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COMPTON SCATTERING MATRIX FOR RELATIVISTIC MAXWELLIAN ELECTRON DISTRIBUTION

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Abstract—The Compton scattering matrix (CSM) describing the scattering of polarized radiation by electrons is reduced to the form of a single integral over electron distribution. Different methods of calculation of this integral for the Maxwellian electrons are proposed. Series expansions of CSM are found in the limiting cases (ultrarelativistic and non-relativistic electrons, and small photon frequencies).

1. INTRODUCTION

Compton scattering is one of the important mechanisms of the energy exchange between the electron and photon gases in the case of small electron density and sufficiently high temperature. The problems of radiative transfer taking Compton scattering into account are reduced in general to the integro-differential equation where the kernel is the Compton scattering kernel (CSK). In the case of small electron and photon energies ($h\nu, kT_e \ll m_e c^2$) this equation in the Fokker–Planck approximation reduces to the diffusion (in frequency) equation well known as Kompaneets equation,[1] which gives a possibility to solve many problems where the evolution of photon spectra due to the interaction with the electron gas is studied.

The first investigation in the general case without any limits on the photon and electron energy gave a presentation of the CSK in the form of a complicated multiple integral which then is calculated by means of Gaussian quadrature, but this gives an incorrect result in a wide interval of scattering angles where the cross-section itself is large. Only a few years ago the first paper appeared where the correct computational scheme for the CSK was presented. The CSK was reduced to a single integral which can be evaluated with a high accuracy.[2] Other methods of calculation of this integral are given in Ref. 3.

When we solve the general relativistic radiative transfer equation taking polarization into account, we meet with the same difficulties as in the unpolarized case, and the problems of the Lorentz transformation of rotation angles are added. The relativistic kinetic equation taking into account induced scattering and polarization was formulated by Nagirner.[4] Fokker–Planck approximation of the general equation was considered by Stark[5] and Wilson.[6] In the frame of double diffusion approximation (in the optical depth and frequency) Sunyaev and Titarchuk[7] calculated frequency dependence of intensity using the solution of Kompaneets equation, and the angular and polarization structure of radiation field was obtained by iteration procedure based on an expansion in scattering orders using Rayleigh matrix [which is quite different from the exact Compton scattering matrix (CSM) for high electron temperature].[8,9]

The analytical expressions of the scattering matrix for the isotropic monoenergetic electrons obtained by Nagirner and Poutanen[10] (see also Refs. 8, 11 where the detailed deduction is presented) gave a possibility to represent the kinetic equation in a form of the radiative polarization transfer equation, where the kernel is a product of the CSM and two usual rotation matrices, and the CSM is presented as a single integral over the electron distribution. If we consider the equation only for intensity (for the unpolarized radiation) then this product reduces to the CSK. The CSM for the power-law distribution of relativistic electrons was deduced by Bonometto et al[12] (see also Ref. 11). The aim of this paper is to give the effective methods of calculation of the CSM averaged over the Maxwellian electron distribution. Methods we are using here were proposed for the calculation of the CSK (unpolarized case) in Refs. 2, 3.

In Sec. 2, the radiative transfer equation for the Compton scattering of polarized radiation is formulated and the general expressions of the scattering matrix for mononenergetic electrons are given. In Sec. 3, we consider different limiting cases of the CSM. The representation of the CSM through the set of integrals, which can be calculated in different ways, is given in Sec. 4. In the following sections we derive power and asymptotic series for the CSM in limiting cases (ultrarelativistic and non-relativistic electrons, and small photon frequencies).

2. THE RADIATIVE TRANSFER EQUATION AND THE SCATTERING MATRIX FOR MONOENERGETIC ISOTROPIC ELECTRONS

The radiation field and the polarization degree at each point $\mathbf{r}$ at time t can be characterized by the vector of the Stokes parameters (occupation number for the scalar case) $\tilde{n} = \tilde{n}(x, \boldsymbol{\omega}, \mathbf{r}, t) = (n_{\mathrm{I}}, n_{\mathrm{Q}}, n_{\mathrm{U}}, n_{\mathrm{V}})^{\mathrm{T}}$. The radiative transfer equation describing propagation of polarized light through the gas of electrons (in the linear approximation) can be written in the following form[8,11]

$$\left(\frac{1}{c}\frac{\partial}{\partial t} + \boldsymbol{\omega}\cdot\nabla\right)\tilde{n} = -n_{\mathrm{e}}\sigma(x)\tilde{n} + \frac{\sigma_0 n_{\mathrm{e}}}{x}\int_0^{\infty} x_1\,\mathrm{d}x_1 \int \mathrm{d}^2\omega_1 \hat{L}(-\chi)\hat{S}(x, x_1, \mu)\hat{L}(\chi_1)\tilde{n}_1, \tag{1}$$

where $x_1, \boldsymbol{\omega}_1\,(x, \boldsymbol{\omega})$ is the energy (in unit of $m_{\mathrm{e}}c^2$) and direction of the incoming (outgoing) photon, $\tilde{n}_1 = \tilde{n}(x_1, \boldsymbol{\omega}_1, \mathbf{r}, t)$, σ_0 is the Thomson cross-section, n_{e} is the electron density, $\mu = \boldsymbol{\omega}\cdot\boldsymbol{\omega}_1$ is the cosine of the scattering angle, and

$$\sigma(x) = \sigma_0\frac{3\pi}{4x^2}\int_1^{\infty} f(\gamma)\,\mathrm{d}\gamma\left[\left(x\gamma + \frac{9}{2} + \frac{2}{x}\gamma\right)\ln\frac{1 + 2x(\gamma + z)}{1 + 2x(\gamma - z)} - 2xz + z\left(x - \frac{2}{x}\right)\ln(1 + 4x\gamma + 4x^2) + 4x^2 z\frac{\gamma + x}{1 + 4x\gamma + 4x^2} - 2\int_{x(\gamma - z)}^{x(\gamma + z)} \ln(1 + 2\xi)\frac{\mathrm{d}\xi}{\xi}\right] \tag{2}$$

is the Compton cross-section. Here γ is the electron energy (in units of $m_{\mathrm{e}}c^2$), $z = \sqrt{\gamma^2 - 1}$, and $f(\gamma)$ is the electron distribution function, which is normalized as follows:

$$4\pi\int_0^{\infty} z^2 f(\gamma)\,\mathrm{d}z = 1. \tag{3}$$

The rotation matrix

$$\hat{L}(\chi) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos 2\chi & \sin 2\chi & 0 \\ 0 & -\sin 2\chi & \cos 2\chi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{4}$$

describes the transformation of the Stokes parameters under the polarization basis rotation.[11,13]

The CSM can be represented as a single integral over the electron energy distribution $f(\gamma)$

$$\hat{S}(x, x_1, \mu) = \frac{3}{8}\int_{\gamma_*}^{\infty} f(\gamma)\,\mathrm{d}\gamma\hat{R}(x, x_1, \mu, \gamma)$$

$$= \begin{bmatrix} S & S_{\mathrm{I}} & 0 & 0 \\ S_{\mathrm{I}} & S_{\mathrm{Q}} & 0 & 0 \\ 0 & 0 & S_{\mathrm{U}} & 0 \\ 0 & 0 & 0 & S_{\mathrm{V}} \end{bmatrix} = \frac{3}{8}\int_{\gamma_*}^{\infty} f(\gamma)\,\mathrm{d}\gamma \begin{bmatrix} R & R_{\mathrm{I}} & 0 & 0 \\ R_{\mathrm{I}} & R_{\mathrm{Q}} & 0 & 0 \\ 0 & 0 & R_{\mathrm{U}} & 0 \\ 0 & 0 & 0 & R_{\mathrm{V}} \end{bmatrix}, \tag{5}$$

where

$$\gamma_* = \gamma_*(x, x_1, \mu) = [x - x_1 + Q(1 + 2/q)^{1/2}]/2,$$

$$q = xx_1(1 - \mu), \quad Q^2 = (x - x_1)^2 + 2q. \tag{6}$$

We assume further the relativistic Maxwellian electron distribution:

$$f(\gamma) = \frac{y}{4\pi K_2(y)} e^{-y\gamma}, \quad y = \frac{m_e c^2}{k_B T_e}, \tag{7}$$

where T_e is the electron temperature, K_n is the Macdonald function of the n-th order. The matrix $\hat{R}(x, x_1, \mu, \gamma)$ is the CSM for the isotropic electron gas of the fixed energy γ.[10,11] The five functions R in Eq. (5) have common parts, and it is convenient to introduce a set of designations, namely:

$$R_a = \frac{x + x_1}{q^2}\left(\frac{\gamma_-}{a_-^3} + \frac{\gamma_+}{a_+^3}\right) + \frac{r}{q}\left(\frac{1}{a_-^3} - \frac{1}{a_+^3}\right) - \frac{2}{q^2}\left(\frac{1}{a_-} - \frac{1}{a_+}\right),$$

$$R_b = \frac{2}{Q} + \frac{q-2}{q}\left(\frac{1}{a_-} - \frac{1}{a_+}\right),$$

$$R_c = \frac{2(x + x_1)}{q^2 r}\left(\frac{\gamma_-}{a_-} + \frac{\gamma_+}{a_+}\right) + \frac{4}{q^2 r}(a_- - a_+). \tag{8}$$

Here

$$a_\pm^2 = \gamma_\pm^2 + r, \quad \gamma_+ = \gamma + x_1, \quad \gamma_- = \gamma - x, \quad r = \frac{1+\mu}{1-\mu}. \tag{9}$$

Then

$$R = R_a + R_b,$$

$$R_I = R_a + R_c,$$

$$R_U = \frac{2}{Q} + \frac{4Q(x + x_1)^2}{q^2 r^2} + 2R_c - \frac{4}{qr}(a_- - a_+) + \frac{2}{q}\left(\frac{1}{a_-} - \frac{1}{a_+}\right)$$
$$- \frac{4}{q^2 r^2}[2(a_-^3 - a_+^3) + 3(x + x_1)(\gamma_- a_- + \gamma_+ a_+)],$$

$$R_Q = R_U + R_a,$$

$$R_V = R_b - qR_a. \tag{10}$$

When the frequencies x and x_1 are small, the quantities $a_\pm$ are very close to each other, and the loss of accuracy can appear in the difference between them. The same phenomenon occurs when μ is close to 1. If $\mu = -1$ then $r = 0$, but the singularity of R_c and R_U is illusory. Expressions for $\hat{R}(x, x_1, \mu, \gamma)$ in these cases are given in Ref. 11. However, we shall use these expressions to obtain different asymptotic and approximate formulae for $\hat{S}(x, x_1, \mu)$.

Matrix $\hat{R}$ is not zero only if the inequality

$$z^2 Q^2 - [\gamma(x_1 - x) - q]^2 \geqslant 0 \tag{11}$$

satisfies. If we fix x, x_1 and μ, then we have from (11) $\gamma \geqslant \gamma_*(x, x_1, \mu)$, where γ_* is the lower limit in the integral (5). Limitations for other quantities can be found in Ref. 11.

The forward scattering ($\mu = 1$) is a special case, and matrix $\hat{R}$ is proportional to the unit matrix:

$$\hat{R}(x, x_1, 1, \gamma) = 4\delta(x - x_1)\ln(\gamma + z). \tag{12}$$

Integration over the electron distribution gives:

$$\hat{S}(x, x_1, 1) = \frac{3}{8\pi}\frac{K_0(y)}{K_2(y)}\delta(x - x_1). \tag{13}$$

From the explicit expressions (10) we can obtain the simple relation between scattering matrices for the direct and inverse processes:

$$\hat{R}(x_1, x, \mu, \gamma) = \hat{R}(x, x_1, \mu, \gamma + x - x_1). \tag{14}$$

The direct consequence of this relationship is the relation between the scattering matrix, averaging over the Maxwellian electron distribution:

$$\hat{S}(x_1, x, \mu) = \exp(y(x - x_1))\hat{S}(x, x_1, \mu). \tag{15}$$

This relation can be useful for checking the accuracy and for reducing the scope of the programmes for calculating of the CSM, because we can confine ourselves to the case $x \geqslant x_1$.

Now we shall consider the method of calculation of the CSM in the general case.

3. GENERAL CASE

Observing that the quantities q, Q, x, x_1 and r do not depend on the electron energy and introducing the designations

$$\delta_n^- = \int_{\gamma_*}^{\infty} (a_-^n - a_+^n)\exp(-y(\gamma - \gamma_*))\,\mathrm{d}\gamma,$$

$$\delta_n^+ = \int_{\gamma_*}^{\infty} (\gamma_- a_-^n + \gamma_+ a_+^n)\exp(-y(\gamma - \gamma_*))\,\mathrm{d}\gamma, \tag{16}$$

we write down the expressions for the redistribution functions averaged over the Maxwellian distribution in the form (note that S_a, S_b, S_c are the integrals from R_a, R_b, R_c, respectively)

$$S_a = \frac{3y\,\mathrm{e}^{-y\gamma_*}}{32\pi K_2(y)}\left[\frac{x + x_1}{q^2}\delta_{-3}^+ + \frac{r}{q}\delta_3^- - \frac{2}{q^2}\delta_{-1}^-\right],$$

$$S_b = \frac{3y\,\mathrm{e}^{-y\gamma_*}}{32\pi K_2(y)}\left[\frac{2}{Qy} + \frac{q-2}{q}\delta_{-1}^-\right],$$

$$S_c = \frac{3y\,\mathrm{e}^{-y\gamma_*}}{32\pi K_2(y)}\left[\frac{2(x + x_1)}{q^2 r}\delta_{-1}^+ + \frac{4}{q^2 r}\delta_1^-\right],$$

$$S_U - 2S_c = \frac{3y\,\mathrm{e}^{-y\gamma_*}}{32\pi K_2(y)}\left[\frac{2}{y}\left(\frac{1}{Q} + \frac{2Q(x + x_1)^2}{q^2 r^2}\right) - \frac{4}{qr}\delta_1^- + \frac{2}{q}\delta_{-1}^- - \frac{12(x + x_i)}{q^2 r^2}\delta_1^+ - \frac{8}{q^2 r^2}\delta_3^-\right]. \tag{17}$$

The expressions for all other functions can be easily obtained from (10). Using the equations

$$\frac{\partial(1/a_\pm)}{\partial\gamma} = -\frac{\gamma_\pm}{a_\pm^3}, \quad \frac{\partial(\gamma_\pm/a_\pm)}{\partial\gamma} = \frac{r}{a_\pm^3}, \quad \frac{\partial a_\pm}{\partial\gamma} = \frac{\gamma_\pm}{a_\pm},$$

$$\frac{\partial(a_\pm\gamma_\pm)}{\partial\gamma} = 2a_\pm - \frac{r}{a_\pm}, \quad \frac{\partial a_\pm^3}{\partial\gamma} = 3a_\pm\gamma_\pm, \tag{18}$$

and integrating $\delta^\pm$ by parts, we express (16) in terms of the other four functions:

$$\delta_{-1}^- = \sigma_0^-, \quad \delta_{-1}^+ = \sqrt{r}\sigma_1^+, \quad \delta_{-3}^- = \frac{qt_*}{\Pi} + \frac{y}{\sqrt{r}}\sigma_1^-,$$

$$\delta_{-3}^+ = \frac{(x + x_1)t_*}{\Pi} - y\sigma_0^+, \quad \delta_1^- = -\frac{Q}{y} + \frac{\sqrt{r}}{y}\sigma_1^-,$$

$$\delta_1^+ = \frac{Q(1+q)(x + x_1)}{yq} + \frac{2(x + x_1)t_*}{y^2} + \frac{\sqrt{r}}{y}\left(\frac{2}{y}\sigma_1^+ - \sqrt{r}\sigma_0^+\right),$$

$$\delta_3^- = -\frac{\Phi_*}{y} - \frac{3(Q^2 + qr)t_*}{y^2} - \frac{6Q}{y^3} + \frac{3\sqrt{r}}{y^2}\left(\frac{2}{y}\sigma_1^- - \sqrt{r}\sigma_0^-\right). \tag{19}$$

Here

$$\Pi = \frac{Q^2 + qr(q+2)}{2q}, \quad \Phi_* = \frac{Q}{2q}[(2q+3)Q^2 + 3rq(q+2)], \tag{20}$$

$$\sigma_n^{\pm} = e^{y\rho_-}L_n^- \pm e^{y\rho_+}L_n^+, \quad \rho_\pm = \gamma_\pm(\gamma_*) = [Qt_* \pm (x + x_1)]/2, \tag{21}$$

$$L_n^{\pm} = \int_{c_\pm}^{\infty} \exp(-y_r p)\frac{p^n\,dp}{\sqrt{1+p^2}}, \quad y_r = \sqrt{r}y, \quad c_\pm = \rho_\pm/\sqrt{r}. \tag{22}$$

The methods of calculation of integrals (22) are given in Refs. 2, 3.

In the same manner as for the scalar CSK studied earlier[2,3] we give in the following sections expansions in a series of the scattering matrix in the limiting cases of weakly-relativistic and ultrarelativistic electron gas, and for small photon energies.

4. ULTRARELATIVISTIC LIMIT

Let us assume that the electron temperature is large, so $y \ll 1$. We assume as well that $y_r = \sqrt{r}y \ll 1$. By making the change of variables $p = (u - 1/u)/2$ in the integrals (22), expanding the exponents in a Taylor series and integrating by parts, one obtains the following expansions

$$L_0^{\pm} = \sum_{k=0}^{\infty} \frac{1}{k!}\left(\frac{y_r}{2u_\pm}\right)^k E_{k+1}\left(\frac{y_r u_\pm}{2}\right), \quad L_1^{\pm} = -\frac{2}{y_r}L_0^{\pm} + \frac{1}{y_r}e^{-y\rho_\pm}, \tag{23}$$

where $u_\pm = (t_* \pm 1)(Q + x + x_1)/2\sqrt{r}$, and E_k is the integral exponential function of order k.

Thus, functions (22), and through (17), the scattering matrix (5) as well, are represented by the quickly converging series.

5. SMALL FREQUENCIES

In this section we consider the case of small frequencies (x, $x_1 \ll 1$) of initial and scattered photons. Let us assume that $x, x_1 \to 0$, and the ratio $x_1/x = X$ is constant. Then for $x \to 0$ the product $x\hat{R}(x, X, x, \mu, \gamma)$ has a finite limit. Expanding it in a Taylor series in x, we have:

$$x\hat{R}(x, X, x, \mu, \gamma) = \sum_{n=0}^{\infty} \hat{R}_n(X, \mu, \gamma)x^n. \tag{24}$$

Expanding powers of $a_\pm$ in a series in Legendre polynomials, we can find formulae for the coefficients of power series (24). The initial expansions are the following:

$$\frac{1}{a_-} = \frac{1}{a_0}\sum_{n=0}^{\infty}\left(\frac{x}{a_0}\right)^n P_n\left(\frac{\gamma}{a_0}\right) \tag{25}$$

and the expansion for $1/a_+$, which can be obtained by replacing x by $-x_1$. Here we denote $a_0^2 = \gamma^2 + r$. The coefficients of the expansion of matrix (10) are easy to find, substituting series, which follow from (25) and (24) into formulae (8) and (10)

$$R_{na}(X, \mu, \gamma) = \frac{1}{a_0^n}\left\{\frac{\chi_n}{X^2w^2a_0^4}P_{n+3} - \frac{1}{Xwa_0^2}[1 - (-X)^{n+1}]\frac{(n+2)(n+3)}{2n+5}P_{n+3,n+1}\right\},$$

$$R_{nb}(X, \mu, \gamma) = \frac{2}{Q_0}\delta_{n0} + \frac{1}{a_0^n}\left\{[1 - (-X)^{n-1}]P_{n-1} - \frac{2}{Xwa_0^2}[1 - (-X)^{n+1}]P_{n+1}\right\},$$

$$R_{nc}(X, \mu, \gamma) = \frac{2}{rX^2w^2a_0^{n+2}}\frac{\chi_n}{2n+5}P_{n+3,n+1},$$

$$R_{nU}(X, \mu, \gamma) = 2R_{nc}(X, \mu, \gamma) + \left[\frac{2}{Q_0} + \frac{4Q_0}{r^2w^2X^2}(1+X)^2\right]\delta_{n0} + \frac{2}{Xw}\frac{1}{a_0^n}\left(\frac{P_{n+1}}{a_0^2} + \frac{2P_{n+1,n-1}}{r(2n+1)}\right)$$

$$\times\left[1 - (-X)^{n+1} - \frac{6\chi_n}{rXw(n+2)(n+3)}\right]. \tag{26}$$

Here for short we use the notations

$$\chi_n = (n+1)[1-(-X)^{n+3}] + (n+3)X[1-(-X)^{n+1}],$$
$$Q_0^2 = (1-X)^2 + 2wX, \quad P_n = P_n(\gamma/a_0), \quad P_{n,m} = P_n - P_m. \tag{27}$$

Note that the coefficients of expansion in the second argument are the same:

$$x\hat{R}(Xx, x, \mu, \gamma) = \sum_{n=0}^{\infty} \hat{R}_n(X, \mu, \gamma)(-x)^n. \tag{28}$$

It is following from the property of the matrix coefficients

$$\hat{R}_n(1/X, \mu, \gamma) = (-1)^n X^{1-n} \hat{R}_n(X, \mu, \gamma). \tag{29}$$

The last equation is easy to verify using explicit expressions (26).

Substituting (24) and (26) into (5) and integrating over γ, we obtain

$$\hat{S}(x, x_1, \mu) = \frac{3y}{32\pi} \frac{e^{-y\gamma_*}}{K_2(y)} \sum_{n=0}^{\infty} x^{n-1} \hat{S}_n(X, \mu). \tag{30}$$

The coefficients of the expansion are deduced from (26) and (24) and expressed by means of the two sets of functions $I_n^{\pm}$

$$S_{na}(X, \mu) = \frac{\chi_n}{X^2 w^2} I_{n+3}^+ - \frac{1}{Xw} \frac{(n+2)(n+3)}{2n+5} [1-(-X)^{n+1}] I_{n+3,n+1},$$

$$S_{nb}(X, \mu) = \frac{2}{yQ_0} \delta_{n0} + [1-(-X)^{n-1}] I_{n-1}^+ - \frac{2}{Xw} [1-(-X)^{n+1}] I_{n+1}^+,$$

$$S_{nc}(X, \mu) = \frac{2}{rX^2 w^2} \frac{\chi_n}{2n+5} I_{n+3,n+1},$$

$$S_{nU}(X, \mu) = 2S_{nc} + \left[\frac{2}{Q_0} + \frac{4Q_0}{r^2 X^2 w^2}(1+X)^2\right] \delta_{n0} + \frac{2}{Xw}\left(I_{n+1}^+ + \frac{2I_{n+1,n-1}}{r(2n+1)}\right)$$
$$\times \left[1-(-X)^{n+1} - \frac{6\chi_n}{rXw(n+2)(n+3)}\right]. \tag{31}$$

The functions mentioned above are the integrals

$$I_n^{\pm} = \int_{\gamma_*}^{\infty} e^{-y(\gamma-\gamma_*)} P_n\left(\frac{\gamma}{a_0}\right) \frac{d\gamma}{a_0^{n\pm 1}}, \quad I_{n,m} = I_n^- - I_m^+. \tag{32}$$

Computations of these functions can be performed by the recursion relations

$$nI_n^+ = \frac{1}{a_*^n} P_{n-1}\left(\frac{\gamma_*}{a_*}\right) - yI_{n-1}^+,$$

$$n(2n-3)I_n^- = \frac{2n-1}{a_*^{n-2}} P_{n-1}\left(\frac{\gamma_*}{a_*}\right) + 2(n-1)I_{n-2}^+ - y(2n-1)I_{n-1}^-, \tag{33}$$

where $a_*^2 = \gamma_*^2 + r$. It is necessary to calculate the first two functions of each set numerically by the same procedure as for integral (22).

The recursion relations give accurate results not for all values of variables. For large y, errors increase very quickly, but in that case integrals (32) can be easily calculated using the Gauss–Laguerre quadrature. The formulae of the next section can be useful as well.

6. NON-RELATIVISTIC LIMIT

When the temperature of the electron gas is low, the value of $y = mc^2/k_B T_e$ is large. In that case it is possible to obtain the expansion in negative powers of y. Let us present here the terms of the order of $1/y^2$ inclusively.

The initial expansions are

$$\frac{1}{a_\pm}=\frac{1}{\alpha^*_\pm}\sum_{n=0}^{\infty}\left(\frac{\gamma_*-\gamma}{\alpha^*_\pm}\right)^n P_n^\pm, \tag{34}$$

where

$$\alpha^*_\pm = a_\pm(\gamma_*)=[(x+x_1)t_*\pm Q]/2, \quad P_n^\pm = P_n(\rho_\pm/\alpha^*_\pm). \tag{35}$$

Substituting (34) into (16), integrating by parts and collecting the terms with equal powers of y, we obtain

$$y\delta_n^\pm \approx \sum_{j=0}^{2}\Delta^\pm_{n,j}/y^j. \tag{36}$$

We express the quantities here by means of the functions introduced earlier:

$$\Delta^-_{-1,0}=\frac{Q}{\Pi}, \quad \Delta^-_{-1,1}=\frac{t_*}{2q\Pi^3}[-Q^4+rq(q-1)Q^2+rqq_*],$$

$$\Delta^-_{-1,2}=\frac{2\Phi_*}{\Pi^3}-\frac{3r\Phi_{**}}{\Pi^5},$$

$$\Delta^+_{-1,0}=\frac{Q(x+x_1)}{q\Pi}, \quad \Delta^+_{-1,1}=\frac{r(x+x_1)t_*}{2q\Pi^3}[(2q+1)Q^2+q_*],$$

$$\Delta^+_{-1,2}=-\frac{3r}{\Pi^5}\frac{Q(x+x_1)}{4q^3}[(2q^2+4q+1)Q^4+2q_*(1+q-q^2)Q^2+q_*^2(1-2q)],$$

$$\Delta^-_{-3,0}=\frac{\Phi_*}{\Pi^3}, \quad \Delta^-_{-3,1}=\frac{3}{\Pi^5}\frac{t_*}{4q^2}[-2(q+1)Q^6+rq(2q^2-2q-7)Q^4+2rqq_*(2q-1)Q^2+rqq_*^2],$$

$$\Delta^-_{-3,2}=\frac{12\Phi_{**}}{\Pi^5}-\frac{15r}{\Pi^7}\frac{Q}{8q^3}[(8q^3+28q^2+28q+7)Q^6+7q_*(2q+3)(2q+1)Q^4$$
$$+7q_*^2(4q+3)Q^2+7q_*^3],$$

$$\Delta^+_{-3,0}=\frac{Q(x+x_1)}{2q^2\Pi^3}[(q+1)Q^2+q_*(1-q)],$$

$$\Delta^+_{-3,1}=\frac{3r(x+x_1)t_*}{4q^2\Pi^5}[(4q^2+6q+1)Q^4+2q_*(3q+1)Q^2+q_*^2]-\frac{2}{r}\Delta^+_{-1,1},$$

$$\Delta^+_{-3,2}=-\frac{15rQ(x+x_1)}{8q^4\Pi^7}[(q+1)(4q^2+8q+1)Q^6+q_*(-4q^3+4q^2+15q+3)Q^4$$
$$+q_*^2(-8q^2+3q+2)Q^2+q_*^3(1-3q)]-\frac{2}{r}\Delta^+_{-1,2},$$

$$\Delta^-_{1,0}=-Q, \quad \Delta^-_{1,1}=-\frac{rqt_*}{\Pi}, \quad \Delta^-_{1,2}=\frac{r\Phi_*}{\Pi^3},$$

$$\Delta^+_{1,0}=Q(x+x_1)\frac{q+1}{q}, \quad \Delta^+_{1,1}=(x+x_1)t_*\left(2-\frac{r}{\Pi}\right),$$

$$\Delta^+_{1,2}=\frac{(x+x_1)Q}{2q^3\Pi^3}[Q^4+rq(3q+5)Q^2+3rqq_*],$$

$$\Delta^-_{3,0}=-\Phi_*, \quad \Delta^-_{3,1}=-3t_*(Q^2+rq), \quad \Delta^-_{3,2}=-3Q\left(2+\frac{r}{\Pi}\right). \tag{37}$$

Here $q_*=rq(q+2)$, Φ_* is defined in Eq. (20), and

$$\Phi_{**}=\frac{Q}{2q^2}[(4q^2+10q+5)Q^4+10q_*(q+1)Q^2+5q_*^2]. \tag{38}$$

Formulae (37) give relative error less that $2 \cdot 10^{-4}$ for $y = 100$. In the limiting case of small photon energies $x \ll 1$ (for example, if we are investigating the broadening of atomic spectral line in the optical or ultraviolet regions due to the scattering by non-relativistic electrons) we can put $x = x_1$ everywhere in Eqs. (37) and keep difference $x - x_1$ only in the expression for γ_* [see Eqs. (6), (17)]. Taking into account terms of order $1/y$, we obtain the deviation of the CSM from the Rayleigh matrix in non-relativistic limit:

$$\hat{S}(x, x_1, \mu) = \frac{3\sqrt{y}(1 - 15/8y)}{32\pi^{3/2}x\sqrt{w}} \exp\left[-y\frac{x - x_1}{2}\left(1 + \frac{x - x_1}{2wx^2}\right)\right]\left\{\hat{\mathscr{R}} + \frac{w}{y}\hat{\mathscr{C}}\right\}, \tag{39}$$

where $w = 1 - \mu$, $\hat{\mathscr{R}}$ is the Rayleigh scattering matrix

$$\hat{\mathscr{R}} = \begin{bmatrix} 1+\mu^2 & \mu^2-1 & 0 & 0 \\ \mu^2-1 & 1+\mu^2 & 0 & 0 \\ 0 & 0 & 2\mu & 0 \\ 0 & 0 & 0 & 2\mu \end{bmatrix}, \tag{40}$$

and $\hat{\mathscr{C}}$ is the correction matrix

$$\hat{\mathscr{C}} = \begin{bmatrix} 2+\mu-5\mu^2 & 4-\mu-5\mu^2 & 0 & 0 \\ 4-\mu-5\mu^2 & -2-3\mu-5\mu^2 & 0 & 0 \\ 0 & 0 & -3-7\mu & 0 \\ 0 & 0 & 0 & 1-3\mu \end{bmatrix}. \tag{41}$$

7. FAST NUMERICAL SCHEME

If we divide the interval of integration $[\gamma_*, \infty)$ in Eq. (5) into subintervals, and then approximate $\exp(-y\gamma)$ by linear function, we can calculate the resulting integral using only elementary function:

$$\int_{\gamma_*}^{\infty} \exp(-y\gamma)\left(\hat{R}(x, x_1, \mu, \gamma) - \hat{1}\frac{2}{Q}\right)d\gamma = \exp(-y\gamma_*)\sum_{i=1}^{3}\frac{2(p_i - p_{i+1})}{(t_{i+1} - t_i)(p_i + p_{i+1})}$$

$$\int_{\gamma_i}^{\gamma_{i+1}}\left[\frac{p_i(\gamma_{i+1} - \gamma) + p_{i+1}(\gamma - \gamma_i)}{\gamma_{i+1} - \gamma_i}\right]\left(\hat{R}(x, x_1, \mu, \gamma) - \hat{1}\frac{2}{Q}\right)d\gamma, \tag{42}$$

where $\hat{1}$ is unit matrix, $\gamma_i = \gamma_* + t_i/y$, $t_i = (0, 0.8, 1.9, 4.7)$, $p_i = \exp(-t_i)$, except $p_4 = 0$. We have used here the same linear approximation that Kershaw *et al.*[2] have done. It gives an exact result if $\hat{R}$ is constant.

To evaluate this integral we need indefinite integrals $\int d\gamma(\hat{R} - \hat{1}\frac{2}{Q})$ and $\int \gamma\, d\gamma(\hat{R} - \hat{1}\frac{2}{Q})$. We shall calculate the integrals for the functions $R_{a,b,c,U}$, and find all others using formulae (10). Preventing round off errors if x, x_1 or/and r are small, we find the following expressions:

$$\int d\gamma R_a = -\frac{2}{q^2}[2b + A_-] + \frac{br}{2v}\left[\frac{u^2 - Q^2}{rq} - 2 - \frac{4}{q}\right] - \frac{bu^2}{q^2 v},$$

$$\int d\gamma\left(R_b - \frac{2}{Q}\right) = \frac{q-2}{q}A_-,$$

$$\int d\gamma R_c = \frac{2}{q^2}[2b + A_-] + \frac{b}{q}\left[2 - \frac{u^2 - Q^2}{rq}\right],$$

$$\int d\gamma\left(R_U - \frac{2}{Q}\right) = \frac{1}{q^2}[2b + A_-] + b\left\{-\frac{3}{q} - 2 - (u^2 + 3v)\left(\frac{u - Q}{rq}\right)^2\right.$$

$$\left. + \left(1 - \frac{3}{2q}\right)\frac{u^2 - Q^2}{rq} + \frac{1}{rq}\left[2v\left(1 - u\frac{u - Q}{rq}\right) - \frac{Q^2}{q}\right]\right\}, \tag{43}$$

$$\int \mathrm{d}\gamma R_{\mathrm{a}}\left(\gamma + \frac{x_1 - x}{2}\right) = -\frac{u^3}{2q^2 v} + \frac{u}{q} + \frac{ur}{4v}\left[\frac{u^2 - Q^2}{rq} - 2 - \frac{4}{q}\right],$$

$$\int \mathrm{d}\gamma \left(R_{\mathrm{b}} - \frac{2}{Q}\right)\left(\gamma + \frac{x_1 - x}{2}\right) = \frac{q-2}{q}(\epsilon A_+ - u),$$

$$\int \mathrm{d}\gamma R_{\mathrm{c}}\left(\gamma + \frac{x_1 - x}{2}\right) = -\frac{u^3 - Q^3}{3rq^2},$$

$$\int \mathrm{d}\gamma \left(R_{\mathrm{U}} - \frac{2}{Q}\right)\left(\gamma + \frac{x_1 - x}{2}\right) = -\left(\frac{2}{q} + 1\right)u$$

$$+ \frac{u^3 - Q^3}{3rq}\left(1 - \frac{2}{q}\right) + \frac{u - Q}{2rq}\left[u(u + Q) + \frac{2}{q}Q^2\right]$$

$$- \frac{1}{10}\left(\frac{u - Q}{rq}\right)^2 [4u^3 + 3u^2 Q + 2uQ^2 + Q^3] + \frac{u}{rq}\left[2v\left(1 - u\frac{u - Q}{rq}\right) - \frac{Q^2}{q}\right],$$

where

$$\epsilon = (x + x_1)/2, \quad b = \frac{2\epsilon}{a_- + a_+},$$

$$v = a_- a_+, \quad u = a_+ - a_- = b(\gamma_- + \gamma_+). \tag{44}$$

Singularity for $r = 0$ is illusory. If r is close to zero we use the following expressions:

$$\frac{1}{rq}\left[2v\left(1 - u\frac{u - Q}{rq}\right) - \frac{Q^2}{q}\right] = 1 + \frac{2}{q} - \frac{u - Q}{2rq}\left[u + Q + \frac{u - Q}{rq}v_\gamma + \frac{8\epsilon^2 u}{qv_\gamma}\right],$$

$$\frac{u - Q}{rq} = \frac{CD}{u + Q}, \tag{45}$$

where $C = 2/[v_\gamma + rq]$, $v_\gamma = v + \gamma_- \gamma_+ + r$, $D = 2\gamma_- \gamma_+ - Q^2 t_*^2/2$.

8. CONCLUSIONS

The Compton scattering matrix is presented as a single integral over the electron distribution. Series expansions for the ultrarelativistic and non-relativistic electrons, and small photon frequencies are given. Formulae presented give a possibility to compute CSM with high accuracy. On the base of these formulae we have written a FORTRAN subroutine to calculate the CSM for the Maxwellian electron distribution. Next step is the solution of the radiative polarization transfer equation, that would be very useful for the interpretation of future X-ray polarimetric observations of various astrophysical objects. As a first application of the developed technique, the polarization of the radiation scattered in the hot optically thin corona around accretion disk was calculated in a single scattering approximation.[9] Results differ drastically from calculations where the Rayleigh matrix was applied.

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