{2}{\sin\Phi_0 z x_1\sqrt{1-\zeta^2}\sqrt{1-\mu^2}} = \frac{2x}{Dq}. \tag{37}$$

After calculating integrals over Φ, the structure of the matrix $\hat{L}(-\chi)\hat{F}\hat{L}(\chi_1)$ simplifies immensely. Indeed,

$$\int_0^{2\pi} \delta(\Gamma)\hat{L}(-\chi)\hat{F}\hat{L}(\chi_1)\mathrm{d}\Phi = \frac{2x}{Dq}\hat{L}(-\chi^0) < \hat{F} > \hat{L}(\chi_1^0), \tag{38}$$

where

$$< \hat{F} >= \begin{pmatrix} F & F_{12} & 0 & 0 \\ F_{21} & F_+ & 0 & 0 \\ 0 & 0 & F_- & 0 \\ 0 & 0 & 0 & B(A+1) \end{pmatrix}. \tag{39}$$

Here we denote

$$\begin{aligned}F_\pm &= \frac{(A+2)^2}{2}\cos 4\Psi_0 \pm \frac{A^2}{2},\\ F_{12} &= F_{21} = A(A+2)\cos 2\Psi_0,\end{aligned} \tag{40}$$

and Ψ_0 is the value of Ψ for $\Phi = \Phi_0$. Thus the matrix $\hat{R}$ can be presented in the form

$$\begin{aligned}&\hat{R}(x_1, \boldsymbol{\omega}_1 \to x, \boldsymbol{\omega})\\ &= \frac{3}{8}\hat{L}(-\chi^0)\int_{\gamma_*}^{\infty} f_\mathrm{e}(\gamma)\mathrm{d}\gamma\hat{R}(x, x_1, \mu, \gamma)\hat{L}(\chi_1^0),\end{aligned} \tag{41}$$

where γ_* is the root of the left hand side of the inequality (35), which is equivalent to

$$\gamma \ge \gamma_* = \gamma_*(x, x_1, \mu) = \left[x - x_1 + Q\sqrt{1 + 2/q}\right]/2, \tag{42}$$

and

$$\hat{R}(x, x_1, \mu, \gamma) = \frac{zx}{\pi q}\int_{\zeta_-}^{\zeta_+}\frac{\mathrm{d}\zeta}{D} < \hat{F} > . \tag{43}$$

We have one integral left to calculate, namely the integration over ζ. This will be done in the next section.

6. Integration over ζ

We show that the variable ζ enters the integrand in (43) only through $\xi = x(\gamma - z\zeta)$ and $\xi_1 = \xi + q$. Indeed, quantities A and B are expressed by Eqs. (9) through ξ and ξ_1 (note that we must take A with the opposite sign). Further, D^2 depends on ξ and ξ_1 as follows:

$$\begin{aligned}D^2 &= -\frac{Q^2}{q^2}\xi^2 + \frac{2d}{q}\xi - a^2\\ &= -\frac{Q^2}{q^2}\xi_1^2 + \frac{2d_1}{q}\xi_1 - a_1^2\\ &= \frac{a_1^2 - a^2}{2q}(\xi + \xi_1) - \frac{Q^2}{q^2}\xi\xi_1 - \frac{a^2 + a_1^2}{2},\end{aligned} \tag{44}$$

where

$$\begin{aligned}a^2 &= (\gamma - x)^2 + r,\ a_1^2 = (\gamma + x_1)^2 + r,\\ d &= (a_1^2 - a^2 - Q^2)/2,\ d_1 = d + Q^2.\end{aligned} \tag{45}$$

Therefore all elements of the matrix $< \hat{F} >$ can be written as a sum of terms involving either ξ or ξ_1. It is very simple to express in such a manner the corner elements on the principal diagonal:

$$F = 2 + \frac{q^2 - 2q - 2}{q}\left(\frac{1}{\xi} - \frac{1}{\xi_1}\right) + \frac{1}{\xi^2} + \frac{1}{\xi_1^2}, \tag{46}$$

$$B(A+1) = 2 + q\left(\frac{1}{\xi} - \frac{1}{\xi_1} - \frac{1}{\xi^2} - \frac{1}{\xi_1^2}\right). \tag{47}$$

Now, using Eqs. (36), (E8) we find the expression for $\sin\Psi_0$:

$$\sin\Psi_0 = \pm\sqrt{-\frac{A}{A+2}\frac{D}{\sqrt{r}}}. \tag{48}$$

Other elements of matrix (39) containing $\cos 2\Psi_0$ and $\cos 4\Psi_0$ can be easily obtained from (44) and (48):

$$\begin{aligned}A(A+2)\cos 2\Psi_0 &= \frac{1}{r}\left[\frac{r - 2a^2}{\xi^2} + \frac{r - 2a_1^2}{\xi_1^2}\right.\\ &\left. - 2\frac{rq + r + Q^2 - a^2 - a_1^2}{\xi\xi_1}\right],\end{aligned} \tag{49}$$

$$\begin{aligned}(A+2)^2\cos^2 2\Psi_0 &= \frac{1}{r^2}\left[\frac{4}{q^2}(rq + Q^2)^2 + \frac{(r - 2a^2)^2}{\xi^2}\right.\\ &+ \frac{(r - 2a_1^2)^2}{\xi_1^2} - 4\frac{rq + Q^2}{q}\left(\frac{r - 2a^2}{\xi} - \frac{r - 2a_1^2}{\xi_1}\right)\\ &\left. - 2\frac{(r - 2a^2)(r - 2a_1^2)}{\xi\xi_1}\right].\end{aligned} \tag{50}$$

Next we choose a new variable of integration. The quantity D, which is in the denominator in the integral (43), involves under a square root the quadratic function on ξ. The limits of variation of ξ can be obtained from (12) and (34). At these limits the quantity D becomes zero. Therefore, it is naturally to do the trigonometric substitution

$$\xi = x(\gamma - z\zeta) = \frac{q}{Q^2}(d + b\cos\alpha). \tag{51}$$

The quantity ξ_1 depends on $\cos\alpha$ as

$$\xi_1 = \frac{q}{Q^2}(d_1 + b\cos\alpha). \tag{52}$$

Then we write (43) as follows

$$\hat{R}(x, x_1, \mu, \gamma) = \frac{1}{\pi Q}\int_0^\pi < \hat{F} > \mathrm{d}\alpha. \tag{53}$$

Furthermore, we use the following equations to integrate over α:

$$\int_0^\pi \frac{\mathrm{d}\alpha}{\xi} = \frac{\pi Q}{aq} \quad \text{and} \quad \int_0^\pi \frac{\mathrm{d}\alpha}{\xi^2} = \frac{\pi Q}{a^3 q^2}d. \tag{54}$$

The integrals for ξ_1 can be obtained from (54) adding the subscript 1 to ξ, a and d.

7. Redistribution functions (RFs)

After the integration over α, we obtain from (53)

$$\hat{R}(x, x_1, \mu, \gamma) = \begin{pmatrix} R & R_\mathrm{I} & 0 & 0 \\ R_\mathrm{I} & R_\mathrm{Q} & 0 & 0 \\ 0 & 0 & R_\mathrm{U} & 0 \\ 0 & 0 & 0 & R_\mathrm{V} \end{pmatrix}, \tag{55}$$

where we introduce five functions as follows:

$$R = \frac{2}{Q} + \frac{q^2 - 2q - 2}{q^2}\left(\frac{1}{a} - \frac{1}{a_1}\right) + \frac{1}{q^2}\left(\frac{d}{a^3} + \frac{d_1}{a_1^3}\right), \tag{56a}$$

$$\begin{aligned} R_\mathrm{I} &= -\frac{1}{rq^2}\left\{\frac{2a^2 - r}{a^3}d + \frac{2a_1^2 - r}{a_1^3}d_1 \right. \\ &\left. + 2\left[Q^2 + r(q+1) - a^2 - a_1^2\right]\left(\frac{1}{a} - \frac{1}{a_1}\right)\right\}, \end{aligned} \tag{56b}$$

$$\begin{aligned} R_\mathrm{U} &= \frac{4}{r^2q^2}\left\{Q^3 + 2Qrq + \frac{r^2q^2}{2Q} + 2(Q^2 + rq)(a - a_1) \right. \\ &+ \left(\frac{1}{a} - \frac{1}{a_1}\right)\left[r(a^2 + a_1^2 - Q^2) - 2a^2a_1^2 - \frac{r^2q}{2}\right] \\ &\left. + \frac{a^2 - r}{a}d + \frac{a_1^2 - r}{a_1}d_1\right\}, \end{aligned} \tag{56c}$$

$$R_\mathrm{Q} = R_\mathrm{U} + \frac{1}{q^2}\left(\frac{d}{a^3} + \frac{d_1}{a_1^3} - \frac{2}{a} + \frac{2}{a_1}\right), \tag{56d}$$

$$R_\mathrm{V} = \frac{2}{Q} + \frac{1}{a} - \frac{1}{a_1} - \frac{1}{q}\left(\frac{d}{a^3} + \frac{d_1}{a_1^3}\right). \tag{56e}$$

Function R was obtained by Aharonian & Atoyan (1981). It is the RF of intensity and has been studied in detail in the works by Kershaw et al. (1986) and Nagirner et al. (1991).

The expressions (56) are useful to obtain various asymptotic and approximate formulae. However, when x and x_1 are small the values a and a_1 are close to each other. Then the loss of accuracy appears in the differences $a - a_1$, $1/a - 1/a_1$ etc. This happens also when μ is close to 1. To prevent this, it is convenient to use different forms of the functions (56). Let

$$u = a_1 - a = \frac{(x + x_1)(2\gamma + x_1 - x)}{a + a_1}, \quad v = aa_1 \tag{57}$$

and

$$\begin{aligned} R_\mathrm{a} &= u\frac{(u^2 - Q^2)(u^2 + 5v)}{2q^2v^3} + u\frac{Q^2}{q^2v^2}, \\ R_\mathrm{b} &= \frac{2}{Q} + \frac{u}{v}\left(1 - \frac{2}{q}\right), \\ R_\mathrm{c} &= \frac{u}{vq}\left(\frac{u^2 - Q^2}{rq} - 2\right). \end{aligned} \tag{58}$$

Using (57) and (58) we obtain

$$\begin{aligned} R &= R_\mathrm{a} + R_\mathrm{b}, \\ R_\mathrm{I} &= R_\mathrm{a} + R_\mathrm{c}, \\ R_\mathrm{U} &= \frac{2}{Q} + 2\frac{u - Q}{rq}\left[\frac{u - Q}{rq}(2Q + u) - 4\right] + \frac{2u}{vq} + 2R_\mathrm{c}, \\ R_\mathrm{Q} &= R_\mathrm{U} + R_\mathrm{a}, \\ R_\mathrm{V} &= R_\mathrm{b} - qR_\mathrm{a}. \end{aligned} \tag{59}$$

Another loss of accuracy will arise when calculating $u - Q$ if γ is close to γ_*. In order to avoid this, we can use the following formulae

$$\begin{aligned} u^2 - Q^2 &= 2rqCD_\mathrm{u}, \\ D_\mathrm{u} &= (\gamma + x_1 - x + \gamma_*)(\gamma - \gamma_*), \\ C &= 2/[\gamma(\gamma + x_1 - x) + r + xx_1\mu + v]. \end{aligned} \tag{60}$$

8. Boundaries and limiting cases

As it was mentioned above, the integrals over azimuth Φ differ from zero (and, therefore, RFs are not zero) only if the inequality (35) satisfies. The limits for the quantities, which RFs depend on, follow from this inequality. These limitations have different forms, depending on what values are fixed. If we fix x, x_1 and μ, then from (35) we have $\gamma \geq \gamma_*(x, x_1, \mu)$ (see Eqs. (41), (42)).

If we calculate RFs (56) for the fixed electron energy $\gamma \geq 1$, then for the fixed initial photon frequency x_1 and the scattering angle arccos μ the frequency of the scattered photon is limited by the roots of Eq. (35): $x_- \leq x \leq x_+$, where

$$x_\pm = x_1\frac{\mu + \gamma(\gamma + x_1)w \pm z\,w\,a_1}{1 + 2\,\gamma\,x_1\,w + x_1^2w^2}, \tag{61}$$

and $w = 1 - \mu$. If the frequency of the scattered photon is fixed, we can obtain the limitations for the frequency of the initial photon, but they are more complex, because the coefficient at x_1^2 in Eq. (35) can be positive and negative. Namely, for $0 \le x\,w \le \gamma - z$ the inequalities $x_1^- \le x_1 \le x_1^+$ must satisfy, where

$$x_1^{\pm} = x\,\frac{\mu + \gamma(\gamma - x)w \pm z\,w\,a}{1 - 2\,\gamma\,x\,w + x^2 w^2}. \qquad (62)$$

If $x\,w$ is limited by $\gamma \pm z$, then $x_1 > x_1^-$.

At last, for the fixed x, x_1, γ values of μ do not fill out the whole interval $[-1,\ 1]$ and $\mu_{\rm m} \le \mu \le \mu_+$, where

$$\mu_{\rm m} = \begin{cases} -1, & |x - x_1| \ge 2\,x\,x_1, \\ -1, & |x - x_1| \le 2\,x\,x_1,\ \ \gamma \ge \gamma_*(x, x_1, -1), \\ \mu_-, & |x - x_1| \le 2\,x\,x_1,\ \ \gamma_{\rm min} \le \gamma \le \gamma_*(x, x_1, -1). \end{cases} \qquad (63)$$

Quantities $\mu_\pm$ are roots of the equation $\gamma_*(x, x_1, \mu) = \gamma$:

$$\mu_- = 1 - \frac{D_{\rm m}}{x\,x_1}, \qquad \mu_+ = 1 - \frac{(x - x_1)^2}{D_{\rm m}}, \qquad (64)$$

where

$$D_{\rm m} = z^2 + \gamma\,(x_1 - x) + z\,\sqrt{z^2 + 2\,\gamma\,(x_1 - x) + (x_1 - x)^2}. \qquad (65)$$

Let us present here the expressions for RFs in such cases, when all operations of Sect. 5 and 6 can not be performed, namely when the co-multiplier of $\cos\Phi$ in the argument of δ-function in Eq. (28) is zero. Then the integral over Φ is equal to $2\,\pi$ and we must calculate the integral over ζ in the other way. We do not present the calculation and give at once the results.

For the case of the scattering by immovable electrons

$$\hat{R}(x, x_1, \mu, \gamma)/z|_{\gamma=1} = 2\delta(x_1 - x - q) \times \begin{pmatrix} \mu^2 - 1 + B & \mu^2 - 1 & 0 & 0 \\ \mu^2 - 1 & \mu^2 + 1 & 0 & 0 \\ 0 & 0 & 2\mu & 0 \\ 0 & 0 & 0 & B\mu \end{pmatrix}, \qquad (66)$$

where $B = 1 - xw + 1/(1 - xw)$. In the Thomson limit ($x \ll 1$) $B = 2$ and matrix $\hat{R}$ is proportional to the Rayleigh matrix. If we pass to the limit $x \to 0$, $\gamma \to 1$ in the expressions (56) – (58) we obtain that the limiting value of $\hat{R}$ is equal to the Rayleigh matrix devided by $x\sqrt{2w}$.

If the energy of the initial photon is formally zero, then the frequency of the scattered photon is zero as well, and the scattering matrix takes the form:

$$\hat{R}(x, 0, \mu, \gamma) = 2\delta(x) \begin{pmatrix} R^{\rm x} & R_{\rm I}^{\rm x} & 0 & 0 \\ R_{\rm I}^{\rm x} & R_{\rm Q}^{\rm x} & 0 & 0 \\ 0 & 0 & R_{\rm U}^{\rm x} & 0 \\ 0 & 0 & 0 & R_{\rm V}^{\rm x} \end{pmatrix}, \qquad (67)$$

where

$$\begin{aligned} R^{\rm x} &= 2\ln(\gamma + z) + w\gamma z[w(\gamma^2 + z^2) - 2], \\ R_{\rm I}^{\rm x} &= -(1 - \mu^2)\gamma z, \\ R_{\rm U}^{\rm x} &= [-3\mu^2 + 2\mu + 3]\ln(\gamma + z) + R_{\rm A}^{\rm x}, \\ R_{\rm Q}^{\rm x} &= 2[2\mu^2 - 1]\ln(\gamma + z) - R_{\rm A}^{\rm x}, \\ R_{\rm V}^{\rm x} &= 2\ln(\gamma + z) - 2w\gamma z, \end{aligned} \qquad (68)$$

and $R_{\rm A}^{\rm x} = \gamma z[2\mu^2 + 2\mu - 4 - w^2(2\gamma^2 - 5)/3]$.

The scattering forward is the special case as well, because the photon frequency does not change:

$$\hat{R}(x, x_1, 1, \gamma) = 4\delta(x - x_1)\ln(\gamma + z). \qquad (69)$$

The scattering matrix in that case degenerates to unit matrix. At last, for $\mu \to -1$ all functions are continuous, function $R_{\rm I}(x, x_1, -1, \gamma) = 0$ and other RFs have finite limits. Formulae (66) – (69) are consistent.

9. Conclusions

We have obtained the analytical expressions for the Compton scattering matrix for an isotropic distribution of the monoenergetic electrons (Eqs. (55) and (56)). In other words, the Klein-Nishina scattering matrix averaged over an isotropic electron distribution is analytically reduced to a single integral over the electron energy (Eq. (41)). The expression for the absorption coefficient is also given (formula (26)).

For the particular case of the Maxwellian electron distribution, methods to compute integrals of type (41) for the function R (see (56a)) are proposed in the works of Kershaw et al. (1986) and Nagirner & Poutanen (1993). Using the results of the present paper and the works mentioned above, it is possible to compute CSM with arbitrary values of x, x_1 and μ. Such calculations are performed by Poutanen (1993). In his work various expansions of the redistribution functions are obtained. Differences between exact CSM and Rayleigh matrix were studied by Poutanen & Vilhu (1993). They have shown that CSM predicts a much smaller degree of polarization for high electron temperature ($kT_{\rm e} > 50$ keV) than the Rayleigh matrix.

For the power-law distribution of ultrarelativistic electrons, the scattering matrix was deduced and investigated by Bonometto et al. (1970). The same results can be easily obtained by expanding (56) in a power series in $1/\gamma$ if $\gamma \gg 1, \gamma \gg x \gg x_1$ and $xx_1 \ll 1$ and integrating over the electron distribution in Eq. (41). Let us show it. We assume that the electron distribution is

$$f_{\rm e}(\gamma) = c_{\rm e} \begin{cases} \gamma^{-1-p}/z, & 1 \ll \gamma_{\rm min} \le \gamma, \\ 0, & \gamma < \gamma_{\rm min}, \end{cases} \qquad (70)$$

where the normalization constant $c_{\rm e}$ can be easily found from Eq. (4). Denoting

$$\hat{S}(x, x_1, \mu) = \frac{3}{8}\int_{\gamma_*}^{\infty} f_{\rm e}(\gamma)\mathrm{d}\gamma \hat{R}(x, x_1, \mu, \gamma), \qquad (71)$$

where now $\gamma_* = \left(2x_1 w/x\right)^{-1/2}$, we obtain (matrices $\hat{S}$ and $\hat{R}$ have the same structure)

$$\begin{aligned} S &= P\left(\frac{1}{p+1} - \frac{2\varepsilon}{p+3} + \frac{2\varepsilon^2}{p+5}\right), \\ S_{\rm I} &= 0, \\ S_{\rm Q} &= -S_{\rm U} = P\frac{\varepsilon^2}{p+5}, \\ S_{\rm V} &= P\left(\frac{1}{p+1} - \frac{2\varepsilon}{p+3}\right). \end{aligned} \qquad (72)$$

Here

$$P = \frac{3c_e}{4x}\left(\max\{\gamma_*, \gamma_{\min}\}\right)^{-(p+1)}, \quad \varepsilon = \min\left\{1, \gamma_*^2/\gamma_{\min}^2\right\}. \quad (73)$$

The solution of the kinetic equation for the Compton scattering (see Eq. (25)) will give rich possibilities to interpretations of X-ray and gamma-ray spectra of various astrophysical objects. However the methods to solve this equation in the general case are not developed yet. Only individual attempts have been made so far.

Nevertheless, using the redistribution functions above, it is possible to compute immediately the polarization of the radiation of optically thin objects in the single scattering approximation. Such computations were made for a 2-phase model of accretion disc by Poutanen & Vilhu (1993).

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Appendix A: some geometrical relations in the Minkowski space

In this appendix we introduce a set of the designations for the four-dimentional quantities and ascertain the relations between them.

Instead of the usual enumeration of the contravariant a^μ or covariant $a_\mu (\mu = 0, 1, 2, 3)$ coordinates of the four-vector $\underline{a}$, we will write $\underline{a} = \{a_0, \boldsymbol{a}\}$, where $a_0 = a^0$ and three-dimensional vector $\boldsymbol{a} = (a^1, a^2, a^3) = -(a_1, a_2, a_3)$. Then the scalar product of two vectors $\underline{a}$ and $\underline{b}$ is $\underline{a}\,\underline{b} = a^\mu b_\mu = a_\mu b^\mu = a_0 b_0 - \boldsymbol{ab}$.

In the frame of reference moving with dimensionless velocity $\boldsymbol{\beta}$ (in units of the speed of light c) relative to the initial frame with the same directions of the coordinate axes the components of the vector $\underline{a}$ are defined according to the Lorentz transformation:

$$a_0' = \frac{a_0 - \boldsymbol{\beta a}}{\sqrt{1-\beta^2}},$$

$$\boldsymbol{a}' = \frac{\boldsymbol{a} - a_0\boldsymbol{\beta}}{\sqrt{1-\beta^2}} + \left(\frac{1}{\sqrt{1-\beta^2}} - 1\right)\frac{\boldsymbol{\beta}\times(\boldsymbol{\beta}\times\boldsymbol{a})}{\beta^2}. \quad (A1)$$

Let us define the vector product of three vectors $\underline{a}, \underline{b}$ and $\underline{c}$ by equation

$$\underline{a}\times\underline{b}\times\underline{c} = \{(\boldsymbol{a}\,\boldsymbol{b}\,\boldsymbol{c}), a_0\boldsymbol{b}\times\boldsymbol{c} + b_0\boldsymbol{c}\times\boldsymbol{a} + c_0\boldsymbol{a}\times\boldsymbol{b}\}. \quad (A2)$$

It is the alternative expressions for the components of the pseudovector $e^{\nu\rho\tau\mu}a_\nu b_\rho c_\tau$, where $e^{\nu\rho\tau\mu}$ is completely antisymmetric unit tensor (cf. Landau & Lifshits 1951). It is easy to prove two identities for this product. The first one expresses the scalar product of two vector products in terms of the determinant of the dyad matrix:

$$(\underline{a}\times\underline{b}\times\underline{c})\cdot(\underline{a}_1\times\underline{b}_1\times\underline{c}_1) = -\det\begin{pmatrix}\underline{a}\,\underline{a}_1 & \underline{a}\,\underline{b}_1 & \underline{a}\,\underline{c}_1\\ \underline{b}\,\underline{a}_1 & \underline{b}\,\underline{b}_1 & \underline{b}\,\underline{c}_1\\ \underline{c}\,\underline{a}_1 & \underline{c}\,\underline{b}_1 & \underline{c}\,\underline{c}_1\end{pmatrix}$$
$$= -\det\left((\underline{a},\underline{b},\underline{c})^{\mathrm{T}}(\underline{a}_1,\underline{b}_1,\underline{c}_1)\right). \quad (A3)$$

The second identity expresses the quadruple vector product of five four-vectors through its co-multipliers and their scalar products:

$$(\underline{a}\times\underline{b}\times\underline{c})\times\underline{d}\times\underline{e} = \underline{a}\,[(\underline{b}\,\underline{d})(\underline{c}\,\underline{e}) - (\underline{b}\,\underline{e})(\underline{c}\,\underline{d})]$$
$$+\ \underline{b}\,[(\underline{a}\,\underline{e})(\underline{c}\,\underline{d}) - (\underline{a}\,\underline{d})(\underline{c}\,\underline{e})] + \underline{c}\,[(\underline{a}\,\underline{d})(\underline{b}\,\underline{e}) - (\underline{a}\,\underline{e})(\underline{b}\,\underline{d})]. \quad (A4)$$

This equation is the generalization of the well-known formula for the double vector product of three 3-dimensional vectors. At last, one can introduce the scalar-vector product of four vectors:

$$(\underline{a}\,\underline{b}\,\underline{c}\,\underline{d}) = (\underline{a}\times\underline{b}\times\underline{c})\underline{d} = d_0(\boldsymbol{a}\,\boldsymbol{b}\,\boldsymbol{c}) - a_0(\boldsymbol{b}\,\boldsymbol{c}\,\boldsymbol{d})$$
$$+\ b_0(\boldsymbol{a}\,\boldsymbol{c}\,\boldsymbol{d}) - c_0(\boldsymbol{a}\,\boldsymbol{b}\,\boldsymbol{d}) = -\det\begin{pmatrix}a^0 & a^1 & a^2 & a^3\\ b^0 & b^1 & b^2 & b^3\\ c^0 & c^1 & c^2 & c^3\\ d^0 & d^1 & d^2 & d^3\end{pmatrix}. \quad (A5)$$

This product satisfies the identity similar to (3)

$$(\underline{a}\,\underline{b}\,\underline{c}\,\underline{d})\cdot(\underline{a}_1\,\underline{b}_1\,\underline{c}_1\,\underline{d}_1) = -\det\left((\underline{a},\underline{b},\underline{c},\underline{d})^{\mathrm{T}}(\underline{a}_1,\underline{b}_1,\underline{c}_1,\underline{d}_1)\right). \quad (A6)$$

It is easy to deduce one more identity

$$\underline{a}(\underline{f}\,\underline{b}\,\underline{c}\,\underline{d}) + \underline{b}(\underline{a}\,\underline{f}\,\underline{c}\,\underline{d}) + \underline{c}(\underline{a}\,\underline{b}\,\underline{f}\,\underline{d}) + \underline{d}(\underline{a}\,\underline{b}\,\underline{c}\,\underline{f}) = \underline{f}(\underline{a}\,\underline{b}\,\underline{c}\,\underline{d}). \quad (A7)$$

Appendix B: polarization bases

Let $\underline{k}$ be a 4-vector of photon momentum. In some frame of reference we introduce four vectors (one time- and three space-like) $\underline{e}_{(\lambda)}$, $\lambda = 0, 1, 2, 3$ with a usual condition of orthonormality

$$\underline{e}_{(\lambda)}\underline{e}_{(\lambda')} = g_{\lambda\lambda'}. \quad (B1)$$

Here $g_{\lambda\lambda'} = \mathrm{diag}(1, -1, -1, -1)$ — the metric tensor of the Minkowski pseudoeuclidean space-time.

These vectors can form usual basis, but in order to be a polarization basis of the photon with the momentum $\boldsymbol{k}$, it has to satisfy the transversality condition

$$\underline{e}_{(1)}\underline{k} = \underline{e}_{(2)}\underline{k} = 0. \quad (B2)$$

Then vector $\underline{k}$ has components only along two vectors and these components are equal to each other because $\underline{k}$ is zero-vector, i.e. $\underline{k}\,\underline{k} = 0$:

$$\underline{k} = \underline{k}\,\underline{e}_{(0)}\cdot[\underline{e}_{(0)} + \underline{e}_{(3)}], \quad \underline{k}\,\underline{e}_{(0)} = -\underline{k}\,\underline{e}_{(3)}. \quad (B3)$$

The transition to another polarization basis is equivalent to the linear transformation

$$\underline{e}'_{(\lambda)} = f_\lambda^{\ \lambda'}\underline{e}_{(\lambda')}. \quad (B4)$$

Let us assume that transformation of bases is only four-dimensional rotation and does not contain reflection. Then only four components of matrix f are independent. Only the transversal components of the vectors $\underline{e}_{(1)}$ and $\underline{e}_{(2)}$ and, therefore, only one parameter of the transformation have a physical sense. This parameter is the rotation angle χ:

$$\cos\chi = -\underline{e}_{(1)}\underline{e}'_{(1)} = -\underline{e}_{(2)}\underline{e}'_{(2)},$$
$$\sin\chi = \underline{e}_{(1)}\underline{e}'_{(2)} = -\underline{e}'_{(1)}\underline{e}_{(2)}. \quad (B5)$$

All the others connect longitudinal and scalar components of the vectors of the polarization basis.

Now we concretize the polarization bases we use. Let some inertial frame of reference E be connected with considered medium. Choose there three-dimensional unit vector $\boldsymbol{l}$ and form the four-vector $\underline{l} = \{\ 0,\ \boldsymbol{l}\ \}$. This vector as well as the vector of the photon momentum $\underline{k} = \{\ k,\ k\boldsymbol{\omega}\ \}$ must be transformed to the other inertial frames by usual rules in accordance with the Lorentz transformation (A1). Now we introduce four vectors

$$
\begin{aligned}
\underline{e}^{\rm ex}_{(0)} &= \underline{\beta} = \frac{\{1,\boldsymbol{\beta}\}}{\sqrt{1-\beta^2}}, \\
\underline{e}^{\rm ex}_{(3)}(\boldsymbol{k}) &= \frac{\underline{k} - (\underline{k}\ \underline{\beta})\underline{\beta}}{(\underline{k}\ \underline{\beta})}, \\
\underline{e}^{\rm ex}_{(1)}(\boldsymbol{k}) &= \frac{(\underline{k}\ \underline{\beta})\underline{l} + (\underline{l}\ \underline{k})\underline{e}^{\rm ex}_{(3)}(\boldsymbol{k})}{\sqrt{(\underline{k}\ \underline{\beta})^2 - (\underline{l}\ \underline{k})^2}}, \\
\underline{e}^{\rm ex}_{(2)}(\boldsymbol{k}) &= \frac{\underline{l} \times \underline{\beta} \times \underline{k}}{\sqrt{(\underline{k}\ \underline{\beta})^2 - (\underline{l}\ \underline{k})^2}}.
\end{aligned} \qquad \text{(B6)}
$$

They form the polarization basis for the photon of momentum $\underline{k}$. In the frame E they are reduced to (note that $\eta = \boldsymbol{l}\ \boldsymbol{\omega}$)

$$
\begin{aligned}
\underline{e}^{\rm ex}_{(0)} &= \{1,0\}, \\
\underline{e}^{\rm ex}_{(3)}(\boldsymbol{k}) &= \{0,\boldsymbol{\omega}\}, \\
\underline{e}^{\rm ex}_{(1)}(\boldsymbol{k}) &= \left\{0, \frac{\boldsymbol{l} - \eta\boldsymbol{\omega}}{\sqrt{1-\eta^2}}\right\}, \\
\underline{e}^{\rm ex}_{(2)}(\boldsymbol{k}) &= \left\{0, \frac{\boldsymbol{\omega} \times \boldsymbol{l}}{\sqrt{1-\eta^2}}\right\}.
\end{aligned} \qquad \text{(B7)}
$$

We use the *external* basis for the description of the polarization in the frame of reference connected with a medium. At the same time, to describe the scattering act, it is convenient to introduce the *internal* basis for which matrix of transformation of the photon polarization characteristics under the scattering is simpliest. For the scattering $\underline{k}, \underline{p} \to \underline{k}_1, \underline{p}_1$ following Berestetskii et al. (1971) we choose the basis in the following manner:

$$
\begin{aligned}
\underline{e}^{\rm in}_{(0)}(\boldsymbol{k}) &= \frac{\xi_1\underline{k} + \xi\underline{k}_1}{\sqrt{2\xi\xi_1(\underline{k}\ \underline{k}_1)}}, \\
\underline{e}^{\rm in}_{(3)}(\boldsymbol{k}) &= \frac{\xi_1\underline{k} - \xi\underline{k}_1}{\sqrt{2\xi\xi_1(\underline{k}\ \underline{k}_1)}}, \\
\underline{e}^{\rm in}_{(1)}(\boldsymbol{k}) &= \frac{\xi_1\underline{k} + \xi\underline{k}_1 - q\underline{p}}{q\,\Delta\, m\, c}, \\
\underline{e}^{\rm in}_{(2)}(\boldsymbol{k}) &= \frac{\underline{p} \times \underline{k} \times \underline{k}_1}{q\,\Delta\, m^3 c^3}.
\end{aligned} \qquad \text{(B8)}
$$

Here the quantities ξ, ξ_1 and q are defined by (7), and

$$
\Delta^2 = 2\,\xi\,\xi_1/q - 1. \qquad \text{(B9)}
$$

The introduced *internal* basis is convenient because if we replace $\underline{k} \leftrightarrow \underline{k}_1$ the first two vectors (0 and 1) do not change at all, whereas two others (2 and 3) only change their signs. Note also that vectors $\underline{e}^{\rm in}_{(1)}(\boldsymbol{k})$ and $\underline{e}^{\rm in}_{(2)}(\boldsymbol{k})$ do not change if we replace in the expressions for them $\underline{p}$ to $\underline{p}_1 = \underline{p} + \underline{k} - \underline{k}_1$. In the laboratory frame of reference where the incoming electron is at rest (i.e. $\underline{p} = \{mc, 0\}$), the components of the vectors become very simple. In particular, in the frame E they are following:

$$
\begin{aligned}
\underline{e}^{\rm in}_{(0)}(\boldsymbol{k}) &= \frac{\{2, \boldsymbol{\omega} + \boldsymbol{\omega}_1\}}{\sqrt{2(1-\mu)}}, \\
\underline{e}^{\rm in}_{(3)}(\boldsymbol{k}) &= \frac{\{0, \boldsymbol{\omega} - \boldsymbol{\omega}_1\}}{\sqrt{2(1-\mu)}}, \\
\underline{e}^{\rm in}_{(1)}(\boldsymbol{k}) &= \frac{\{1+\mu, \boldsymbol{\omega} + \boldsymbol{\omega}_1\}}{\sqrt{1-\mu^2}}, \\
\underline{e}^{\rm in}_{(2)}(\boldsymbol{k}) &= \frac{\{0, \boldsymbol{\omega} \times \boldsymbol{\omega}_1\}}{\sqrt{1-\mu^2}}.
\end{aligned} \qquad \text{(B10)}
$$

If we add proportional to $\underline{k}$ term (unessential for the description of polarization) to $\underline{e}^{\rm in}_{(1)}$, then this vector will have only space components:

$$
\underline{e}^{\rm in}_{(1)}(\boldsymbol{k}) - \sqrt{\frac{1+\mu}{1-\mu}}\ \frac{\underline{k}}{k} = \frac{\{0, \boldsymbol{\omega}_1 - \mu\boldsymbol{\omega}\}}{\sqrt{1-\mu^2}}. \qquad \text{(B11)}
$$

Note that vectors $\underline{e}_{(1)}$ and $\underline{e}_{(2)}$ of the *internal* and *external* bases have the analogous structure.

Appendix C: transformations of the Stokes parameters

Various reasons lead to transformation of parameters describing the polarization state of radiation field. The scattering changes the polarization state of photons and so the polarization parameters change, as well. The parameters are transformed also when we rotate the vectors of the polarization basis. We limit ourselves with these two kinds of transformation and consider them separately.

C.1. Transformation under the scattering

The simpliest case of changing the photon's polarization state occurs when the photon and the electron before and after scattering are in so called *pure* states, i.e. in the definite quantum states with complete polarization. Then it is possible to write the transformation of the components of the two-dimensional vector-potential:

$$
\begin{pmatrix} A_1' \\ A_2' \end{pmatrix} = \check{T} \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}. \qquad \text{(C1)}
$$

Here the elements of the matrix $\check{T}$ may be complex. The density matrix of the correspondent electromagnetic field is transformed with the same matrix $\check{T}$:

$$
\check{n}' = \check{T}\,\check{n}\,\check{T}^{\dagger} \qquad \text{(C2)}
$$

or

$$
n'_{(a_1 a_1')} = T_{a_1 a} T^*_{a_1' a'} n_{(aa')}. \qquad \text{(C3)}
$$

The Stokes vectors corresponding to the polarization matrices in (C2) are transformed with 4×4 matrix:

$$\tilde{n}' = \hat{F}\tilde{n}, \tag{C4}$$

and 16 real elements of the matrix $\hat{F}$ are expressed through 4 complex elements of matrix $\check{T}$ (see e.g. Dolginov et al. 1979; Hovenier et al. 1986).

If the interacting particles are not in the *pure* states, for example, if their polarization states are some mixtures of quantum states, then the equations connecting the density matrices before and after scattering are to be averaged over all *pure* states. The transformation coefficients in (C3), after such averaging, are not anymore the products of the elements of one matrix and the Eq. (C3) takes more general form:

$$n'_{(a_1 a'_1)} = T_{a_1 a a'_1 a'} n_{(aa')}. \tag{C5}$$

Here the transformation coefficients are the components of the two-dimensional tensor of rank 4:

$$T_{a_1 a a'_1 a'} = \overline{T_{a_1 a} T^*_{a'_1 a'}}. \tag{C6}$$

Briefly one can write the relation (C5) by means of tensor designation

$$\check{n}' = \overset{\diamond}{\mathrm{T}}\ \check{n}. \tag{C7}$$

Transformation formula (C4) for the Stokes parameters does not change its form. The elements of the matrix $\hat{F}$ can be expressed through the averaged products of the elements of the matrix $\check{T}$ (i.e. through the elements of tensor (C6)) by formulae analogous to mentioned above, namely:

$$\begin{aligned}
\begin{pmatrix} F_{\mathrm{II}} \\ F_{\mathrm{IQ}} \end{pmatrix} &= \tfrac{1}{2}\left[T_{1111} + T_{2121} \pm (T_{1212} + T_{2222})\right], \\
\begin{pmatrix} F_{\mathrm{QI}} \\ F_{\mathrm{QQ}} \end{pmatrix} &= \tfrac{1}{2}\left[T_{1111} - T_{2121} \pm (T_{1212} - T_{2222})\right], \\
\begin{pmatrix} F_{\mathrm{IU}} \\ F_{\mathrm{QU}} \end{pmatrix} &= \mathrm{Re}\,(T_{1112} \pm T_{2221}), \\
\begin{pmatrix} F_{\mathrm{IV}} \\ F_{\mathrm{QV}} \end{pmatrix} &= \mathrm{Im}\,(T_{1112} \mp T_{2221}), \\
\begin{pmatrix} F_{\mathrm{UI}} \\ F_{\mathrm{UQ}} \end{pmatrix} &= \mathrm{Re}\,(T_{1121} \pm T_{2212}), \\
\begin{pmatrix} F_{\mathrm{UU}} \\ F_{\mathrm{VV}} \end{pmatrix} &= \mathrm{Re}\,(T_{1122} \pm T_{1221}), \\
\begin{pmatrix} F_{\mathrm{UV}} \\ F_{\mathrm{VU}} \end{pmatrix} &= -\mathrm{Im}\,(T_{1221} \mp T_{1122}), \\
\begin{pmatrix} F_{\mathrm{VI}} \\ F_{\mathrm{VQ}} \end{pmatrix} &= -\mathrm{Im}\,(T_{1121} \pm T_{2212}).
\end{aligned} \tag{C8}$$

Thus, the transformation law (C4) for the Stokes parameters is more general than (C3) and it is more convenient. Let us consider now the transformations of the second type.

C.2. Transformation under the basis rotation

As it was mentioned in the Appendix B, when bases are transformed, only the angle χ connecting the transversal components of the vectors $\underline{e}_{(1)}$ and $\underline{e}_{(2)}$ has a physical meaning. When we choose a new polarization basis, $\check{n}$ is transformed as an usual matrix:

$$n'_{(a'b')} = f_a^{\ a'} n_{(ab)} f_b^{\ b'}, \tag{C9}$$

or by means of the matrix designation

$$\check{n}' = \check{f}^{-1}(\chi)\check{n}\check{f}(\chi), \tag{C10}$$

where quantities $f_a^{\ a'}$ form the matrix

$$\check{f}(\chi) = \begin{pmatrix} \cos\chi & \sin\chi \\ -\sin\chi & \cos\chi \end{pmatrix}, \quad \check{f}^{-1}(\chi) = \check{f}(-\chi) = \check{f}^{\mathrm{T}}(\chi). \tag{C11}$$

This matrix connects the transversal components of the potential vectors as well. The corresponding transformations of the Stokes parameters $\tilde{n}' = \hat{L}\tilde{n}$ may be obtained by the same way as above. The resulting transformation matrix (Chandrasekhar 1960) has the form:

$$\hat{L}(\chi) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos 2\chi & \sin 2\chi & 0 \\ 0 & -\sin 2\chi & \cos 2\chi & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{C12}$$

Appendix D: relationships between vectors of different bases

In the text of the paper there were introduced several three-dimensional bases in the frame E. The first basis $\boldsymbol{l}_1, \boldsymbol{l}_2, \boldsymbol{l}$ connects only with the frame E, i.e. with the electron gas as a whole. The second one is formed by the polarization bases together with a photon momentum vector. Namely, the *external* basis $\boldsymbol{e}_1^{\mathrm{ex}}(\boldsymbol{\omega}), \boldsymbol{e}_2^{\mathrm{ex}}(\boldsymbol{\omega}), \boldsymbol{\omega}$ for the photon $\boldsymbol{\omega}$ (Eq. (3)) and the analogous basis for the photon $\boldsymbol{\omega}_1$, and *internal* bases (14) for the same photons. To obtain the expressions for the scalar products of both three-dimensional and four-dimensional vectors, we need the relations connecting the vector coordinates in mentioned bases. We give such relations in this Appendix.

The directions of the photon propagations before and after scattering can be expressed by the vectors $\boldsymbol{l}_1, \boldsymbol{l}_2, \boldsymbol{l}$ in the following manner

$$\begin{aligned}
\boldsymbol{\omega} &= \sqrt{1-\eta^2}\,(\cos\varphi\,\boldsymbol{l}_1 + \sin\varphi\,\boldsymbol{l}_2) + \eta\,\boldsymbol{l}, \\
\boldsymbol{\omega}_1 &= \sqrt{1-\eta_1^2}\,(\cos\varphi_1\,\boldsymbol{l}_1 + \sin\varphi_1\,\boldsymbol{l}_2) + \eta_1\,\boldsymbol{l},
\end{aligned} \tag{D1}$$

and the *external* polarization basis for the photon $\boldsymbol{\omega}$

$$\begin{aligned}
\boldsymbol{e}_1^{\mathrm{ex}}(\boldsymbol{\omega}) &= \frac{\boldsymbol{l} - \eta\,\boldsymbol{\omega}}{\sqrt{1-\eta^2}} = \sqrt{1-\eta^2}\,\boldsymbol{l} - \eta\,(\cos\varphi\,\boldsymbol{l}_1 + \sin\varphi\,\boldsymbol{l}_2), \\
\boldsymbol{e}_2^{\mathrm{ex}}(\boldsymbol{\omega}) &= \frac{\boldsymbol{\omega} \times \boldsymbol{l}}{\sqrt{1-\eta^2}} = (\sin\varphi\,\boldsymbol{l}_1 - \cos\varphi\,\boldsymbol{l}_2),
\end{aligned} \tag{D2}$$

and analogously for the photon $\boldsymbol{\omega}_1$, if we replace η, φ to η_1, φ_1. The cosine of the scattering angle is given by (11). The vectors of the *internal* basis for $\boldsymbol{\omega}$ in the laboratory frame are defined by (14) and analogously for $\boldsymbol{\omega}_1$. Note that $\boldsymbol{e}_2(\boldsymbol{\omega}_1) = -\boldsymbol{e}_2(\boldsymbol{\omega})$. At the same time, though the four-vectors $\underline{e}^{\rm in}_{(1)}(\boldsymbol{k}) = \underline{e}^{\rm in}_{(1)}(\boldsymbol{k}_1)$, the correspondent to them space-vectors in the laboratory frame differ, because in order to reduce four-vectors into the three-dimensional ones we must add to them the different vectors (proportional to $\underline{k}$ and $\underline{k}_1$, correspondently).

It is easy to obtain the relations

$$\begin{aligned} \boldsymbol{e}_1(\boldsymbol{\omega}_1) &= -\mu\, \boldsymbol{e}_1(\boldsymbol{\omega}) + \sqrt{1-\mu^2}\, \boldsymbol{\omega}, \\ \boldsymbol{\omega}_1 &= \sqrt{1-\mu^2}\, \boldsymbol{e}_1(\boldsymbol{\omega}) + \mu\, \boldsymbol{\omega} \end{aligned} \tag{D3}$$

and similar equations with replacing $\boldsymbol{\omega} \leftrightarrow \boldsymbol{\omega}_1$.

The basis $\boldsymbol{e}_1(\boldsymbol{\omega}),\ \boldsymbol{e}_2(\boldsymbol{\omega})$ is the most important for us, and we call it the main basis.

Now we give coordinates of the electron momentum in the frame E in different bases. Let its expression in the basis $\boldsymbol{l}_1, \boldsymbol{l}_2, \boldsymbol{l}$ be the following

$$\boldsymbol{\Omega} = \sqrt{1-\kappa^2}\,(\cos\psi\, \boldsymbol{l}_1 + \sin\psi\, \boldsymbol{l}_2) + \kappa\, \boldsymbol{l}. \tag{D4}$$

Then the cosines of its angles with $\boldsymbol{\omega}$ and $\boldsymbol{\omega}_1$ are

$$\begin{aligned} \zeta &= \boldsymbol{\omega}\boldsymbol{\Omega} = \eta\,\kappa + \sqrt{1-\eta^2}\,\sqrt{1-\kappa^2}\,\cos(\psi-\varphi), \\ \zeta_1 &= \boldsymbol{\omega}_1\boldsymbol{\Omega} = \eta_1\,\kappa + \sqrt{1-\eta_1^2}\,\sqrt{1-\kappa^2}\,\cos(\psi-\varphi_1). \end{aligned} \tag{D5}$$

The azimuth of the vector $\boldsymbol{\Omega}$ in the main basis was denoted in Eq. (15) as Φ. Then

$$\begin{aligned} \zeta_1 &= \mu\,\zeta + \sqrt{1-\mu^2}\,\sqrt{1-\zeta^2}\,\cos\Phi, \\ \kappa &= \eta\,\zeta + \frac{\sqrt{1-\zeta^2}}{\sqrt{1-\mu^2}}\,[(\eta_1-\mu\,\eta)\cos\Phi + C_l\,\sin\Phi]. \end{aligned} \tag{D6}$$

Many indentities are following from the given formulae, but we omit them. Let us give here the expressions for the four mixed products:

$$\begin{aligned} C_l &= (\boldsymbol{l}\,\boldsymbol{\omega}\,\boldsymbol{\omega}_1) = \sqrt{1-\eta^2}\,\sqrt{1-\eta_1^2}\,\sin(\varphi-\varphi_1), \\ C_\Omega &= (\boldsymbol{\Omega}\,\boldsymbol{\omega}\,\boldsymbol{\omega}_1) = \sqrt{1-\mu^2}\,\sqrt{1-\zeta^2}\,\sin\Phi, \\ C_\omega &= (\boldsymbol{l}\,\boldsymbol{\omega}\,\boldsymbol{\Omega}) = \frac{\sqrt{1-\zeta^2}}{\sqrt{1-\mu^2}}\,[C_l\,\cos\Phi - (\eta_1-\mu\,\eta)\,\sin\Phi], \\ C_{\omega_1} &= (\boldsymbol{l}\,\boldsymbol{\omega}_1\,\boldsymbol{\Omega}) = -C_l\,\zeta \\ &+ \frac{\sqrt{1-\zeta^2}}{\sqrt{1-\mu^2}}\,[\mu\,C_l\,\cos\Phi - (\eta-\mu\,\eta_1)\,\sin\Phi]. \end{aligned} \tag{D7}$$

Their squares are expressed by means of introduced cosines:

$$\begin{aligned} C_l^2 &= 1-\eta^2-\eta_1^2-\mu^2+2\,\eta\,\eta_1\,\mu, \\ C_\Omega^2 &= 1-\zeta^2-\zeta_1^2-\mu^2+2\,\zeta\,\zeta_1\,\mu, \\ C_\omega^2 &= 1-\eta^2-\zeta^2-\kappa^2+2\,\eta\,\zeta\,\kappa, \\ C_{\omega_1}^2 &= 1-\eta_1^2-\zeta_1^2-\kappa^2+2\,\eta_1\,\zeta_1\,\kappa. \end{aligned} \tag{D8}$$

These products satisfy also the identity

$$C_l\,\boldsymbol{\Omega} - C_\Omega\,\boldsymbol{l} + C_{\omega_1}\,\boldsymbol{\omega} - C_\omega\,\boldsymbol{\omega}_1 = 0. \tag{D9}$$

From this equation we may obtain many others. Note at last that in the main basis vector $\boldsymbol{l}$ is presented as:

$$\boldsymbol{l} = \frac{\eta_1-\mu\,\eta}{\sqrt{1-\mu^2}}\,\boldsymbol{e}_1(\boldsymbol{\omega}) + \frac{C_l}{\sqrt{1-\mu^2}}\,\boldsymbol{e}_2(\boldsymbol{\omega}) + \eta\,\boldsymbol{\omega}. \tag{D10}$$

Appendix E: rotation angles of the polarization basis

To describe the scattering of the photon and electron of momenta $\underline{k}, \underline{p}$ before and $\underline{k}_1, \underline{p}_1$ after scattering, we need the set of the polarization bases. It is the *external* basis $\underline{e}^{\rm ex}_{(\mu)}(\boldsymbol{k})$ defined by Eq. (B6), and $\underline{e}^{\rm ex}_{(\mu)}(\boldsymbol{k}_1)$ for the photon of momentum $\underline{k}_1$, and also two *internal* bases: $\underline{e}^{\rm in}_{(\mu)}(\boldsymbol{k})$ defined by (B8) and $\underline{e}^{\rm in}_{(\mu)}(\boldsymbol{k}_1)$, which have two identical vectors (0 and 1). Two others (2 and 3) differ by a sign. In this Appendix we give the rotation angles between the *internal* and *external* vectors for each photon. First of all, we find these angles in the frame E for the scattering by the electron at rest. Let denote χ^0 the rotation angle of the *internal* polarization basis for photon $\boldsymbol{\omega}$ to the *external*:

$$\begin{aligned} \boldsymbol{e}_1(\boldsymbol{\omega}) &= \cos\chi^0\,\boldsymbol{e}_1^{\rm ex}(\boldsymbol{\omega}) + \sin\chi^0\,\boldsymbol{e}_2^{\rm ex}(\boldsymbol{\omega}), \\ \boldsymbol{e}_2(\boldsymbol{\omega}) &= -\sin\chi^0\,\boldsymbol{e}_1^{\rm ex}(\boldsymbol{\omega}) + \cos\chi^0\,\boldsymbol{e}_2^{\rm ex}(\boldsymbol{\omega}). \end{aligned} \tag{E1}$$

Finding the scalar product of vectors (3) and (14) one obtains

$$\begin{aligned} \cos\chi^0 &= (\eta_1-\mu\,\eta)/\sqrt{1-\eta^2}\,\sqrt{1-\mu^2}, \\ \sin\chi^0 &= -C_l/\sqrt{1-\eta^2}\,\sqrt{1-\mu^2}. \end{aligned} \tag{E2}$$

In the same manner for the photon $\boldsymbol{\omega}_1$:

$$\begin{aligned} \cos\chi_1^0 &= (\eta-\mu\,\eta_1)/\sqrt{1-\eta_1^2}\,\sqrt{1-\mu^2}, \\ \sin\chi_1^0 &= C_l/\sqrt{1-\eta_1^2}\,\sqrt{1-\mu^2}, \end{aligned} \tag{E3}$$

that is obtained from (E2) by replacing $\boldsymbol{\omega} \leftrightarrow \boldsymbol{\omega}_1$.

Let us now find the transformation angles for the scattering by electron of an arbitrary momentum in the arbitrary frame of reference. For the photon $\underline{k}$ we have:

$$\begin{aligned} -\cos\chi &= \underline{e}_1^{\rm in}(\boldsymbol{k})\,\underline{e}_1^{\rm ex}(\boldsymbol{k}) = \underline{e}_2^{\rm in}(\boldsymbol{k})\,\underline{e}_2^{\rm ex}(\boldsymbol{k}) \\ &= \frac{\xi\,[(\underline{k}\,\underline{\beta})\,(\underline{l}\,\underline{k}_1) - (\underline{l}\,\underline{k})\,(\underline{k}_1\,\underline{\beta})] - q\,[(\underline{k}\,\underline{\beta})\,(\underline{p}\,\underline{l}) - (\underline{l}\,\underline{k})\,(\underline{p}\,\underline{\beta})]}{\sqrt{(\underline{k}\,\underline{\beta})^2 - (\underline{l}\,\underline{k})^2}\,q\,\Delta\,m\,c}, \\ -\sin\chi &= \underline{e}_1^{\rm in}(\boldsymbol{k})\,\underline{e}_2^{\rm ex}(\boldsymbol{k}) = -\underline{e}_2^{\rm in}(\boldsymbol{k})\,\underline{e}_1^{\rm ex}(\boldsymbol{k}) \\ &= \frac{q\,(\underline{l}\,\underline{\beta}\,\underline{k}\,\underline{p}) - \xi\,(\underline{l}\,\underline{\beta}\,\underline{k}\,\underline{k}_1)}{\sqrt{(\underline{k}\,\underline{\beta})^2 - (\underline{l}\,\underline{k})^2}\,q\,\Delta\,m\,c}. \end{aligned} \tag{E4}$$

These expressions in the frame E are simplified:

$$\begin{aligned} \cos\chi &= \frac{\gamma\,(\eta_1-\mu\,\eta) - z\,[(1-\mu)\,\kappa + (\eta_1-\eta)\,\zeta]}{\sqrt{1-\eta^2}\,(1-\mu)\,\Delta}, \\ \sin\chi &= -\frac{(\gamma - z\,\zeta)\,C_l - z\,(1-\mu)\,C_\omega}{\sqrt{1-\eta^2}\,(1-\mu)\,\Delta}, \end{aligned} \tag{E5}$$

where now

$$\Delta^2 = \frac{[2(\gamma - z\zeta)(\gamma - z\zeta_1) - (1-\mu)]}{1-\mu} = -\frac{A+2}{A}. \qquad \text{(E6)}$$

For the photon ω_1 all expressions are obtained by replacing $\omega \leftrightarrow \omega_1$. In particular, in the frame E

$$\cos\chi_1 = \frac{\gamma(\eta - \mu\eta_1) - z[(1-\mu)\kappa + (\eta - \eta_1)\zeta_1]}{\sqrt{1-\eta_1^2}(1-\mu)\Delta},$$
$$\sin\chi_1 = \frac{(\gamma - z\zeta_1)C_l + z(1-\mu)C_{\omega_1}}{\sqrt{1-\eta_1^2}(1-\mu)\Delta}. \qquad \text{(E7)}$$

Substituting the expressions for κ, C_ω and C_{ω_1} to the (E5) and (E7) and denoting

$$\cos\Psi = [(\gamma - z\zeta)\sqrt{r} - z\sqrt{1-\zeta^2}\cos\Phi]/\Delta,$$
$$\sin\Psi = z\sqrt{1-\zeta^2}\sin\Phi/\Delta \qquad \text{(E8)}$$

one can find that in the frame E

$$\chi = \chi^0 - \Psi, \quad \chi_1 = \chi_1^0 + \Psi. \qquad \text{(E9)}$$

Note that in order to verify the basic identity between sines and cosines of this Appendix we must use the relations of the Appendices A and D.

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