Compton scattering by Maxwellian electrons: frequency and directional redistribution of radiation

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A relativistic kinetic equation (RKE) describing Compton scattering is formulated. Methods are proposed for calculating the frequency and directional redistribution functions for radiation scattered by relativistic Maxwellian electrons. The resdistribution function is represented by series expansions in three limiting cases: a) low frequencies ($h\nu \ll mc^2$); b) nonrelativistic electrons ($kT \ll mc^2$); and, c) ultrarelativistic electrons ($kT \gg mc^2$). A method is developed for calculating the redistribution function in the general case. Explicit expressions are obtained for the redistribution function in the case of scattering by weakly relativistic electrons and are used to derive the limiting forms of the RKE.

1. Introduction. Compton scattering, i.e., scattering of radiation by electrons with a change in frequency, plays a large role in the formation of the spectra of a number of astrophysical objects. Investigators have been interested in Compton scattering for a long time. Scattering of x rays or loner-wavelength radiation by an almost nonrelativistic gas, when the photon frequency shift in each scattering is small, has been studied in a large number of works (Zel'dovich, 1975; Syunyaev and Titarchuk, 1980; Pozdnyakov et al., 1982). Scattering of low frequency radiation by relativistic electrons with power-law (Blumental and Gould, 1970) and Maxwellian (Poutanen, 1990) energy distributions has also been investigated.

In several papers (for example, Agaronyan and Atoyan, 1981; Arutyunyan and Dzhrbashyan, 1985; Ribberfors, 1975) concerning Compton scattering, no restrictions were imposed on the photon and electron energies. In these works expressions were obtained for the frequency and directional redistribution function for radiation undergoing Compton scattering by an isotropic electron gas. A detailed derivation of these expression and citations of preceding works are given in the paper by Nagirner et al. (1991).

In the present paper we propose methods for calculating the redistribution function in the case of scattering of radiation by a Maxwellian electron gas with different ratio of the gas temperature and photon frequencies. This problem has been solved correctly for a quite wide range of photon energies and for any electron temperature only by Kershaw et al. (1986). We shall extend these limits and increase the accuracy of the calculation of the redistribution function. We shall also give explicit expression for the redistribution function in the case of scattering by an almost nonrelativistic electron gas and we shall derive from them the limiting forms of the relativistic kinetic equation (RKE).

In Sec. 2 we give the general form of the RKE and we present expression for the intensity of singly-scattered radiation and for the redistribution function in the case of monoenergetic electrons. In Sec. 3 we propose a new method for calculating the integrals over the Maxwellian electron velocity distribution function. In Sec. 4 we consider scattering by an ultrarelativistic gas. In Sec. 5 we give a method for calculating the redistribution function for low-frequency radiation scattered by relativistic electrons. In the last sections, we present for nonrelativistic electrons the redistribution function in the form of an asymptotic series, we give explicit expressions for the redistribution function in the case of scattering by cold electrons, and we derive in terms of these expressions the RKE for this case.

2. Redistribution function. We assume that the electron gas is in equilibrium and nondegenerate. Then the electron momentum distribution in the coordinate system which we call E and which is embedded in the case is determined by the relativistic Maxwellian distribution with some temperature T. In this case, in the same system E, the relativistically invariant and dimensionless frequency and directional redistribution function in the case of Compton scattering is given by the formula (Kershaw et al., 1986; Nagirner et al., 1991)

$$R(x, x_1, \mu) = \frac{3y}{32\pi K_2(y)} \int_{\gamma_*}^{\infty} e^{-y\gamma} R(x, x_1, \mu, \gamma)\, d\gamma, \tag{1}$$

where $x_1 = h\nu_1/mc^2$ and $x = h\nu/mc^2$ are dimensionless photon frequency before and after scattering; $y = mc^2/kT$ is a parameter of the Maxwellian distribution; μ is the cosine of the scattering angle; and, $K_2(y)$ is a modified Bessel function. The lower limit in the integral (1) is

$$\gamma_* = \gamma_*(x, x_1, \mu) = (x - x_1 + Qt_*)/2. \tag{2}$$

Here $Q^2 = (x - x_1)^2 + 2q$, $t_* = (1 + 2/q)^{1/2}$, $q = xx_1w$, $w = 1 - \mu$, and the function of four arguments, which appears in the integrand and is proportional to the redistribution function in the case of scattering by electrons with fixed energy γmc^2 and an isotropic momentum distribution, is given by the expression

$$R(x, x_1, \mu, \gamma) = \frac{2}{Q} + \frac{q^2 - 2q - 2}{q^2}\left(\frac{1}{a} - \frac{1}{a_1}\right) + \frac{x + x_1}{q^2}\left(\frac{\gamma - x}{a^3} + \frac{\gamma + x_1}{a_1^3}\right) + \frac{r}{q}\left(\frac{1}{a^3} - \frac{1}{a_1^3}\right), \tag{3}$$

where $a^2 = (\gamma - x)^2 + r$, $a_1{}^2 = (\gamma + x_1)^2 + r$, $r = (1 + \mu)/w$. A detailed derivation of the redistribution functions (1) and (3) is given by Nagirner et al. (1991). The expression for $R(x, x_1, \mu, \gamma)$ reduces to the following form:

$$R(x, x_1, \mu, \gamma) = \frac{2}{Q} + \frac{u}{v}\left[1 - \frac{2}{q} + \frac{u^2(5v + u^2) - Q^2(3v + u^2)}{2q^2v^2}\right], \tag{4}$$

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where $u = a_1 - a = (2\gamma + x_1 - x)(x + x_1)/(a + a_1)$, $v = aa_1$. There is no loss of accuracy in using Eq. (4) for calculations, as happens, for example, for small x and x_1, if Eq. (3) is used directly.

If we fix x_1, μ, and γ, then the redistribution function is determined by Eq. (3) for values of x such that $x^- \le x \le x^+$, where $x^\pm$ are determined from the condition $\gamma = \gamma_*(x^\pm, x_1, \mu)$:

$$x^\pm = x_1 \frac{\mu + \gamma(\gamma + x_1) w \pm zwa_1}{1 + 2\gamma x_1 w + x_1^2 w^2}, \quad \gamma^2 = z^2 + 1. \tag{5}$$

For fixed x, μ, and γ the limits for x_1 are given by Eq. (5), if in Eq. (5) the substitutions $x_1 \Leftrightarrow -x$ and a_1 by a are made. Finally, the expressions for the limits of μ are as follows:

$$\mu^\pm = \frac{1}{xx_1}\left[z^2 + \gamma(x_1 - x) \pm z\sqrt{(\gamma + x_1 - x)^2 - 1}\right]. \tag{6}$$

It is easy to verify that the redistribution function (3) is symmetric with respect to frequency:

$$R(x_1, x, \mu, \gamma) = R(x, x_1, \mu, \gamma + x - x_1), \tag{7}$$

and the complete redistribution function (1) has the symmetry property, following from Eqs. (2) and (7),

$$R(x_1, x, \mu) = e^{-y(x_1 - x)} R(x, x_1, \mu) \tag{8}$$

which reflects the equilibrium nature of the electron distribution.

We now find in terms of the redistribution function (1) the RKE describing the change in the frequency and directional distribution of the photon gas in the case of Compton scattering:

$$\frac{1}{c}\frac{\partial n}{\partial t} + \omega\nabla n = \frac{\sigma_T n_e}{x}\int_0^\infty x_1 dx_1 \int d^2\omega_1 \times [R(x, x_1, \mu)(1 + n)n_{11} - R(x_1, x, \mu)n(1 + n_{11})]. \tag{9}$$

Here $n = n(\mathbf{r}, t, x\omega)$, $n_{11} = n(\mathbf{r}, t, x_1, \omega_1)$ are the average occupation numbers of the photon states and are related to the intensity of the radiation by the relation $I = 2h\nu^3 n/c^2$; ω is a unit direction vector; $\mu = \omega\cdot\omega_1$; n_e is the electron density in the system E; and, σ_T is the Thomson scattering cross section. If the radiation intensity is determined by the Planck function, then the property (8) of the redistribution function ensures that the collision integral in the RKE (9) vanishes.

Let radiation with intensity $I_0(x, \omega)$ be incident on a unit volume of the electron gas. Then the singly-scattered radiation in the system E will be characterized by the intensity

$$I(x, \omega) = n_e\sigma_T x^2 \int_0^\infty \frac{dx_1}{x_1^2}\int d^2\omega_1 R(x, x_1, \mu) I_0(x, \omega_1). \tag{10}$$

We now proceed to study the function $R(x, x_1, \mu)$.

3. General case. Following Kershaw et al. (1986), the redistribution function (1) can be represented in the form

$$R(x, x_1, \mu) = \frac{3}{32\pi}\frac{e^{-y t_*}}{K_2(y)}\left\{\frac{2}{Q} + y\frac{2t_*}{q} + y\frac{e^{y\rho_-}}{q^2}(\Lambda_- L_0^- + qbL_1^-) + y\frac{e^{y\rho_+}}{q^2}(\Lambda_+ L_0^+ - qbL_1^+)\right\}, \tag{11}$$

where $\rho_\pm = (Qt_* \pm (x + x_1))/2$, $\Lambda_\pm = \pm(2 + 2q - q^2) - y(x + x_1)$. The quantities $L_n^\pm$ are integrals of the form

$$L_n^\pm = \int_{c_\pm}^\infty e^{-bp}\frac{p^n dp}{\sqrt{1 + p^2}}, \quad n = 0, 1. \tag{12}$$

Here $c_\pm = \rho \pm l\sqrt{r}$, $b = y\sqrt{r}$. We note that the lower limit of the integral c_+ is always positive and c_- is negative only for $x > 1$, $x_1 > 1$, and $\mu_- \le \mu \le \mu_+$, $\mu_\pm = (1 \pm [(x^2 - 1)(x_2^2 - 1)]^{1/2})/xx_1$. Thus $c_- > 0$ for the entire photon spectrum except for γ-rays. For this reason, we confine our attention first to the case $c_- \ge 0$.

In order to calculate the integrals (12), we represent the root in the expression for $L_n^\pm$ as a contour integral using Cauchy's formula

$$\frac{1}{\sqrt{1 + p^2}} = \frac{1}{2\pi i}\int\frac{dz}{z - p}\frac{1}{\sqrt{1 + z^2}}. \tag{13}$$

We take the integration contour to be two rectilinear sections along the imaginary axis from ∞i to i and from $-i$ to $-\infty i$ and a semicircle passing through the points i, -1, and $-i$. The sum of the integrals along the rectilinear sections is $1/2(1 + p^2)^{1/2}$, and we make in the integral over the semicircle the substitution of variables $z = -\sin^2\varphi + i\cos\varphi(1 + \sin^2\varphi)^{1/2}$ (φ ranges from 0 to π). This gives

$$\frac{p^n}{+ p^2} = \frac{2}{\pi}\int_0^\pi \frac{d\varphi}{p - z}\frac{z^{n+1}}{z - 1} + \delta_{nl}, \quad n = 0, 1, \tag{14}$$

where δnl is the Kronecker delta. Substituting these expressions into Eq. (12), we obtain

$$L_n^\pm = \frac{2}{\pi}\int_0^\pi \frac{z^{n+1}}{z - 1}e^{-bz}E_1(b(c_\pm - z))\,d\varphi + \frac{e^{-y\rho_\pm}}{b}\delta_{nl}, \quad n = 0, 1. \tag{15}$$

Here E_1 is the exponential integral. The expression for the quantities $L_n^\pm$ presented by Kershaw et al. (1986) were derived by representing the roots (13) in terms of continued fractions, and are obtained if the integrals over φ are represented as a sum according to the quadrature formula of rectangles. The representation of $L_n^\pm$ in the form (15) makes it possible to use Gauss' formula and to calculate them with high accuracy for small orders of the quadrature formulas.

Now let $c_- < 0$. The integrals (12) can be represented as a sum from c_- to 0 and from 0 to ∞. The integral over the second interval can be expressed in terms of the Struve and Neumann functions (see Gradshtein and Ryzhik, 1962)):

$$\int_0^\infty e^{-bp}\frac{p^n dp}{\sqrt{1 + p^2}} = \frac{\pi}{2}[H_n(b) - Y_n(b)] - \delta_{nl}, \quad n = 0, 1. \tag{16}$$

In calculating the first integral, we replace the expression (14) for the roots by

$$\frac{p^n}{\sqrt{1 + p^2}} = \frac{2}{\pi}\int_0^\pi \frac{d\varphi}{p + z}\frac{(-z)^{n+1}}{z - 1} - \delta_{nl}, \quad n = 0, 1, \quad p < 0, \tag{17}$$

where z can be expressed in terms of φ just as in Eq. (14). Thus

$$\int_{c_-}^0 e^{-bp}\frac{p^n dp}{\sqrt{1 + p^2}} = \frac{2}{\pi}\int_0^\pi \frac{(-z)^{n+1}}{z - 1}e^{bz}[E_1(b(c_- + z)) - E_1(bz)]\,d\varphi + \frac{1 - e^{-y\rho_-}}{b}\delta_{nl}. \tag{18}$$

Methods for calculating the functions H_n and Y_n are given by Luke (1980). Methods for calculating the function $E_1(z)$ itself and related functions are given in the Appendix.

If x, $x_1 << 1$, or $b >> 1$, then in calculations using Eq. (11) there is a loss of accuracy. For this reason we shall examine these cases separately. In the next section, in order to complete the picture, we present a convenient method, given by Kershaw et al. (1986), for calculating the redistribution function in the case of scattering by ultrarelativistic electrons.

4. Ultrarelativistic limit. We assume that $b = y\sqrt{r} << 1$. Then we make the substitution $p = (u - 1/u)/2$ in the expression for $L_n^{\pm}$. Expanding $e^{b/2u}$ in a Taylor series and integrating by parts, we obtained instead of Eq. (12)

$$L_0^{\pm} = \sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{b}{2u_{\pm}}\right)^k E_{k+1} \left(\frac{bu_{\pm}}{2}\right), \tag{19}$$

$$L_1^{\pm} = -\frac{2}{b} \sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{b}{2u_{\pm}}\right)^k E_{k+1} \left(\frac{bu_{\pm}}{2}\right) + \frac{e^{-yp_{\pm}}}{b},$$

where $u_{\pm} = (t_* \pm 1)(Q + x + x_1)/2\sqrt{r}$. Thus the redistribution function will be given in the form of a rapidly converging series:

$$R(x, x_1, \mu) = \frac{3e^{-y t_*}}{32\pi K_2(y)} \left\{\frac{2}{Q} + \frac{2yt_*}{q} + \Omega_+ + \Omega_-\right\},$$
$$\Omega_{\pm} = \frac{ye^{yp_{\pm}}}{q^2} \sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{b}{2u_{\pm}}\right)^k E_{k+1} \left(\frac{bu_{\pm}}{2}\right) (\Lambda_{\pm} \pm 2nq). \tag{20}$$

No loss of computational accuracy occurs with these formulas.

In the next section we present a method for calculating the redistribution function of low-frequency radiation scattered by relativistic electrons.

5. Low frequencies. We take as low frequencies $x << 1$, i.e., $h\nu << mc^2$. To simplify the derivation we assume that x, $x_1 \to 0$, $x_1/x = \zeta = \text{const}$. Since $dx_1 = x d\xi$, as follows from Eq. (7), the product $xR(x, \xi x, \mu, \gamma)$ has a finite limit as $x \to 0$. We expand it in powers of x:

$$xR(x, \xi x, \mu, \gamma) = \sum_{n=0}^{\infty} R_n(\xi, \mu, \gamma)\, x^n. \tag{21}$$

An expression for r_0 was derived by Arutyunyan and Dzhrbashyan (1985). It is easy to expand the function with interchanged frequency arguments. The expansion coefficients will be the same as before:

$$xR(x\xi, x, \mu, \gamma) = \sum_{n=0}^{\infty} (-x)^n R_n(\xi, \mu, \gamma). \tag{22}$$

A formula for R_n is obtained by expanding $1/a$ and $1/a_1$ for small values of x and x_1 in Legendre polynomials:

$$R_n(\xi, \mu, \gamma) = \frac{2}{Q_0}\delta_{n0} + \frac{1}{a_0^n}\Big\{\left[1 + \beta_n/\xi\right] P_{n-1}(\Gamma)$$
$$+ [1 + \beta_n \xi^3] \frac{n+1}{w^2 \xi^2 a_0^4} P_{n+3}(\Gamma) + \frac{1 + \beta_n \xi}{w \xi a_0^2} \Big[\frac{n^2 + n - 4}{2n + 5}$$
$$\times P_{n+1}(\Gamma) + (n + 3) P_{n+3}(\Gamma) \left(\frac{1}{wa_0^2} - \frac{n+2}{2n+5}\right)\Big]\Big\}, \tag{23}$$

where $Q_0^2 = (1 - \xi)^2 + 2w\xi$, $a_0^2 = \gamma^2 + r$, $\Gamma = \gamma/a_0$, $\beta_n = (-\xi)^n$. The property of the expansion coefficients

$$R_n(1/\xi, \mu, \gamma) = (-1)^n \xi^{1-n} R_n(\xi, \mu, \gamma) \tag{24}$$

can be derived from Eq. (22) or from the explicit expression (23). The fact that the redistribution function R_0 for low frequencies x and x_1 depends only on the ratio $\xi = x_1/x$ was employed by Arutyunyan and Nikogosyan (1980) and Nishimura et al. (1986) in order to derive some solutions of the kinetic equation.

In order to calculate the function $R(x, x_1, \mu, \gamma)$ itself, the expansion (21) is no more advantageous than Eq. (4), but it does make it possible to calculate easily the redistribution function averaged over a Maxwellian distribution. We substitute Eq. (21) into Eq. (1). Integrating over γ, we obtain

$$R(x, x_1, \mu) = \frac{3ye^{-y\gamma_*}}{32\pi K_2(y)} \sum_{n=0}^{\infty} x^{n-1} R_n(\xi, \mu). \tag{25}$$

Here

$$R_n(\xi, \mu) = \frac{2}{yQ_0}\delta_{n0} + (1 + \beta_n/\xi) I_{n-1,n} + (1 + \beta_n \xi^3) \frac{n+1}{w^2\xi^2} I_{n+3,n+4}$$
$$+ \frac{1 + \beta_n \xi}{w\xi}\left[\frac{n^2 + n - 4}{2n + 5} I_{n+1,n+2} + (n + 3)\left(\frac{I_{n+3,n+4}}{w} - \frac{n+2}{2n+5} I_{n+3,n+2}\right)\right], \tag{26}$$

and the integrals

$$I_{n,n\pm 1} = \int_{\gamma_*}^{\infty} e^{-y(\gamma - \gamma_*)} P_n(\Gamma)/a_0^{n\pm 1}\, d\gamma. \tag{27}$$

These integrals can be calculated with the help of the recurrence relations

$$I_{n,n+1} = [-yI_{n-1,n} + P_n(\Gamma_*)/a_*^n]/n,$$
$$I_{n,n-1} = [2(n - 1) I_{n-2,n-1} - y(2n - 1) I_{n-1,n-2}$$
$$+ (2n - 1) P_n(\Gamma_*)/a_*^{n-2}]/(n(2n - 3)), \tag{28}$$

where $a_*^2 = \gamma_*^2 + r$, $\Gamma_* = \gamma_*/a_*$. If $b >> 1$ or $y\gamma_* >> 1$, then I_{01} and I_{10} can be calculated with the help of the Gauss–Laguerre quadrature formula; otherwise we use Eq. (15) to calculate I_{01} and I_{10}. It should be noted that for $y > 100$, a loss of accuracy occurs in Eq. (28) during subtraction and the recurrence relations cannot be used. Then the original formulas (1) and (4) should be used.

6. Nonrelativistic limit. We now give an expansion in powers of $1/y$ of the photon redistribution function in the case when the photons are scattered by a nonrelativistic electron gas. We assume that y is large (the temperature $T << mc^2/k = 5.93 \cdot 10^9$ K). We expand $1/a$ and $1/a_1$ in Legendre polynomials:

$$\frac{1}{a} = \frac{1}{a_-}\sum_{n=0}^{\infty}\left(\frac{\gamma_* - \gamma}{a_-}\right)^n P_n^-, \quad \frac{1}{a_1} = \frac{1}{a_+}\sum_{n=0}^{\infty}\left(\frac{\gamma_* - \gamma}{a_+}\right)^n P_n^+, \tag{29}$$

where $a_{\pm} = [(x + x_1)t_* \pm Q]/2$, $P_n^{\pm} = P_n(\rho_{\pm}/a_{\pm})$. Substituting Eq. (19) into Eqs. (3) and (1) and integrating over γ, we obtain (Kershaw et al., 1986)

$$R(x, x_1, \mu) = \frac{3e^{-y\gamma_*}}{32\pi K_2(y)} \sum_{n=0}^{\infty} y^{-n} R_n^y, \tag{30}$$

$$R_n^y = \frac{2}{Q}\delta_{n0} - \frac{n!}{q^2}\Big\{\frac{\Theta_n P_n^- + (n + 1)\, qt_* P_{n+1}^-}{(-a_-)^{n+1}}$$
$$+ \frac{-\Theta_n P_n^+ + (n + 1)\, qt_* P_{n+1}^+}{(-a_+)^{n+1}}\Big\}, \tag{31}$$

where $\Theta_n = q^2 - 2 + (n - 1)q$. The asymptotic series (30) must be summed, as usual, until the average term is smaller than the preceding term. If $x, x_1 << 1$, then a loss of computational accuracy occurs with Eq. (31). Subtracting terms proportional to Θ_n analytically, we obtain an expression for R_n^y without this drawback:

$$R_n^y = \frac{2}{Q}\delta_{n0} - \frac{n!}{q^2}\{\Theta_n\left[\frac{\Delta_p^n}{(-a_-)^{n+1}} + \frac{2QqS_nP_n^+}{Q_*}\right] + (n+1)qt_*\left[\frac{P_{n+1}^-}{(-a_-)^{n+1}} + \frac{P_{n+1}^-}{(-a_+)^{n+1}}\right]\}. \tag{32}$$

Here $Q_* = Q^2 + q_*(q + 2)$, $q_* = rq$, and the difference $\Delta_p^n = P_n^- - P_c^+$ is calculated using the recurrence relation

$$(n+1)\Delta_p^{p+1} = (2n+1)[p_-\Delta_p^n/a_- + P_n^+\Delta_p^1] - n\Delta_p^{n-1} \tag{33}$$

with $\Delta_p^0 = 0$, $\Delta p_1 = -2rq^2t_*/Q_*$. The quantities S_n are given by the following recurrence relation:

$$S_0 = 1, \quad S_{n+1} = -S_n/a_+ + (-a_-)^{-(n+1)}. \tag{34}$$

If the redistribution function need not be calculated to high accuracy, then the first three terms in the expansion (30) are sufficient. These terms give the following relative error (ε) in the calculation of $R(x, x_1, \mu)$: if $y = 10$, then $\varepsilon = 0.1$ with $\mu = -1$, $\varepsilon = 0.015$ with $\mu = 0.0$, and $\varepsilon = 0.003$ with $\mu = 0.8$; if $y = 100$, then $\varepsilon < 2\cdot10^{-4}$; for higher values of y, ε behaves as $1/y^2$.

Next we show, with the help of the representation (30), how the limiting forms of the RKE are obtained for an almost nonrelativistic Maxwellian electron distribution.

7. Scattering by cold electrons. For large y, the function $R(x, x_1, \mu)$ is not small only for those values of x, x_1, and μ for which γ is close to 1. We represent the difference $\gamma_* - 1$ in the form

$$\gamma_*(x, x_1, \mu) - 1 = \frac{(q + x - x_1)^2}{d^2}, \tag{35}$$

where $d^2 = q(Qt_* + 2 + x_1 - x)$. The equality $q = x_1 - x$ means that scattering electrons are stationary. In this case

$$x_1 = \frac{x}{1 - xw}, \quad 1 - \mu = w = \frac{1}{x} - \frac{1}{x_1}, \tag{36}$$

so that $R(x, x_1, \mu)$ is a delta function. We take this into account explicitly. In the limit $y \to \infty$, we have the following expansion of the exponential function in delta functions:

$$\sqrt{\frac{y}{\pi}}\,e^{-y\chi^2} = \sum_{n=0}^{\infty}\frac{\delta^{(2n)}(\chi)}{n!\,(4y)^n} \sim \delta(\chi) + \frac{1}{4y}\delta''(\chi), \tag{37}$$

which can be easily verified by Fourier transforming both sides of the equation (or performing a two-sided Laplace transform; both sides of the equation (or performing a two-sided Laplace transform here and below we take into account only terms of order $1/y$). In calculating the x_1 or μ integrals of the products containing a delta function and its derivatives, we employ the formula

$$\int_{-\infty}^{+\infty}\rho(y)\,\delta^{(n)}(\chi(y))\,dy = (-1)^n\left[\frac{1}{\chi'(y)}\frac{d}{dy}\right]^n\left[\frac{\rho(y)}{\chi'(y)}\right]\Big|_{y=y_0}, \quad n = 0, 1, \ldots. \tag{38}$$

Here it is assumed that $\chi(y)$ vanishes only at $y = y_0$ (and the derivatives of $\chi(y)$ do not vanish anywhere), and that the function itself and its derivatives are finite everywhere.

The substitutions (36) greatly simplify the functions (32):

$$R_0^y|_{x_1=x+q} = \frac{F_*}{\sqrt{q(q+2)}}, \quad F_* = w_0 + wq, \quad w_0 = 1 + \mu^2,$$
$$R_1^y|_{x_1=x+q} = \frac{wF_{1*}}{\sqrt{q}\,(q+2)^{3/2}},$$
$$F_{1*} = 2(2 + \mu - 5\mu^2) + \mu(2 - 3\mu + 3\mu^2)q + \mu(1 - \mu)q^2, \tag{39}$$

since $Q^2 = q(q + 2)$, $Q_* = 2q(q + 2)/w$, $d = \sqrt{2}Q$. From the quantities (39) we must eliminate either μ, if the integral is over μ, or x_1, if the integral of the delta function is calculated over x_1 according to Eq. (36). The quantity q must be replaced either by $x_1 - x$ or by $x^2w/(1 - xw)$. In the first case, x_1 ranges from x to $x/(1 - 2x)$ for $x \leq 1/2$ and to ∞ for $x > 1/2$; in the second case w ranges from 0 to $\min\{2, 1/x\}$. Using the asymptotic expression $c^{-y}/K_2(y) \sim (2y/\pi)^{1/2}(1 - 15/8y)$ for $y >> 1$, we obtain from Eq. (30), to order $1/y$,

$$R(x, x_1, \mu) = \frac{3}{16\pi}\{\delta(x_1 - x - q)\left[F_*\left(1 - \frac{15}{8y}\right) + \frac{wF_{1*}}{y(q+2)}\right] + \frac{R_0^y}{4\sqrt{2}\,y}\delta''\left(\frac{x_1 - x - q}{d}\right)\}. \tag{40}$$

If the expression (40) is substituted into the RKE, then the term with δ'' leads to very complicated expressions. For this reason we shall not do this here. At $y = \infty$, i.e., in the case of scattering by stationary electrons, the kinetic equation is quite simple:

$$\frac{1}{c}\frac{\partial n}{\partial t} + \omega\nabla n = \frac{3\sigma_T n_e}{16\pi}\int\frac{x_1^2}{x^2}d^2\omega_1\,[n_{11}(1 + n)F_*|_{x_1=x/(1-xw)} - n(1 + n_{11})F_*|_{x_1=x/(1+xw)}]. \tag{41}$$

For the case when $n = n(t, x)$ does not depend on either **r** or ω, i.e., the radiation field is homogeneous and isotropic, an equation of the type (41), as well as the differential equation of the Fokker–Planck type following from it, were derived for low frequencies by Ross et al. (1978) [see also (Nagirner, 1984)].

We now consider the case when the energies of the nonrelativistic electrons and soft photons are of the same order of magnitude ($\sim kT$).

8. Nonrelativistic limit and low frequencies. We set $\tilde{x} = h\nu/kT$, i.e., $x = h\nu/mc^2 = \tilde{x}/y$. Then

$$q = \tilde{q}/y^2, \quad q_* = \tilde{q}_*/y^2, \quad \tilde{q} = \widetilde{xx}_1 w, \quad \tilde{q}_* = \widetilde{xx}_1(2 - w),$$
$$Q = \tilde{Q}/y, \quad \tilde{Q}^2 = (\tilde{x} - \tilde{x}_1)^2 + 2\tilde{q}, \quad Q_* \sim (\tilde{Q}^2 + 2\tilde{q}_*)/y^2. \tag{42}$$

In the low-frequency limit, different ratios of the characteristic energy lost (or acquired) by a photon in a single scattering event and the characteristic frequency scale over which n varies significantly must be distinguished. If the broadening of an atomic line in the visible or ultraviolet region is studied in the case of scattering by nonrelativistic electrons, then the profile of the absorption coefficient in the line is obviously much narrower than the average frequency shift of the photon scattered by an electron, and the recoil effect and the difference between $\tilde{x}$ and $\tilde{x}_1$ can be neglected, i.e., we can set $\tilde{x}_1 = \tilde{x} = x_0$ everywhere except the difference $\tilde{x}_1 - \tilde{x}$. Dropping corrections of order $1/y$, we obtain the redistribution function

$$R(x, x_1, \mu)\, dx_1 \sim \frac{3w_0\sqrt{y}}{32\pi^{3/2}x_0\sqrt{w}} \exp\left(-\frac{y(\tilde{x}-\tilde{x}_1)^2}{4wx_0^2}\right) d\tilde{x}_1. \quad (43)$$

Such a redistribution function was obtained by Hummer and Mihalas (1967) [see also Mihalas (1982)], who averaged it over angles, and also obtained a redistribution function close to (43) corresponding not to a Rayleigh but rather a spherical scattering phase function (i.e., containing the factor $1/4\pi$ instead of $3w_0/16\pi$).

Conversely, if one is calculating the formation of the continuous spectrum over a range much wider than the photon frequency shift occurring in each scattering event, and the spectrum itself is smooth enough, then the integral RKE can be replaced by a differential equation in the frequency. Substituting (42) in Eq. (40), expanding once again all functions in powers of $1/y$, and confining attention to terms of order $1/y$, we obtain the following expression for the redistribution function in this approximation:

$$\tilde{R}(\tilde{x}, \tilde{x}_1, \mu) = R(x, x_1, \mu)/y, \quad (44)$$

where

$$\tilde{R}(\tilde{x}, \tilde{x}_1, \mu) = \frac{3}{16\pi}\{[w_0(1 + w\tilde{x}/y) - 2\mu(1 + 4\mu - 3\mu^2)/y]\,\delta(\tilde{x}_1 - \tilde{x}) + ww_0\tilde{x}/y\,[\tilde{x}\delta''(\tilde{x}_1 - \tilde{x}) + (\tilde{x} + 2)\,\delta'(\tilde{x}_1 - \tilde{x})]\}, \quad (45)$$

and a not too complicated expression is obtained after the effect of δ'' is calculated.

Substituting the redistribution function (45) into the RKE and using the rule (38), we arrive at the equation

$$\frac{1}{c}\frac{\partial n}{\partial t} + \omega\nabla n = \sigma_T n_e \frac{3}{16\pi}\int d^2\omega_1 \Big\{ (n_1 - n)\Big[w_0\Big(1 - \frac{2w\tilde{x}}{y}\Big/y\Big) + \frac{2}{y}(1 - 2\mu - 3\mu^2 + 2\mu^3)\Big] + \frac{ww_0\tilde{x}}{y}[\tilde{x}(n_1'' + n'(1+2n)) + 4(n_1' + n_1(1+n))]\Big\}. \quad (46)$$

Here $n = n(\mathbf{r}, t, \omega, \tilde{x})$ and $n_1 = n(\mathbf{r}, t, \omega, \tilde{x}_1)$, and the derivatives are calculated with respect to the frequency $\tilde{x}$.

When the radiation field is homogeneous and isotropic, the equation (46) becomes the well-known equation derived by Kompaneets (1956),

$$\frac{1}{c}\frac{\partial n}{\partial t} = \frac{n_e\sigma_T}{y}\frac{1}{\tilde{x}^2}\frac{\partial}{\partial\tilde{x}}\Big[\tilde{x}^4\Big(n + n^2 + \frac{\partial n}{\partial\tilde{x}}\Big)\Big], \quad (47)$$

which corresponds to the directionally averaged redistribution function (45)

$$\tilde{R}(\tilde{x}, \tilde{x}_1) = \{(1 + (\tilde{x} - 2)/y)\,\delta(\tilde{x}_1 - \tilde{x}) + \tilde{x}\,[\tilde{x}\delta''(\tilde{x}_1 - \tilde{x}) - (\tilde{x} + 2)\,\delta'(\tilde{x}_1 - \tilde{x})]\}/4\pi. \quad (48)$$

Equations like (46) and (47) have been derived by a more traditional method, namely, by means of Taylor expansions in powers of n (Babuel-Peyrissac and Rouvillois, 1969). We, however, have obtained explicit expressions for the redistribution function for these cases.

9. Conclusions. We have presented well-known representations, and given new ones, for redistribution functions in the case of scattering of radiation by a Maxwellian electron gas, which make it possible to calculate the redistribution function for any values of the parameters. If the electron temperature is such that $y \geq 10$, then the redistribution function can be calculated using Eqs. (1) and (4) with the help of the Gauss–Laguerre formula. If the gas is ultrarelativistic, i.e., $y << 1$, then the expansion (20) must be used. If both frequencies are small, then the redistribution function is best calculated using the series (25). In the case $y >> 1$, the calculations can be performed using Eqs. (30)–(32). Finally, the expression (11) must be used in the intermediate case. These ranges of values of x, x_1, and y overlap. A program for calculating the redistribution function to a fixed degree of accuracy has been written on the basis of the formulas presented in this paper.

For limiting values of x, x_1, and y the redistribution function was represented by expansions which can be used not only to calculate the function but also to investigate different quantities analytically. As an example, we demonstrated how the limiting forms of the kinetic equations can be obtained with the help of such representations.

In studying different astrophysical objects in which Compton scattering plays a large role (for example, the accretion disks of binary x-ray sources and the nuclei of active galaxies), the polarization of the x-rays from these sources could become an important source of additional information, so that it is of great interest to investigate the scattering matrix of polarized radiation scattered by a relativistic electron gas. The scattering matrix for radiation scattered by an isotropic monoenergetic electron gas was derived by us in Nagirner and Poutanen, 1991). The next step in the construction of a reliable apparatus for interpreting polarization observations in x rays is to transfer the methods given in the present paper to a matrix describing the frequency, directional, and polarization redistribution of radiation. This will be done in a separate work.

APPENDIX

In order to calculate the value of the exponential integral $E_1(z)$ with complex argument $z = c + id$, the complex plane is divided into two regions: In the first region $|z| \leq 8$, and in the second region $|z| > 8$. In these two cases we employed rational approximations and the Padé approximations, respectively. We calculated the coefficients for approximations of order 20 (Luke, 1980). Such approximations give high computational accuracy over the entire complex plane, except for the region near the negative real axis within the annulus $8 < |x|$. In this region, $E_1(z)$ can be represented by the from

$$E_1(z) = E_1(c) - i\pi\,\mathrm{sign}\,(\arg z) + \int_z^c e^{-z'}\frac{dz'}{z'}. \quad (49)$$

The function E_1 with negative real argument can be calculated in terms of its expansions in Chebyshev polynomials [see Luke (1980)]. We write the integral on the right-hand side in the form of the series

$$\int_z^c e^{-z'}\frac{dz'}{z'} = e^{-c}\sum_{n=0}^{\infty}\left(-i\frac{d}{c}\right)^{n+1} J_n,$$
$$J_n = \int_0^1 u^n e^{-iud}\,du, \quad d = i(c-z). \tag{50}$$

The value of the integral J_n can be calculated using the recurrence relations

$$J_0 = i(e^{-id}-1)/d, \quad J_n = i(e^{-id} - nJ_{n-1})/d, \tag{51}$$

if d is not very small. In the case $|d| < 1$, the quantities J_n can be calculated with the help of the series

$$J_n = \sum_{k=0}^{\infty}\frac{(-id)^k}{k!\,(k+n+1)}. \tag{52}$$

We now consider the function $e^x[E_1(z+\Delta z) - E_1(z)]$, employed in the expression (18). For small values of Δz ($|\Delta z| < 0.01$), loss of accuracy can occur on subtraction. If $|z|$ is not too small ($|z| > 0.1$), then this can be avoided by representing this function in the form

$$e^z[E_1(z+\Delta z) - E_1(z)] = \Delta E_e(z) - E_e(z+\Delta z)(e^{-z}-1), \tag{53}$$

where $E_e(z) = e^z E_1(z)$, $\Delta E_e(z) = E_e(z+\Delta z) - E_e(z)$. We write the difference $\Delta E_e(z)$ in the form of the series

$$\Delta E_e(z) = \sum_{n=1}^{\infty}(\Delta z)^n d_n,$$
$$d_0 = E_e(z), \quad d_{n+1} = \frac{1}{n+1}\left(\frac{1}{z^{n+1}} - d_n\right). \tag{54}$$

If, however, $|z| \le 0.1$, then the function $\Delta E_1(z) = E_1(z + \Delta z) - E_1(z)$ is best calculated using the series

$$\Delta E_1(z) = -\ln\left(1+\frac{\Delta z}{z}\right) + \Delta z\sum_{n=1}^{\infty}(-1)^n\frac{c_n}{n!\,n},$$
$$c_1 = 1, \quad c_{n+1} = (z+\Delta z)c_n + z^n. \tag{55}$$

We have thus examined the question of calculating the function $E_1(z)$ and the functions associated with it for any value of z.

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