RADIATIVE PROCESSES
in
ASTROPHYSICS

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Chapter 1

Introduction

Most of the information we obtain about the Universe reaches us in form of the electromagnetic (EM) radiation. The only important exceptions are the measurements in situ at the surfaces of a few planets and their satellites (e.g. Earth and Moon, Mars, Venus, Saturn's rings and moons Enceladus and Titan, Jupiter), measurements of the particles in the solar system, cosmic rays and neutrinos at the Earth surface. In some near future gravitational waves may also be detected and used to probe processes happening in vicinities of relativistic objects.

The vast majority of what we know about the Universe, we learnt by collecting and studying of various forms of EM radiation. In the ancient times, people could only use light in the optical part of the spectrum, while in the 20th century the astrophysical instruments have been developed to study radiation in the radio, infrared, ultraviolet (UV), X-ray and gamma-ray part of the EM spectrum. The radio astronomy appeared only after the World War II when the radars were build to detect emission in radio band. To observe UV, X-ray and gamma-ray photons one needs to make either balloon flights to the height of 10–20 km, or even better to send an observatory to space. Therefore, very active developments in that field were only possible starting from the 1960s when the first rockets and satellites were built. We have learnt that some of the objects emit a large fraction of their radiation in the energy bands invisible from the ground (i.e. Earth surface). An example of such an object is shown in Fig. 1.1.

To understand the physics of such objects, we need to obtain the data simultaneously in various energy band of the EM spectrum. To interpret the EM radiation we receive, we need to know the microphysics, i.e. the radiative processes that give rise to this radiation as well as the processes that change this radiation on the way to our detectors. Much of the information about stars, interstellar

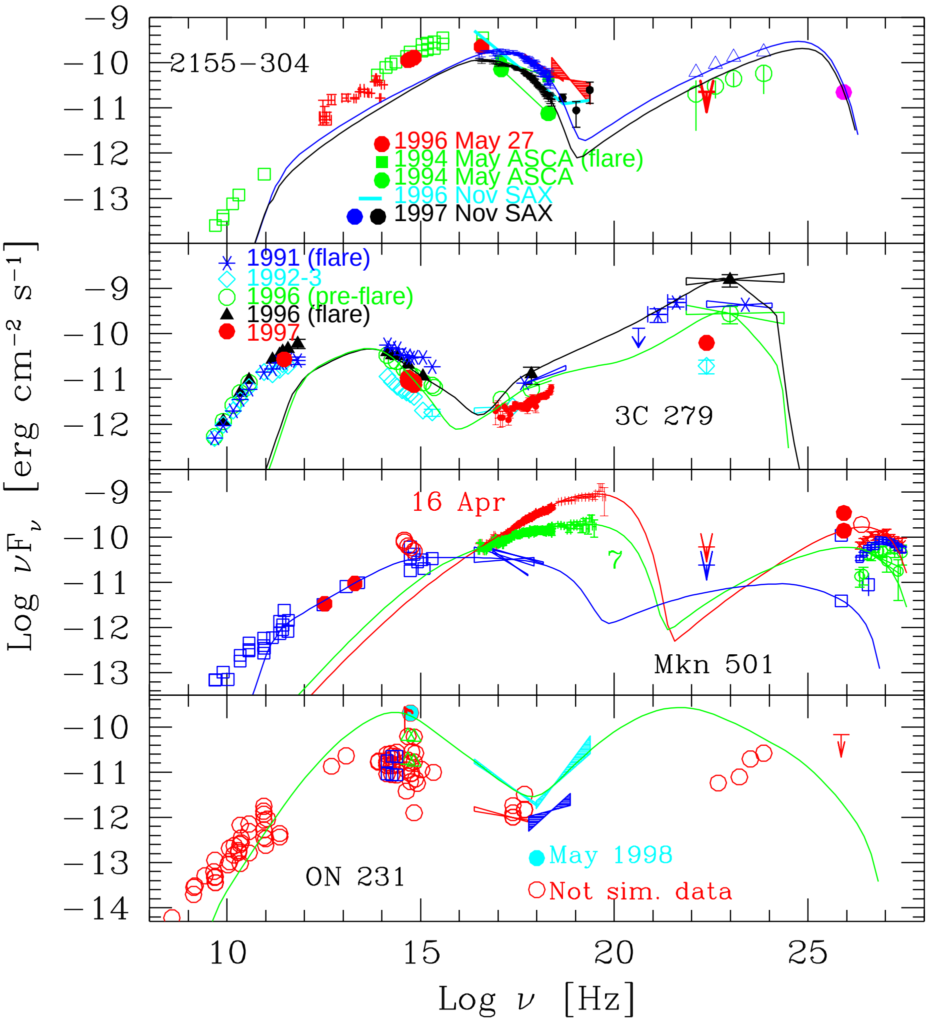
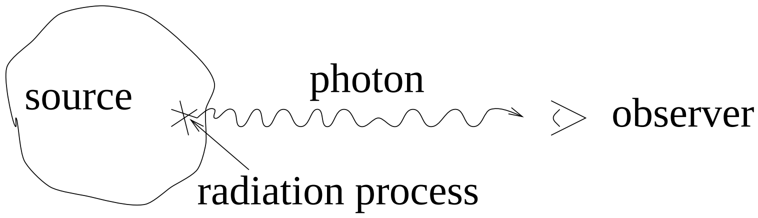


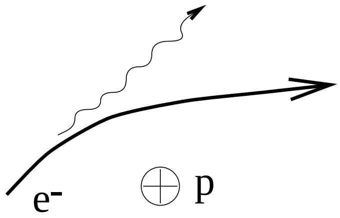
Figure 1.1: Some sources emit radiation from radio to γ-rays. In order to understand the physics of such objects, one requires to perform coordinated multifrequency observations. Here examples of the spectral energy distributions of a few active galactic nuclei are shown. The spectra extend from the radio to the TeV energies (teraelectron volt, i.e. 10^{12} eV). The energy of the most energetic photons exceeds the energy of photons in the visual band by a factor 10^{12}.

medium, and galaxies comes from studying their line spectra. However, many objects do not produce strong spectral lines, and we have to study their continuum (i.e. weakly dependent on wavelength) spectra. The most important radiative processes that produce these continua will be studied during our course.



We first start with the basic description and general concepts of the radiation field. We then introduce the radiative transfer equation that describes how the radiation changes when it passes through the medium in the source as well as on the way to us. Next we consider microscopic description of the radiation field as its relation to the Maxwell equations that describe electromagnetic fields. Further we consider radiation of the moving charges. We then discuss radiative processes giving rise to continuum spectra. All of them involve motion of charged particles: bremsstrahlung (or free-free emission) involves interaction between electrons and other charged species (e.g. protons), cyclotron or synchrotron radiation which is produced by electrons in the magnetic field, and Compton scattering which is a result of scattering of photons by the electrons. In the order of appearance we will study:

1. Bremsstrahlung. Radiation of an electron in the electric field of a proton or another charged particle. Important work on that process appeared in the beginning of the 20th century. It is important in HII regions, planetary nebulae, stars, accreting white dwarfs, and clusters of galaxies (see Fig. 1.2).



2. Synchrotron radiation. Theory was developed mostly in the 2nd half of the 20th century (Ginzburg, Syrovatsky), while some important papers have already

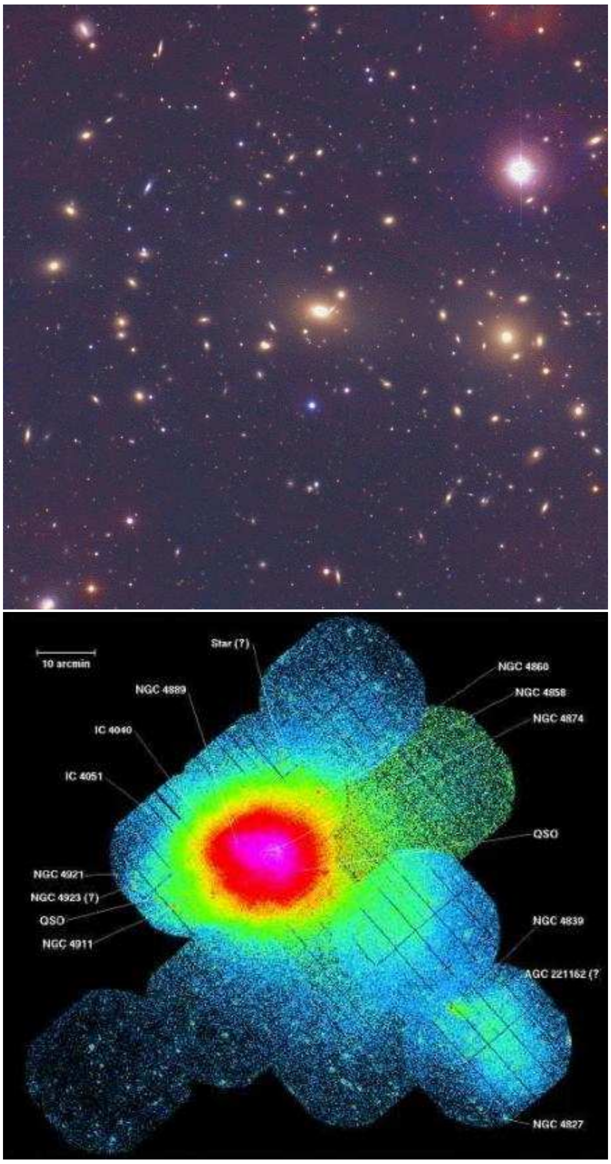


Figure 1.2: Clusters of galaxies, the largest bound structures in the Universe, are filled with hot ($T \sim 10^7 - 10^8$ K) X-ray emitting gas. Upper panel shows the image of Coma cluster taken in the optical band, where you can see hundreds of galaxies. Lower panel, is the XMM image of the Coma cluster in the X-ray, where the glowing gas filling the whole intergalactic space is seen.

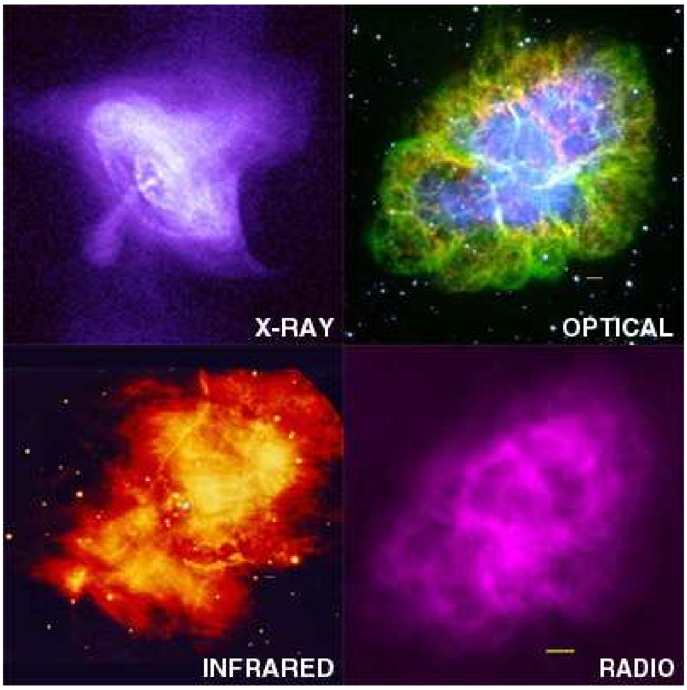
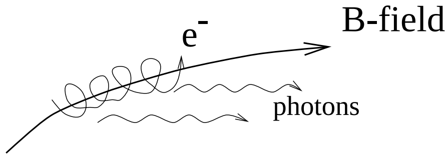


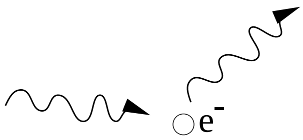
Figure 1.3: A detailed view of the Crab Nebula produced by a supernovae seen on Earth in the year 1054. Here are pictures of the Crab at X-ray (Chandra), optical (Palomar), infrared (Keck), and radio (VLA) wavelengths (from NASA/CXC/SAO). Most of the diffuse emission is produced by synchrotron radiation of electrons spiraling in the magnetic field as was first proposed by Iosif Shklovsky in 1953.

appeared in the 1910s (Shott 1912). It found important applications with the start of radioastronomy. The process is important in pulsar nebulae (see Fig. 1.3) and jets from active galaxies (see Fig. 1.1, where the lower energy bump is believed to be produced by synchrotron radiation).

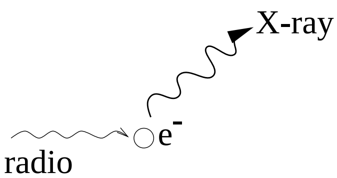


3. Coherent Compton scattering (Thomson scattering). The theory of radiation transport where Thomson scattering played a dominant role was developed

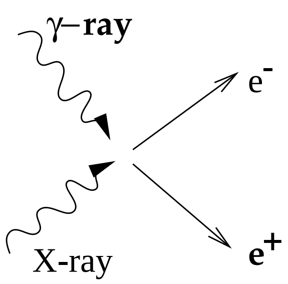
in the 1930s to 1950s by Chandrasekhar, Ambartsumyan, and Sobolev. It is important as a source of additional opacity in planetary and stellar atmospheres, e.g. white dwarfs.



4. Non-coherent Compton Scattering. Development in the 1940s-50s because of its importance for the nuclear bomb research (Kompaneets). Important applications in the physics of early Universe (Sunyaev, Zeldovich, Illarionov) and astrophysics of relativistic objects (Sunyaev, Titarchuk, Lightman). The theory was further developed in the 1970s–1990s when appeared the need to interpret the data from X-ray satellites. It is a dominating radiative process shaping the spectra from accreting black holes (both stellar-mass and super-massive), neutron stars, and jet from active galactic nuclei (e.g. high-energy bump in Fig. 1.1).



5. $e^{\pm}$ pair production. At the extreme environments of compact objects such as pulsars and accreting black holes, high-energy photons interact with low-energy photons producing electron-positron pairs. This process was studied in 1980s (by Svensson, Lightman, Zdziarski) in connection with interpretation of the data from active galaxies. With the development of γ-ray astronomy (including TeV astronomy) it is becoming even more clear how important it is for the correct interpretation of the data from pulsars, jets from active galactic nuclei, and gamma-ray bursts.



Chapter 2

Basics of radiative transfer

2.1 Definitions

- In vacuum the photon wavelength and frequency are related as $\lambda\nu = c$. Photon energy is related to the frequency through Planck constant: $E = h\nu$.
- Specific intensity I_ν [units: erg cm^{-2} s^{-1} Hz^{-1} str^{-1}]. The energy passing through the area dA normal to the direction of photon propagation within the solid angle dΩ during the time interval dt in the frequency interval dν is:

$$\mathrm{d}E = I_\nu \,\mathrm{d}\Omega\, \mathrm{d}A\, \mathrm{d}t\, \mathrm{d}\nu. \tag{2.1}$$

- Energy flux F_ν [units: erg cm^{-2} s^{-1} Hz^{-1}]. The energy passing through the area dA during the time interval dt in the frequency interval dν is:

$$\mathrm{d}E = F_\nu \,\mathrm{d}A\, \mathrm{d}t\, \mathrm{d}\nu. \tag{2.2}$$

  The flux can be obtained by integrating the intensity over solid angles and taking into account the projection effect:

$$F_\nu = \int I_\nu \cos\theta \,\mathrm{d}\Omega, \tag{2.3}$$

  where θ is the angle between the normal to the area and the direction of the ray.

  The energy passing per unit time through the sphere of radius r from the isotropic source in steady-state in the absence of sinks on the way is

$$L = 4\pi r^2 F(r) = 4\pi r_1^2 F(r_1). \tag{2.4}$$

This is just the energy conservation law. The consequence of that is the so called inverse square law:

$$F(r) = \frac{L}{4\pi r^2} \propto r^{-2}. \tag{2.5}$$

- (Specific) momentum flux (pressure) [units: dyn cm^{-2} Hz^{-1}]:

$$p_\nu = \frac{1}{c} \int I_\nu \cos^2\theta \, \mathrm{d}\Omega \tag{2.6}$$

because the photon momentum is $h\nu/c$. Another $\cos\theta$ comes from the projection of the transferred momentum to the normal to the area.

- (Specific) energy density [units: erg cm^{-3} Hz^{-1}]:

$$u_\nu = \frac{1}{c} \int I_\nu \, \mathrm{d}\Omega = \frac{4\pi}{c} J_\nu, \tag{2.7}$$

where

$$J_\nu = \frac{1}{4\pi} \int I_\nu \, \mathrm{d}\Omega \tag{2.8}$$

is the mean intensity. Total energy density [units: erg cm $^{-3}$]:

$$u = \int u_\nu \, \mathrm{d}\nu = \frac{4\pi}{c} \int J_\nu \, \mathrm{d}\nu. \tag{2.9}$$

- Radiation pressure for isotropic radiation field.
Consider a reflecting enclosure containing isotropic radiation field. Each photon transfers twice its normal component of the momentum on reflection:

$$p_\nu = \frac{2}{c} \int I_\nu \cos^2\theta \, \mathrm{d}\Omega. \tag{2.10}$$

which agrees with previous formula (2.6) as here we integrate only over half a sphere, 2π steradians. Assuming isotropy, i.e. $J_\nu = I_\nu$, we get:

$$p = \frac{2}{c} \int J_\nu \mathrm{d}\nu \int \cos^2\theta \, \mathrm{d}\Omega = \frac{1}{3} u. \tag{2.11}$$

2.2 The Equation of Radiative Transfer

- The intensity is conserved along a ray

$$\frac{\mathrm{d}I_\nu}{\mathrm{d}s} = 0 \tag{2.12}$$

unless there is absorption or emission

$$\frac{\mathrm{d}I_\nu}{\mathrm{d}s} = -\alpha_\nu I_\nu + j_\nu, \tag{2.13}$$

where α_ν [cm^{-1}] is the absorption coefficient and j_ν [$\mathrm{erg\ cm^{-3}\ Hz^{-1}\ str^{-1}}$] is the emission coefficient. This is the equation of radiative transfer. The absorption coefficient can be represented as the product of number density n of absorbing particles [cm^{-3}] and the effective cross-section σ_ν [cm^2] of an individual particle:

$$\alpha_\nu = n\sigma_\nu. \tag{2.14}$$

Alternatively

$$\alpha_\nu = \rho\kappa_\nu, \tag{2.15}$$

where ρ is the density [$\mathrm{g\ cm^{-3}}$] and κ_ν [$\mathrm{cm^2\ g^{-1}}$] is the opacity (or mass absorption coefficient).

- Defining the optical depth [units: none]

$$\mathrm{d}\tau_\nu = \alpha_\nu\,\mathrm{d}s \tag{2.16}$$

and the source function

$$S_\nu = \frac{j_\nu}{\alpha_\nu}, \tag{2.17}$$

we can rewrite the radiative transfer equation

$$\frac{\mathrm{d}I_\nu}{\mathrm{d}\tau_\nu} = -I_\nu + S_\nu. \tag{2.18}$$

- This has the *formal* solution

$$I_\nu(\tau_\nu) = I_0 e^{-\tau_\nu} + \int_0^{\tau_\nu} e^{-(\tau_\nu - \tau'_\nu)} S_\nu(\tau'_\nu)\mathrm{d}\tau'_\nu. \tag{2.19}$$

Evaluating this expression is not trivial and can often only be done numerically. The biggest problem is that the source function itself often depends on the intensity of radiation. Thus the equation we have at hand is *integro-differential*, not differential.

If S_ν is known and is independent of location, we get

$$I_\nu(\tau_\nu) = I_0 e^{-\tau_\nu} + S_\nu(1 - e^{-\tau_\nu}) = S_\nu + e^{-\tau_\nu}(I_0 - S_\nu). \tag{2.20}$$

Thus when $\tau_\nu \ll 1$, we obtain $I_\nu \approx I_0 + \tau_\nu S_\nu$, while when $\tau_\nu \gg 1$ one gets $I_\nu \approx S_\nu$.

- Radiation force.
  If medium absorbs radiation, then the radiation exerts the force on the medium, because the radiation carries momentum. The vector of radiation flux is

$$\vec{F}_\nu = \int I_\nu \vec{n} \mathrm{d}\Omega \tag{2.21}$$

where $\vec{n}$ is the unit vector along the direction of the ray. As the photon has momentum $h\nu/c$, the momentum transferred per unit area per unit length per unit time is

$$\vec{\mathcal{F}} = \frac{1}{c} \int \alpha_\nu \vec{F}_\nu \mathrm{d}\nu. \tag{2.22}$$

One can understand this relation from the following: the amount of energy passing through area dA in frequency interval dν in solid angle dΩ is $I_\nu \mathrm{d}A \cos\theta \mathrm{d}\nu \mathrm{d}\Omega$ (where θ is the angle the photon momentum makes with the normal of dA). To get the corresponding momentum along the normal to multiply this by $\cos\theta$ and divide this by c. The probability that photons are absorbed within the length ds is $\alpha_\nu \mathrm{d}s/\cos\theta$. Now integrate over solid angle (and remember that the volume is $\mathrm{d}V = \mathrm{d}A\mathrm{d}s$), we get equation (2.22) gives the force per unit volume. The force per unit mass is

$$\vec{f} = \frac{1}{c} \int \kappa_\nu \vec{F}_\nu \mathrm{d}\nu. \tag{2.23}$$

Note that there is no additional $\cos\theta$ factor here (which we had when computing radiation pressure), because we assume that *volume* can absorb photons. If we consider a problem of e.g. radiation force on a *surface*, i.e. solar sail, this factor appears.

2.3 Thermal Emission

- Thermal emission is radiation emitted by material in thermal equilibrium (TE); blackbody radiation (BB) is thermal emission that is in TE itself.
- BB is independent of material, shape, color, direction, flavor, etc. It only depends on temperature and wavelength (or frequency)

$$I_\nu = f(\nu, T) \equiv B_\nu(T). \tag{2.24}$$

- Kirchhoff's Law: material emitting thermal radiation has

$$S_\nu = B_\nu(T) \tag{2.25}$$

  and therefore

$$j_\nu = \alpha_\nu B_\nu(T). \tag{2.26}$$

- Thermal radiation has $S_\nu = B_\nu(T)$ and blackbody radiation has $I_\nu = B_\nu(T)$. Thermal radiation becomes blackbody radiation for $\tau \gg 1$.
- From thermodynamic arguments follows Stefan-Boltzmann Law: energy density

$$u(T) = aT^4 \tag{2.27}$$

  where $a = 7.56\times10^{-15}$ erg cm^{-3} K^{-4} is the radiation constant. Since $u = \frac{4\pi}{c}B$ and the flux $F = \pi B$, we get

$$F = \sigma_{\rm SB} T^4 \tag{2.28}$$

  where $\sigma_{\rm SB} = ac/4 = 5.67 \times 10^{-5}$ erg cm^{-2} K^{-4} s^{-1} is the Stefan-Boltzmann constant.
- The Planck spectrum

$$B_\nu(T) = \frac{2h\nu^3}{c^2}\frac{1}{e^{h\nu/kT} - 1}. \tag{2.29}$$

  or

$$B_\lambda(T) = \frac{2hc^2}{\lambda^5}\frac{1}{e^{hc/\lambda kT} - 1}, \tag{2.30}$$

  and

$$B_\lambda(T)\,{\rm d}\lambda = B_\nu(T)\,{\rm d}\nu. \tag{2.31}$$

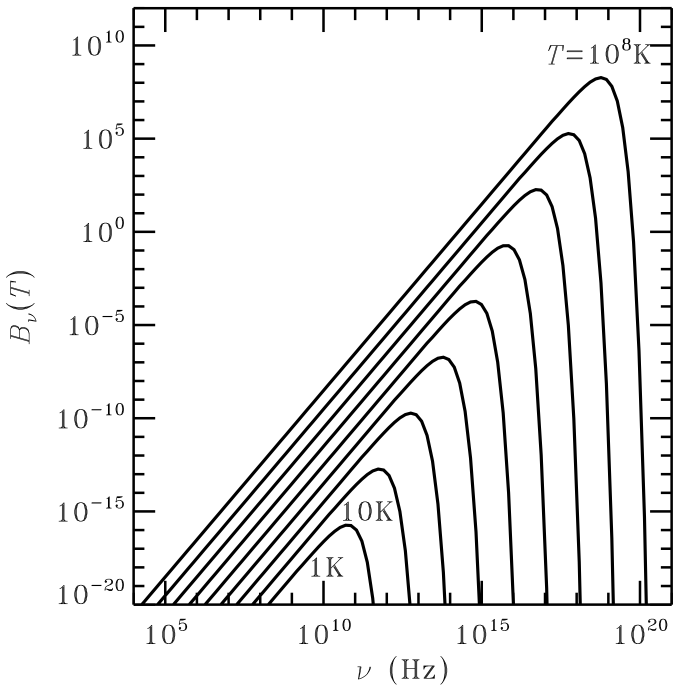


Figure 2.1: The Planck function for various temperatures.

- Limiting behavior: Rayleigh-Jeans limit $h\nu \ll kT$:

$$I_\nu^{RJ}(T) = 2\frac{\nu^2}{c^2}kT. \tag{2.32}$$

- Limiting behavior: Wien limit $h\nu \gg kT$:

$$I_\nu^{W}(T) = \frac{2h\nu^3}{c^2}\exp(-h\nu/kT). \tag{2.33}$$

- Planck function is monotonic in T, if $T_1 > T_2$, then $B_\nu(T_1) > B_\nu(T_2)$ for all ν. This can be easily checked by showing that $\partial B_\nu(T)/\partial T > 0$.

- Wien displacement law. Planck function reaches the maximum at

$$\nu_{\rm max} \approx 5.88 \times 10^{10} T, \tag{2.34}$$

where ν is in Hz and T in K, or

$$\lambda_{\rm max} \approx 0.290/T \tag{2.35}$$

with λ in cm and and T in K. Note that $\lambda_{\rm max}\nu_{\rm max} \neq c$.

- Relation to fundamental constants. Let us integrate the Planck function

$$\int_0^\infty B_\nu(T)\,{\rm d}\nu = \frac{2h}{c^2}\left(\frac{kT}{h}\right)^4 \int_0^\infty \frac{x^3}{e^x - 1}{\rm d}x. \tag{2.36}$$

The integral over x is $\pi^4/15$. Therefore, we have

$$B = \int_0^\infty B_\nu(T)\,{\rm d}\nu = \frac{2\pi^4 k^4}{15 c^2 h^3} T^4 = \sigma_{\rm SB} T^4/\pi. \tag{2.37}$$

- Characteristic temperatures: brightness temperature $T_{\rm b}$ – a measure of the source intensity (or brightness). Defined so that at a given ν: $I_\nu = B_\nu(T_{\rm b})$. Often used in radioastronomy, where the Plack function is replaced with its Rayleigh-Jean limit $B_\nu \propto T_{\rm RJ}$.

- Characteristic temperatures: color temperature $T_{\rm c}$ – a measure of the spectral shape. Defined as the temperature for a which a black body spectrum has a maximum at the same frequency or wavelength as the measured peak in the observed spectrum (which may not look like a blackbody curve at all).

- Characteristic temperatures: effective temperature $T_{\rm eff}$ – a measure of the magnitude of the flux. Defined as the temperature for which a blackbody would have the same flux as the measured flux $F = \sigma_{\rm SB} T^4$. The actual measured spectrum does not need to look like a blackbody.

2.4 Einstein Coefficients

Linking Kirchhoff's Law (macroscopic) with microscopic properties.

- For a two-level system with levels $E_2 > E_1$, $E_2 - E_1 = h\nu_0$, and degeneracies g_1 and g_2, define
  1. probability for spontaneous emission (s^{-1}) = A_{21}.
  2. probability for absorption = $B_{12}\overline{J}$, where $\overline{J} = \int J_\nu \phi(\nu) d\nu$ and $\phi(\nu)$ is the profile function. It describes the finite width around the frequency ν_0, where absorption can take place. For a slowly varying average intensity J_ν (like the Planck function), $\phi(\nu)$ can be approximated as a δ-function, and $\overline{J} = J_{\nu_0}$.
  3. probability for stimulated emission = $B_{21}\overline{J}$.

- In thermodynamic equilibrium (TE), the number of transitions per unit time per unit volume from state 1 to state 2 should be exactly balanced by the opposite transitions:

$$n_1 B_{12}\overline{J} = n_2 A_{21} + n_2 B_{21}\overline{J}. \tag{2.38}$$

Find $\overline{J}$:

$$\overline{J} = \frac{A_{21}/B_{21}}{(n_1/n_2)/(B_{12}/B_{21}) - 1}. \tag{2.39}$$

In TE from the Boltzmann formula we get:

$$\frac{n_1}{n_2} = \frac{g_1 \exp(-E/kT)}{g_2 \exp[-(E + h\nu_0)/kT]} = \frac{g_1}{g_2} \exp(h\nu_0/kT). \tag{2.40}$$

Thus

$$\overline{J} = \frac{A_{21}/B_{21}}{(g_1 B_{12}/g_2 B_{21}) \exp(h\nu_0/kT) - 1}. \tag{2.41}$$

We also know that in TE $J_\nu = B_\nu$, so we get the relation between the Einstein coefficients:

$$A_{21}/B_{21} = \frac{2h\nu^3}{c^2}, \tag{2.42}$$

$$g_1 B_{12} = g_2 B_{21}. \tag{2.43}$$

These properties do not depend on the temperature of the gas but are the properties of the atoms only. Thus they are valid not only for the TE, but always.

- The macroscopic emission and absorption coefficients can be written in terms of the microscopic Einstein coefficients as

$$j_\nu = \frac{h\nu}{4\pi} n_2 A_{21} \phi(\nu) \tag{2.44}$$

and

$$\alpha_\nu = \frac{h\nu}{4\pi}(n_1 B_{12} - n_2 B_{21})\phi(\nu) \tag{2.45}$$

Here, absorption also includes stimulated emission (as negative absorption).

- The source function is

$$S_\nu = \frac{2h\nu^3}{c^2}\left(\frac{n_1 g_2}{n_2 g_1} - 1\right)^{-1}. \tag{2.46}$$

- Thermal emission.
When the matter is thermal equilibrium with itself (but not necessarily with radiation), we have

$$\frac{n_1}{n_2} = \frac{g_1}{g_2}\exp(h\nu/kT), \tag{2.47}$$

and

$$\frac{n_2 g_1}{n_1 g_2} = \exp(-h\nu/kT) < 1. \tag{2.48}$$

The matter is said in local thermodynamic equilibrium (LTE). In this case

$$\alpha_\nu = \frac{h\nu}{4\pi} n_1 B_{12}[1 - \exp(-h\nu/kT)]\phi(\nu), \tag{2.49}$$

$$S_\nu = B_\nu(T). \tag{2.50}$$

- Non-thermal emission. Inverted populations, masers.
In real astrophysical circumstances, the matter does not need to be in LTE, i.e.

$$\frac{n_1}{n_2} \neq \frac{g_1}{g_2}\exp(h\nu/kT). \tag{2.51}$$

The extreme case of such a non-thermal population is an inverted population when

$$\frac{n_1}{g_1} < \frac{n_2}{g_2}. \tag{2.52}$$

In such a case, the absorption coefficient is negative, $\alpha_\nu < 0$. The intensity of radiation thus increases exponentially along the ray. This is *maser* - *m*icrowave *a*mplification by *s*timulated *e*mission of *r*adiation, it is similar to *laser* - *l*ight *a*mplification...

2.5 Scattering

In addition to emission and absorption, the photons can also be scattered in the medium. A volume elements emits photons due to the scattering with the rate completely dependent on the amount of radiation falling on the element. Let us consider a simple case of coherent (or elastic, monochromatic, i.e. with no frequency shift) isotropic (i.e. scattering probability is the same in all directions) scattering. We denote the 'absorption' coefficient for scattering as $\alpha_{\rm sc}$ (it might be also frequency dependent).

- Pure scattering. The radiation energy 'absorbed' by the volume element $\mathrm{d}V = \mathrm{d}A\ \mathrm{d}s$ ($\mathrm{d}s = c\mathrm{d}t$) in time $\mathrm{d}t$ is:

$$\int \mathrm{d}\Omega\ I_\nu \mathrm{d}A\ \mathrm{d}t\ \alpha_{\rm sc}\mathrm{d}s, \tag{2.53}$$

  where $I_\nu \mathrm{d}\Omega\ \mathrm{d}A\ \mathrm{d}t$ is the amount of energy passing through area element $\mathrm{d}A$ in time $\mathrm{d}t$ within solid angle $\mathrm{d}\Omega$ and $\alpha_{\rm sc}\mathrm{d}s$ is the probability of scattering with the volume. For isotropic scattering this energy is emitted in all direction $4\pi j_\nu\ \mathrm{d}V\ \mathrm{d}t$. Thus

$$j_\nu = \alpha_{\rm sc}\frac{1}{4\pi}\int I_\nu \mathrm{d}\Omega = \alpha_{\rm sc} J_\nu, \tag{2.54}$$

$$S_\nu = \frac{j_\nu}{\alpha_{\rm sc}} = J_\nu, \tag{2.55}$$

  and

$$\frac{\mathrm{d}I_\nu}{\mathrm{d}s} = -\alpha_{\rm sc}(I_\nu - J_\nu). \tag{2.56}$$

  This is integro-differential equation.

- Mean free path (for scattering only).
  The average distance a photon can travel without being scattered. Let $\mathrm{d}\tau = \alpha_{\rm sc}\mathrm{d}s$ is the optical depth for both processes. The probability that a photon propagates optical depth τ is $\exp(-\tau)$. The mean optical depth is then

$$\langle\tau\rangle = \int_0^\infty \tau \exp(-\tau)\ \mathrm{d}\tau = 1. \tag{2.57}$$

  The mean free path is then

$$l = \frac{\langle\tau\rangle}{\alpha_{\rm sc}} = \frac{1}{\alpha_{\rm sc}}. \tag{2.58}$$

- Random walk.
Scattering can be viewed as a random walk of photons within the medium. Let the displacement of the photon at step i is $\vec{r_i}$. The net displacement after N steps is
$$\vec{R} = \vec{r_1} + \vec{r_2} + ... + \vec{r_N}. \tag{2.59}$$
Let us find the mean square displacement:
$$\vec{l_*^2} = \langle \vec{R}^2 \rangle = \langle \vec{r_1}^2 \rangle + \langle \vec{r_2}^2 \rangle + ... + \langle \vec{r_N}^2 \rangle + 2\langle \vec{r_1} \cdot \vec{r_2} \rangle + 2\langle \vec{r_1} \cdot \vec{r_3} \rangle ... \tag{2.60}$$
The terms involving scalar products give zero, because for isotropic scattering the average of cosine of the scattering angle is zero. Each term involving square of a displacement gives approximately the square of mean free path l^2 (to be more exact, the mean square of the free path, $\langle r_i^2 \rangle = 2l^2$). Therefore
$$l_*^2 \approx N\, l^2, \tag{2.61}$$
$$l_* \approx \sqrt{N}\, l. \tag{2.62}$$

- Mean number of scatterings.
A photon injected in the medium of size R should on average travel that distance in order to escape the medium, i.e. $R \approx \sqrt{N}\, l$. Thus, the mean number of scatterings is
$$N \approx (R/l)^2 = (R\alpha_{\rm sc})^2 = \tau^2, \quad \tau \gg 1, \tag{2.63}$$
where τ is the optical thickness of the medium. When τ is small, photons mostly escape without interactions. The probability that they interact within the medium is $1 - \exp(-\tau) \approx \tau \ll 1$, and
$$N \approx \tau, \quad \tau \ll 1. \tag{2.64}$$
A more general formula working for any τ is then
$$N \approx \tau + \tau^2, \quad \text{or} \quad N \approx \max[\tau, \tau^2]. \tag{2.65}$$

- Escape time.
This is the time it takes for a photon to diffuse from the medium:
$$t_{\rm esc} = \frac{Nl}{c} = \frac{NR/\tau}{c} = \begin{cases} \dfrac{R}{c}\tau, & \tau \gg 1, \\ \dfrac{R}{c}, & \tau \ll 1. \end{cases} \tag{2.66}$$
Note that it grows as τ, not as the number of scatterings $\propto \tau^2$, because for large τ the distance travelled between scatterings decreases as $1/\tau$.

2.6 Scattering and absorption

- Combined scattering and absorption (in a thermal medium)

$$\frac{\mathrm{d}I}{\mathrm{d}s} = -(\alpha_\nu + \alpha_{\mathrm{sc}})(I_\nu - S_\nu) \tag{2.67}$$

with

$$S_\nu = \frac{\alpha_\nu B_\nu + \alpha_{\mathrm{sc}} J_\nu}{\alpha_\nu + \alpha_{\mathrm{sc}}} \tag{2.68}$$

- Mean free path (for absorption and scattering).
The average distance a photon can travel without being absorbed or scattered. The extinction coefficient $\alpha_\nu + \alpha_{\mathrm{sc}}$ and the optical depth for both processes is $\mathrm{d}\tau_\nu = (\alpha_\nu + \alpha_{\mathrm{sc}})\mathrm{d}s$. The mean free path is then

$$l_\nu = \frac{\langle\tau_\nu\rangle}{\alpha_\nu + \alpha_{\mathrm{sc}}} = \frac{1}{\alpha_\nu + \alpha_{\mathrm{sc}}}. \tag{2.69}$$

- A chance that after the free path the photon will be absorbed is $= \epsilon_\nu = \alpha_\nu/(\alpha_\nu + \alpha_{\mathrm{sc}})$; chance that it will be scattered $= 1 - \epsilon_\nu = \alpha_{\mathrm{sc}}/(\alpha_\nu + \alpha_{\mathrm{sc}})$. The quantity $1 - \epsilon_\nu$ is called the single-scattering albedo. Source function is then

$$S_\nu = (1 - \epsilon_\nu)J_\nu + \epsilon_\nu B_\nu. \tag{2.70}$$

- Thermalization length.
A photon is created by thermal emission of an atom. It scatters many times, but at some point it can get absorbed by some other atom. The total path between creation and absorption is called thermalization length. Because the probability of getting absorbed in each interaction act (i.e. in the end of each free path) is ϵ, a photon on average has $N = 1/\epsilon$ scatterings before absorption. Thus we have

$$l_*^2 = \frac{l^2}{\epsilon}, \quad l_* = \frac{l}{\sqrt{\epsilon}}, \tag{2.71}$$

and

$$l_* = \frac{1}{\sqrt{\alpha_\nu(\alpha_\nu + \alpha_{\mathrm{sc}})}} \tag{2.72}$$

- Effective optical thickness of the medium.

$$\tau_* = \sqrt{\tau_a(\tau_a + \tau_s)}, \tag{2.73}$$

where $\tau_a = \alpha_\nu R$ and $\tau_s = \alpha_{sc} R$ are the optical thickness of the medium of size R for absorption and scattering separately. If $\tau_* \gg 1$, the medium is effectively optically thick. The radiation field is then close to thermalization with the matter and $I_\nu \approx B_\nu$, $S_\nu \approx B_\nu$.

2.7 Radiative diffusion

Solving the equation of radiative transfer in near-homogenous media. We will use plane-parallel geometries, with $\mu = \cos\theta$.

- Rosseland approximation.
Consider a star. Assume that the medium is near-homogeneous and that the opacity is large so that the intensity is close to the Planck function. We can rewrite the radiative transfer equation in the following form:

$$\mu \frac{\mathrm{d}I_\nu}{\mathrm{d}z} = -(\alpha_\nu + \alpha_{sc})(I_\nu - S_\nu), \tag{2.74}$$

where z is the vertical coordinate and $\mu = \cos\theta$, and θ is the polar (zenith) angle. Rewrite RTE:

$$I_\nu = S_\nu - \mu \frac{1}{\alpha_\nu + \alpha_{sc}} \frac{\mathrm{d}I_\nu}{\mathrm{d}z}. \tag{2.75}$$

Inside the star, intensity is close to the Planck function and the source function too. Thus we can approximate in the rhs $I = B$ and $S = B$:

$$I_\nu = B_\nu - \mu \frac{1}{\alpha_\nu + \alpha_{sc}} \frac{\mathrm{d}B_\nu}{\mathrm{d}z}. \tag{2.76}$$

Find the flux:

$$\begin{aligned} F_\nu(z) &= \int I_\nu \cos\theta \mathrm{d}\Omega = 2\pi \int_{-1}^{1} I_\nu(z,\mu)\mu \mathrm{d}\mu = -\frac{2\pi}{\alpha_\nu + \alpha_{sc}} \frac{\mathrm{d}B_\nu}{\mathrm{d}z} \int_{-1}^{1} \mu^2 \mathrm{d}\mu \\ &= -\frac{4\pi}{3(\alpha_\nu + \alpha_{sc})} \frac{\mathrm{d}B_\nu(T)}{\mathrm{d}z} = -\frac{4\pi}{3(\alpha_\nu + \alpha_{sc})} \frac{\mathrm{d}B_\nu(T)}{\mathrm{d}T} \frac{\mathrm{d}T}{\mathrm{d}z}. \end{aligned} \tag{2.77}$$

Then the total flux at height z is

$$F(z) = \int_0^\infty F(\nu)\mathrm{d}\nu = -\frac{4\pi}{3}\frac{\mathrm{d}T}{\mathrm{d}z}\int_0^\infty (\alpha_\nu + \alpha_{\mathrm{sc}})^{-1}\frac{\mathrm{d}B_\nu(T)}{\mathrm{d}T}\mathrm{d}\nu. \tag{2.78}$$

Define the Rosseland mean opacity as

$$\frac{1}{\alpha_R} = \frac{\int_0^\infty (\alpha_\nu + \alpha_{\mathrm{sc}})^{-1}\frac{\partial B_\nu}{\partial T}\mathrm{d}\nu}{\int_0^\infty \frac{\partial B_\nu}{\partial T}\mathrm{d}\nu}. \tag{2.79}$$

Since

$$\int_0^\infty \frac{\partial B_\nu}{\partial T}\mathrm{d}\nu = \frac{\partial}{\partial T}\int_0^\infty B_\nu \mathrm{d}\nu = \frac{\partial B(T)}{\partial T} = \sigma_{SB}4T^3/\pi \tag{2.80}$$

we get

$$F(z) = -\frac{16\sigma_{SB}T^3}{3\alpha_R}\frac{\partial T}{\partial z} = \frac{4\pi}{3}\frac{\partial B}{\partial \tau_R}, \tag{2.81}$$

where $\mathrm{d}\tau_R = -\alpha_R \mathrm{d}z$ This means that the energy flux deep inside a star, for example, only depends on the temperature gradient and a single, weighted mean of all opacities.

- Eddington approximation.
  Now assume that in a near-homogenous medium the intensity is almost isotropic, but no longer assume that total opacity is large. Expanding the intensity into first-order terms of μ:

$$I_\nu(\tau, \mu) = a_\nu(\tau) + b_\nu(\tau)\mu. \tag{2.82}$$

We can evaluate the three moments of the intensity (equivalent with the mean intensity, the flux, and the pressure) as (suppressing the ν subscripts for clarity)

$$J \equiv \frac{1}{2}\int_{-1}^{1} I \,\mathrm{d}\mu = a, \tag{2.83}$$

$$H \equiv \frac{1}{2}\int_{-1}^{1} I\mu \,\mathrm{d}\mu = b/3, \tag{2.84}$$

$$K \equiv \frac{1}{2}\int_{-1}^{1} I\mu^2 \,\mathrm{d}\mu = a/3 \tag{2.85}$$

The latter equation for $K = J/3$ is the Eddington approximation (equivalent with expansion to first order in μ). The RTE:

$$\mu \frac{\mathrm{d}I}{\mathrm{d}\tau} = I - S, \tag{2.86}$$

where $\mathrm{d}\tau = -(\alpha + \alpha_{\mathrm{sc}})\mathrm{d}z$ (note minus sign). Integrating over μ we get:

$$\frac{\mathrm{d}H}{\mathrm{d}\tau} = J - S. \tag{2.87}$$

Multiplying (2.86) by μ before integrating, we get using Eddington approximation:

$$\frac{\mathrm{d}K}{\mathrm{d}\tau} = H = \frac{1}{3}\frac{\mathrm{d}J}{\mathrm{d}\tau}. \tag{2.88}$$

The last two equations can be combined to give second-order equation for J (*radiative diffusion equation*) which we can hope to solve:

$$\frac{1}{3}\frac{\partial^2 J}{\partial \tau^2} = \epsilon(J - B). \tag{2.89}$$

where we used $S = (1 - \epsilon)J + \epsilon B$ from equation (2.70). If we have the temperature structure of the medium, i.e. $B(T)$, we can solve this equation for J taking proper boundary conditions. We thus get J, then using (2.70) we obtain S_ν and then I_ν by applying the formal solution of the RTE.

- Introducing optical depth

$$\tau_* = \sqrt{3\epsilon}\,\tau = \sqrt{3\tau_a(\tau_a + \tau_s)}, \tag{2.90}$$

we get a different form of the diffusion equation

$$\frac{\partial^2 J}{\partial \tau_*^2} = J - B. \tag{2.91}$$

- Two-stream approximation.
  In the Eddington approximation, let's approximate $I_\nu(\mu, z)$ with I_ν along two directions only, $\mu = \pm 1/\sqrt{3}$.

$$I^+ = I(\tau, \mu = 1/\sqrt{3}), \tag{2.92}$$
$$I^- = I(\tau, \mu = -1/\sqrt{3}). \tag{2.93}$$

The expression for $J, H,$ and K now become

$$J = \frac{1}{2}(I^+ + I^-), \tag{2.94}$$

$$H = \frac{1}{2\sqrt{3}}(I^+ - I^-), \tag{2.95}$$

$$K = \frac{1}{6}(I^+ + I^-) = J/3, \tag{2.96}$$

Our choice of $\mu = \pm 1/\sqrt{3}$ is explained by the fact that the Eddington approximation still holds, $K = J/3$. In other words, if the intensity is near-isotropic, the intensity can be approximated by taking into account only the the intensity along angles $\mu = \pm 1/\sqrt{3}$.

With some more algebra we solve equation (2.94), (2.95) using (2.88):

$$I^+ = J + \frac{1}{\sqrt{3}}\frac{\partial J}{\partial \tau}, \tag{2.97}$$

$$I^- = J - \frac{1}{\sqrt{3}}\frac{\partial J}{\partial \tau}. \tag{2.98}$$

This gives us our two boundary conditions for J and $\partial J/\partial\tau$, if we know what I^+ and I^- are at a given locations τ_1 and τ_2 in the source. For example, if no radiation is entering from outside we have:

$$I^-(\tau = \tau_1) = 0, \quad I^+(\tau = \tau_2) = 0. \tag{2.99}$$

The Eddington approximation is often used for stellar atmospheres. In that case, the inner boundary conditions (inside the star), can be written as

$$F = 4\pi H = \frac{4\pi}{3}\frac{\partial B(T)}{\partial \tau} = \frac{4\pi}{3}\frac{\partial J(\tau)}{\partial \tau} \tag{2.100}$$

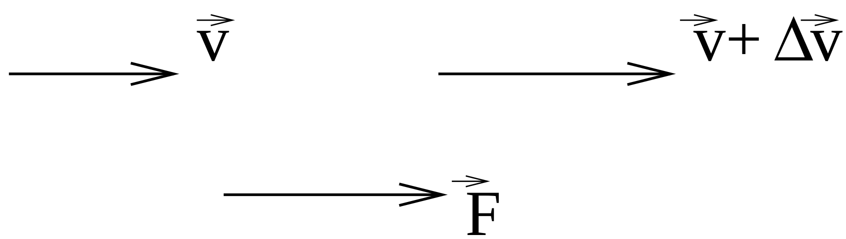
at very large τ where the temperature structure is known.

Chapter 3

Radiation fields

3.1 Definitions of electric and magnetic fields

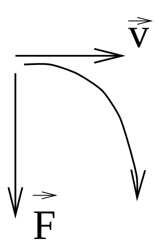
The electric field $\vec{E}$ and the magnetic field $\vec{B}$ can be defined through their effect on a charge q at location $\vec{r}$ moving with velocity $\vec{v}$. This is the operational definition (by doing experiments). Consider behaviour of a charged particle of charge q:



If acceleration is parallel to the velocity then the force is parallel the velocity. Let us define the electric field as force per unit charge:

$$\vec{F} = q\vec{E}. \tag{3.1}$$

If, on the other hand, the acceleration is perpendicular to the velocity, then the force is also perpendicular to the velocity.



We define then the magnetic field as follows:

$$\vec{F} = q\frac{\vec{v}}{c} \times \vec{B}. \tag{3.2}$$

The total force is called **Lorentz force**:

$$\vec{F} = q\left(\vec{E} + \frac{\vec{v}}{c} \times \vec{B}\right). \tag{3.3}$$

3.2 Work performed by the field

The work performed moving *one* particle per unit time is:

$$\vec{F} \cdot \frac{\mathrm{d}\vec{r}}{\mathrm{d}t} = \vec{F} \cdot \vec{v} = q\left(\vec{E} \cdot \vec{v} + \vec{v} \cdot \frac{\vec{v}}{c} \times \vec{B}\right) = q\,\vec{E} \cdot \vec{v}, \tag{3.4}$$

where $\mathrm{d}\vec{r}$ is the displacement and $\vec{F} \cdot \mathrm{d}\vec{r}$ is the work.

Let us write Newton's law for a non-relativistic particle:

$$\vec{F} = \frac{\mathrm{d}\vec{p}}{\mathrm{d}t} = \frac{\mathrm{d}m\vec{v}}{\mathrm{d}t}. \tag{3.5}$$

The work per unit time is then

$$q\vec{v} \cdot \vec{E} = \vec{F} \cdot \vec{v} = m\vec{v} \cdot \frac{\mathrm{d}\vec{v}}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\frac{mv^2}{2} = \frac{\mathrm{d}}{\mathrm{d}t}\epsilon_{\mathrm{mech}}. \tag{3.6}$$

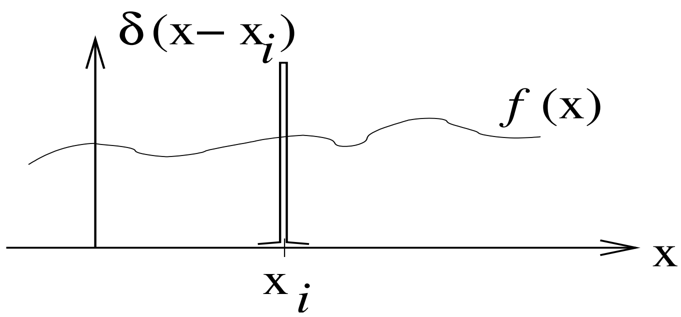
Thus the work per unit time done by the field on the particle is equal to the change of the kinetic (mechanic) energy per unit time.

Now let us consider a collection of particles. We can define the charge density as:

$$\rho_e(\vec{r}, t) = \sum_i q_i \delta\left(\vec{r} - \vec{r}_i(t)\right), \tag{3.7}$$

and similarly the current density:

$$\vec{j}_e(\vec{r}, t) = \sum_i q_i \vec{v}_i \delta\left(\vec{r} - \vec{r}_i(t)\right). \tag{3.8}$$



Notice that

$$\int \delta(\vec{r} - \vec{r}_i(t))\, \mathrm{d}^3 r = 1. \tag{3.9}$$

The proper total charge is given by the integral over the volume:

$$q = \int \rho\, \mathrm{d}^3 r = \sum_i \int q_i\, \delta(\vec{r} - \vec{r}_i(t))\, \mathrm{d}^3 r, \tag{3.10}$$

and similarly the proper total current is

$$q\vec{v} = \int \vec{j}\, \mathrm{d}^3 r = \sum_i \int q_i \vec{v}_i\, \delta(\vec{r} - \vec{r}_i(t))\, \mathrm{d}^3 r. \tag{3.11}$$

The work per unit time done by the fields at position $\vec{r}$ per unit volume is then:

$$\sum_i \delta(\vec{r} - \vec{r}_i(t))\ q_i \vec{v}_i \cdot \vec{E} = \vec{j}_e \cdot \vec{E}. \tag{3.12}$$

3.3 Maxwell's equations in differential form

Coulomb's law

$$\nabla \cdot \vec{D} = 4\pi\rho. \tag{3.13}$$

Guilbert's "law". No magnetic charges (=monopoles).

$$\nabla \cdot \vec{B} = 0. \tag{3.14}$$

Faraday's law of induction

$$\nabla \times \vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t}. \tag{3.15}$$

Maxwell's generalization of Ampere's law:

$$\nabla \times \vec{H} = \frac{4\pi}{c}\vec{j} + \frac{1}{c}\frac{\partial \vec{D}}{\partial t}, \tag{3.16}$$

where $\frac{1}{c}\frac{\partial \vec{D}}{\partial t}$ is called the displacement current.

The fields are related as

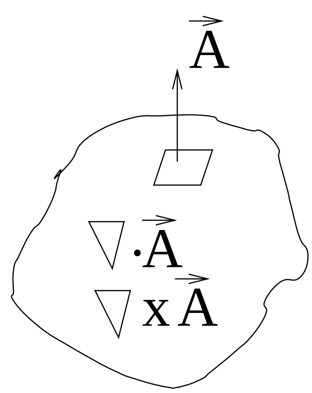
$$\vec{B} = \mu \vec{H}, \qquad \vec{D} = \epsilon \vec{E}, \tag{3.17}$$

where ϵ and μ are the dielectric constant and magnetic permeability of the medium. In vacuum $\epsilon = \mu = 1$.

Notes

1) Note the invariance for $\vec{E} \to \vec{B}$ and $\vec{B} \to -\vec{E}$ if $\rho = 0$, $\vec{j} = 0$ and $\epsilon = \mu = 1$.

2) if $\nabla \cdot \vec{A}$ and $\nabla \times \vec{A}$ are known, then $\vec{A}$ is uniquely specified (to arbitrary constant). Helmholtz theorem (ch 1.15 in Arfken 2nd ed.)



Thus if ρ, $\vec{j}$ (the sources of the field) are known, then $\vec{E}, \vec{B}$ uniquely determined through Maxwell's equations.

Definitions:

$$\nabla = (\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}),$$

$$\text{div } A = \nabla \cdot \vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z},$$

$$\text{curl } A = \nabla \times \vec{A} = \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}, \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}, \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right).$$

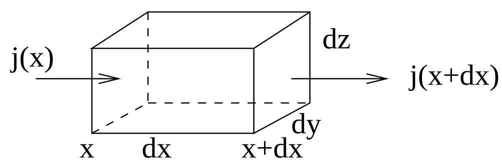
3.4 Conservation law for electric charge (Benjamin Franklin)

The conservation law for electric charge follows directly from Maxwell's equations. Take $\nabla\cdot$ on Ampere's law:

$$\underbrace{\nabla\cdot(\nabla\times\vec{H})}_{=0} = \nabla\cdot\frac{4\pi}{c}\vec{j} + \frac{1}{c}\frac{\partial}{\partial t}\underbrace{(\nabla\cdot\vec{D})}_{4\pi\rho} \tag{3.18}$$

$$\Longrightarrow \qquad \nabla\cdot\vec{j} + \frac{\partial\rho}{\partial t} = 0 \tag{3.19}$$

Let us prove the conservation law in a simple way. Let us a consider a current (flow) of charges through a volume of size dx dy dz, with the flow in the x direction only.



The net flux out of volume is equal to the rate of change of the charge in volume:

$$\underbrace{[j(x+\mathrm{d}x) - j(x)]}_{\frac{\partial j}{\partial x}\mathrm{d}x}\,\mathrm{d}y\mathrm{d}z = -\frac{\partial\rho}{\partial t}\mathrm{d}x\mathrm{d}y\mathrm{d}z. \tag{3.20}$$

Thus we get

$$\frac{\partial j}{\partial x} + \frac{\partial\rho}{\partial t} = 0.$$

3.5 Energy density, flux of electromagnetic field, energy conservation

Consider the work done per unit volume and unit time. Using Ampere's law we get

$$\vec{j} = \frac{1}{4\pi}\left[c(\nabla\times\vec{H}) - \frac{\partial\vec{D}}{\partial t}\right] \tag{3.21}$$

and

$$\vec{j} \cdot \vec{E} = \frac{1}{4\pi}\left[c\vec{E} \cdot (\nabla \times \vec{H}) - \vec{E} \cdot \frac{\partial \vec{D}}{\partial t}\right]. \quad (3.22)$$

The first term in the brackets (see problem set 2):

$$\vec{E} \cdot (\nabla \times \vec{H}) = \vec{H} \cdot (\nabla \times \vec{E}) - \nabla \cdot (\vec{E} \times \vec{H}). \quad (3.23)$$

Substitute $\nabla \times \vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t}$ from Faraday's law:

$$\vec{j} \cdot \vec{E} = \frac{1}{4\pi}\left[-\frac{1}{2\mu}\frac{\partial \vec{B}^2}{\partial t} - \frac{\epsilon}{2}\frac{\partial \vec{E}^2}{\partial t} - c\nabla \cdot (\vec{E} \times \vec{H})\right] \quad (3.24)$$

We obtain thus Poynting's theorem (conservation of energy)

$$\vec{j} \cdot \vec{E} + \frac{1}{8\pi}\frac{\partial}{\partial t}\left(\epsilon E^2 + \frac{1}{\mu}B^2\right) = -\nabla \cdot \left(\frac{c}{4\pi}\vec{E} \times \vec{H}\right). \quad (3.25)$$

We can define the energy density of the electromagnetic field as

$$U_{\text{field}} = \frac{1}{8\pi}\left(\epsilon E^2 + \frac{B^2}{\mu}\right). \quad (3.26)$$

The Poynting flux

$$\vec{S} = \frac{c}{4\pi}(\vec{E} \times \vec{H}) \quad (3.27)$$

determines the energy flux of the electromagnetic field.

Equation (3.25) allows a simple interpretation: the change of mechanical energy and field energy is equal to the minus of the divergence of flux.

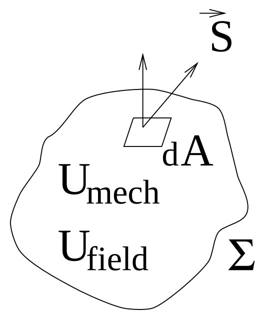
We can integrate over the volume

$$\int_V \mathrm{d}V\, \vec{j} \cdot \vec{E} + \frac{\partial}{\partial t}\int_V \mathrm{d}V\, U_{\text{field}} = -\int_V \mathrm{d}V\, \nabla \cdot \vec{S}. \quad (3.28)$$

Using Gauss's theorem one gets

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\epsilon_{\text{mech}}^{\text{volume}} + \epsilon_{\text{field}}^{\text{volume}}\right) = -\int_\Sigma \mathrm{d}\vec{A} \cdot \vec{S}. \quad (3.29)$$

Thus the change of the total energy in the volume is equal to the inward (sign $-$) energy flux through surface.



If surface area $\Sigma \to \infty$ then the electrostatic and magnetostatic fields (check your old textbooks) depend on distance as $E \propto 1/r^2$ and $H \propto 1/r^2$. Therefore, the Poynting flux

$$\vec{S} \propto \vec{E} \times \vec{H} \propto \frac{1}{r^4}. \tag{3.30}$$

The total energy escaping to infinity is then

$$\int \vec{S} \cdot \mathrm{d}\vec{A} \propto \frac{1}{r^4} r^2 \to 0. \tag{3.31}$$

We shall later find that the time-dependent fields depend on distance as $\propto \frac{1}{r}$, therefore the energy escaping to infinity

$$\int \vec{S} \cdot \mathrm{d}\vec{A} \propto \frac{1}{r^2} r^2 \to \text{finite}. \tag{3.32}$$

This way the radiation escapes to infinity.

3.6 Maxwell's equations in vacuum

Vacuum mean that there is no charges, no currents. $\epsilon = \mu = 1$.

$$\rho = 0, \quad \vec{j} = 0.$$

There exists a trivial solution: $\vec{E} = \text{const}$ and $\vec{B} = \text{const}$. Why would non-trivial solutions exist?

Faraday's induction law

$$\nabla \times \vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t}.$$

Thus time varying $\vec{B}$-field gives rise to $\vec{E}$-field that in turn gives rise to $\vec{B}$-field through Ampere's law

$$\nabla \times \vec{B} = \frac{1}{c}\frac{\partial \vec{E}}{\partial t}.$$

Thus continuously varying $\vec{E}$ and $\vec{B}$ fields can generate each other for ever even though $\rho = 0$, $\vec{j} = 0$, i.e. no sources for the fields. It was hard to accept in 1860s. Waves in vacuum required introduction of aether. It is now less surprising if the radiation is considered as particles (photons).

3.6.1 Wave equation in vacuum

Looking at non-trivial solutions, waves carrying energy & momentum. Take curl of Faraday's law:

$$\underbrace{\nabla \times (\nabla \times \vec{E})} = -\frac{1}{c}\frac{\partial}{\partial t}\nabla \times \vec{B} = -\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\vec{E}. \tag{3.33}$$

$$= \nabla(\underbrace{\nabla \cdot \vec{E}}_{=0}) - \nabla^2 \vec{E}$$

(see exercise 2.3). Thus we get the homogeneous wave equations:

$$\frac{\partial^2 \vec{E}}{\partial t^2} - c^2 \nabla^2 \vec{E} = 0, \tag{3.34}$$

and by symmetry

$$\frac{\partial^2 \vec{B}}{\partial t^2} - c^2 \nabla^2 \vec{B} = 0. \tag{3.35}$$

Here $\nabla^2 \equiv \Delta$ is Laplacian operator.

3.6.2 Solution of the wave equation

The general solution of the wave equations

$$\vec{E} = \vec{a}_1 E_0 e^{i(\vec{k}\cdot\vec{r}-\omega t)}, \quad \vec{B} = \vec{a}_2 B_0 e^{i(\vec{k}\cdot\vec{r}-\omega t)}, \tag{3.36}$$

where $\vec{a}_1$ and $\vec{a}_2$ are the unit vectors, E_0 and B_0 are complex constants, $\vec{k} = k\vec{n}$ and ω are the wave-vector and frequency. Such a solution represent waves traveling in the $\vec{n}$ direction. The most general solution of Maxwell equations can then be constructed by superposition of wave of various frequencies and traveling in different directions.

Substituting (3.36) into Maxwell's equation, we get:

$$i\vec{k}\cdot\vec{a}_1E_0 = 0, \qquad i\vec{k}\cdot\vec{a}_2B_0 = 0,$$
$$i\vec{k}\times\vec{a}_1E_0 = \frac{i\omega}{c}\vec{a}_2B_0, \qquad i\vec{k}\times\vec{a}_2B_0 = -\frac{i\omega}{c}\vec{a}_1E_0. \tag{3.37}$$

The top two equations tell us that $\vec{a}_1$ and $\vec{a}_2$ are both perpendicular to $\vec{k}$. With that knowledge, the bottom two equations tell us that $\vec{a}_1$ and $\vec{a}_2$ are perpendicular to each other. Thus $\vec{a}_1$, $\vec{a}_2$ and $\vec{k}$ for the right-hand triad of mutually perpendicular vectors.

We thus can get the relation between E_0 and B_0:

$$E_0 = \frac{\omega}{kc}B_0, \quad B_0 = \frac{\omega}{kc}E_0, \tag{3.38}$$

so that

$$\omega^2 = c^2k^2 \tag{3.39}$$

and

$$E_0 = B_0. \tag{3.40}$$

Taking k and ω positive we get

$$\omega = ck. \tag{3.41}$$

The phase velocity of the waves is

$$v_{\mathrm{ph}} = \omega/k = c \tag{3.42}$$

and the group velocity is also c:

$$v_{\mathrm{gr}} = \frac{\partial\omega}{\partial k} = c. \tag{3.43}$$

We can now compute the energy flux of these waves. Since $\vec{E}$ and $\vec{B}$ vary sinusoidally with time, the Poynting vector fluctuate. We can take the time average, which is normally measured. It is shown in problem set 2 that for two quantities

$$A(t) = \mathcal{A}e^{i\omega t}, \quad B(t) = \mathcal{B}e^{i\omega t}, \tag{3.44}$$

the time average of their product of their real parts is

$$\langle Re A(t)\, Re B(t)\rangle = \frac{1}{2} Re(\mathcal{A}\mathcal{B}^*) = \frac{1}{2} Re(\mathcal{A}^*\mathcal{B}), \tag{3.45}$$

where $*$ denotes complex conjugate. Thus we get

$$\langle S \rangle = \frac{c}{8\pi} Re(E_0 B_0^*) = \frac{c}{8\pi}|E_0|^2 = \frac{c}{8\pi}|B_0|^2, \tag{3.46}$$

where we used $E_0 = B_0$.

3.7 Radiation spectrum

From the time variation of the electrical field (and, analogously, the magnetic field) follows the spectrum of the radiation. The spectrum is the amount of energy per unit area per unit time per unit frequency interval, and is most easily derived through a Fourier transformation. Let us consider a pulse of radiation that passes by an observer. For a pulse of radiation, $E(t) \to 0$ and $B(t) \to 0$ as $t \to \pm\infty$. We only consider the E-field along one axis, $E(t) = \hat{a} \cdot \vec{E}(t)$. The Fourier transform and its inverse are now defined as

$$\hat{E}(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} E(t)\, e^{i\omega t} \mathrm{d}t, \tag{3.47}$$

$$E(t) = \int_{-\infty}^{\infty} \hat{E}(\omega)\, e^{-i\omega t} \mathrm{d}\omega. \tag{3.48}$$

The quantity $\hat{E}(\omega)$ contains the full frequency information of $E(t)$. The amount of energy $\mathrm{d}W$ passing through a surface element $\mathrm{d}A$ per time $\mathrm{d}t$ is given by the Poynting vector $\vec{S}$:

$$\frac{\mathrm{d}W}{\mathrm{d}A\, \mathrm{d}t} = |\vec{S}(t)| = \frac{c}{4\pi} E^2(t). \tag{3.49}$$

The total energy per unit area is

$$\frac{\mathrm{d}W}{\mathrm{d}A} = \frac{c}{4\pi} \int_{-\infty}^{\infty} E^2(t) \mathrm{d}t. \tag{3.50}$$

From Parseval's theorem [1] we get

$$\frac{\mathrm{d}W}{\mathrm{d}A} = c\int_0^\infty |\hat{E}(\omega)|^2 \mathrm{d}\omega, \tag{3.51}$$

because $|\hat{E}(\omega)|^2 = |\hat{E}(-\omega)|^2$. Thus we can define the energy per unit area per unit frequency (for the entire pulse):

$$\frac{\mathrm{d}W}{\mathrm{d}A\,\mathrm{d}\omega} = c|\hat{E}(\omega)|^2. \tag{3.52}$$

If pulses repeat on the time-scale T, we can introduce power per unit time:

$$\frac{\mathrm{d}W}{\mathrm{d}A\,\mathrm{d}t\,\mathrm{d}\omega} = \frac{c}{T}|\hat{E}(\omega)|^2. \tag{3.53}$$

In this expression we need to measure the emission pulse over lengths of time T that is sufficiently long to sample all relevant frequencies ω ($\omega T > 1$), but short compared to the duration of the whole signal (the properties of $E(t)$ remain approximately constant, i.e. process is stationary). The fact that the time variation of the electrical field and its spectrum are related through a Fourier transform makes it very convenient to derive a spectral shape from the characteristics of $E(t)$.

As an example, let us consider a pulse described by an exponential:

$$E(t) = e^{-t/\tau}, \qquad t > 0, \tag{3.54}$$

where τ is the decay constant. Compute the Fourier transform

$$\hat{E}(\omega) = \frac{1}{2\pi}\int_0^\infty e^{-t/\tau} e^{i\omega t}\mathrm{d}t = \frac{1}{2\pi}\frac{1}{\frac{1}{\tau} - i\omega}, \tag{3.55}$$

and therefore the spectrum is

$$|\hat{E}(\omega)|^2 = \frac{1}{4\pi^2}\frac{1}{\frac{1}{\tau} - i\omega}\frac{1}{\frac{1}{\tau} + i\omega} = \frac{1}{4\pi^2}\frac{1}{\frac{1}{\tau^2} + \omega^2} = \begin{cases} \text{const} = \frac{\tau^2}{4\pi^2}, & \omega \ll 1/\tau, \\ \propto \omega^{-2}, & \omega \gg 1/\tau. \end{cases} \tag{3.56}$$

Other examples are shown in Fig. 3.1.

---

[1] The Parseval's theorem for Fourier pairs is stated as:

$$\int_{-\infty}^{\infty} E^2(t)\mathrm{d}t = 2\pi \int_{-\infty}^{\infty} |\hat{E}(\omega)|^2 \mathrm{d}\omega.$$

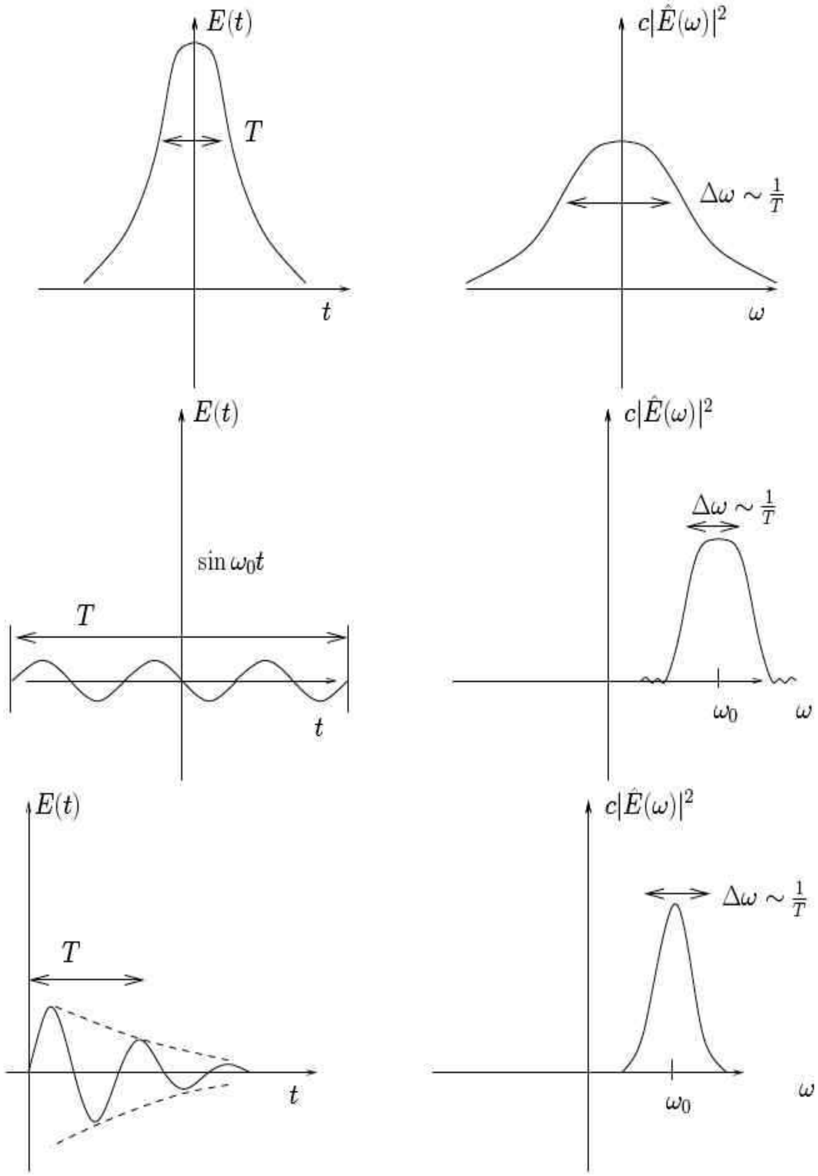


Figure 3.1: EM pulses (left) and associated radiation spectra (right) for three pulse shapes. Top: a pulse of duration T has a spectrum stretching over a bandwidth of $\sim 1/T$. Middle: A periodic signal with frequency ω_0 for a total duration of time T will have a spectrum of width $\sim 1/T$ centered on a frequency ω_0. Bottom: A similar periodic signal with a decay time of T (damped oscillator) will produce a spectrum of bandwidth $\sim 1/T$ centered on a frequency ω_0 , but without the higher and lower frequency 'wiggles' found in the previous example.

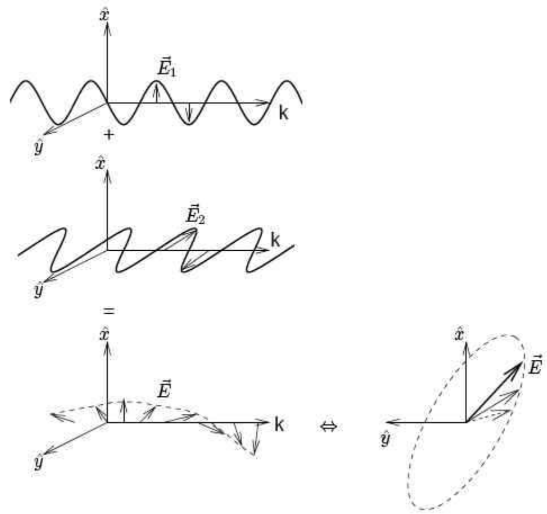


Figure 3.2: Any (single-frequency) $\vec{E}$ wave can be decomposed into two orthogonal waves with amplitudes $\vec{E}_1$ and $\vec{E}_2$ and the same frequency (but arbitrary phase difference). The resulting composite $\vec{E}$ traces out an ellipse.

3.8 Stokes parameters

Consider a plane wave in the z-direction. The Fourier decomposition:

$$\vec{E}_+(z - ct) = \int_{-\infty}^{\infty} \vec{e}_+(k) e^{ik(z-ct)} \mathrm{d}k. \tag{3.57}$$

Here k - wavenumber; $\omega \equiv kc$ - angular frequency; $\omega = 2\pi\nu$, ν is frequency.

So far we only considered oscillation in one plane (linearly polarized). Most general wave is superposition of oscillations in 2 perpendicular planes. It is convenient to consider $\vec{e}_+$ complex. $\vec{e}_+$ should be transverse to $\hat{z}$:

$$\vec{e}_+(k) = \hat{x}\mathcal{E}_x(k) e^{i\phi_x(k)} + \hat{y}\mathcal{E}_y(k) e^{i\phi_y(k)}, \tag{3.58}$$

where $\mathcal{E}_{x,y}$ are real amplitudes; $\phi_{x,y}$ are phases. ($e^{ix} = \cos x + i \sin x$).

The Fourier components are the real parts of $\vec{e}_+(k) e^{ik(z-ct)}$:

$$\vec{E}_k = \hat{x}\mathcal{E}_x(k) \cos[k(z - ct) + \phi_x(k)] + \hat{y}\mathcal{E}_y(k) \cos[k(z - ct) + \phi_y(k)]. \tag{3.59}$$

Thus an arbitrary polarized monochromatic wave (i.e. a given k) is described by 4 real parameters (instead of just one intensity I_ν). It is inconvenient to use

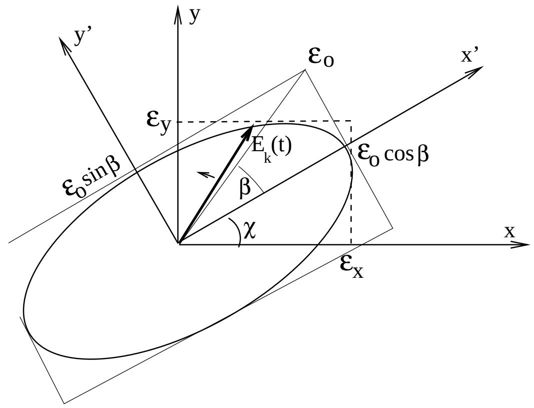


Figure 3.3: Geometry for elliptically polarized wave propagating in z-direction.

$\mathcal{E}_x, \mathcal{E}_y, \phi_x, \phi_y$ since they have different units. Stokes (1852) found a convenient set of 4 parameters describing polarized light.

Let us consider fixed position, say, $z = 0$:

$$\vec{E}_k(t) = \hat{x}E_x(t) + \hat{y}E_y(t) = \hat{x}\mathcal{E}_x \cos(\omega t - \phi_x) + \hat{y}\mathcal{E}_y \cos(\omega t - \phi_y). \tag{3.60}$$

(The four parameters are $\mathcal{E}_x, \mathcal{E}_y, \phi_x, \phi_y$.) Here real amplitudes

$$\begin{aligned} E_x(t) &= \mathcal{E}_x(\cos\omega t \cos\phi_x + \sin\omega t \sin\phi_x), \\ E_y(t) &= \mathcal{E}_y(\cos\omega t \cos\phi_y + \sin\omega t \sin\phi_y). \end{aligned} \tag{3.61}$$

The $\vec{E}_k$ vector traces an ellipse: elliptically polarized wave. The principal axis of the ellipse has a tilt χ (polarization angle) with respect to the x-axis.

In the new coordinate system x', y' which is rotated by angle χ relative to the old x, y system, the ellipse equation is given by the following relations:

$$E_1(t) = \mathcal{E}_0 \cos\beta \cos\omega t, \qquad E_2(t) = \mathcal{E}_0 \sin\beta \sin\omega t,$$

where we introduced the ellipticity parameter β. Rotation is counter clockwise for $0 < \beta < \pi/2$ (left handed polarization). $\beta = \pm\pi/4$ means circular polarization, while for $\beta = 0$ or $\beta = \pi/2$ the polarization is linear.

Since the coordinates in two systems are related as

$$E_x = E_1 \cos\chi - E_2 \sin\chi, \quad E_y = E_1 \sin\chi + E_2 \cos\chi,$$

we get

$$\begin{aligned} E_x(t) &= \mathcal{E}_0(\cos\beta\cos\chi\cos\omega t - \sin\beta\sin\chi\sin\omega t), \\ E_y(t) &= \mathcal{E}_0(\cos\beta\sin\chi\cos\omega t + \sin\beta\cos\chi\sin\omega t). \end{aligned} \tag{3.62}$$

Identifying coefficients in front of $\cos\omega t$ and $\sin\omega t$ in equations (3.61) and (3.62), we get:

$$\begin{aligned} \mathcal{E}_x\cos\phi_x &= \mathcal{E}_0\cos\beta\cos\chi, \\ \mathcal{E}_x\sin\phi_x &= -\mathcal{E}_0\sin\beta\sin\chi, \\ \mathcal{E}_y\cos\phi_y &= \mathcal{E}_0\cos\beta\sin\chi, \\ \mathcal{E}_y\sin\phi_y &= \mathcal{E}_0\sin\beta\cos\chi, \end{aligned} \tag{3.63}$$

where we have three new parameters $\mathcal{E}_0, \beta$ and χ describing completely 100% polarized monochromatic wave instead of the previous four: $\mathcal{E}_x, \mathcal{E}_y, \phi_x, \phi_y$. Among the phases, it is only the difference $\phi_y - \phi_x$ that matters. Stokes (1852) defined 4 practical quantities to characterise a wave (Stokes parameters):

$$\begin{aligned} I &= \mathcal{E}_x^2 + \mathcal{E}_y^2 = \mathcal{E}_0^2, \\ Q &= \mathcal{E}_x^2 - \mathcal{E}_y^2 = \mathcal{E}_0^2\cos 2\beta\cos 2\chi, \\ U &= 2\mathcal{E}_x\mathcal{E}_y\cos(\phi_y - \phi_x) = \mathcal{E}_0^2\cos 2\beta\sin 2\chi, \\ V &= 2\mathcal{E}_x\mathcal{E}_y\sin(\phi_y - \phi_x) = \mathcal{E}_0^2\sin 2\beta. \end{aligned} \tag{3.64}$$

Here we again have 4 parameters, but they are not independent since for a completely polarized wave:

$$I^2 = Q^2 + U^2 + V^2.$$

Sometimes alternatively one uses 3 parameters:

$$\begin{aligned} \mathcal{E}_0 &= \sqrt{I}, \\ \sin 2\beta &= V/I, \\ \tan 2\chi &= U/Q, \end{aligned} \tag{3.65}$$

where β is the ellipticity parameter and χ the polarization angle.

One of the possible ways of presenting polarization is on the Poincare sphere (see Fig. 3.8).

Light is normally not monochromatic and not 100% polarized. Different part of the object have different polarizations and different phases. Therefore, in reality

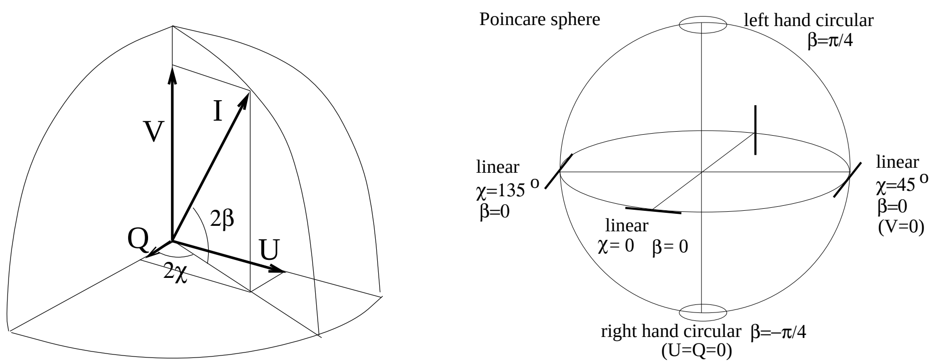


Figure 3.4: Representation of Stokes parameters using the Poincare sphere.

$I^2 \geq Q^2+U^2+V^2$ and four parameters are needed to characterize the polarization. An important property of Stokes parameters that they are additive for a superposition of independent waves (i.e. those that do not have permanent phase relations).

$$I = \sum I^{(k)},\ Q = \sum Q^{(k)},\ U = \sum U^{(k)},\ V = \sum V^{(k)}. \tag{3.66}$$

Therefore any Stokes vector can be represented as a sum of one unpolarized (first term) and one completely polarized (second term) parts:

$$\begin{pmatrix} I \\ Q \\ U \\ V \end{pmatrix} = \begin{pmatrix} I - \sqrt{Q^2+U^2+V^2} \\ 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} \sqrt{Q^2+U^2+V^2} \\ Q \\ U \\ V \end{pmatrix}. \tag{3.67}$$

Polarization degree is then defined as

$$\Pi = \sqrt{Q^2+U^2+V^2}/I. \tag{3.68}$$

One can also define linear $\Pi_{\rm lin} = \sqrt{Q^2+U^2}/I$ and circular polarizations $\Pi_{\rm cir} = V/I$.

Chapter 4

Radiation from moving charges

4.1 Electromagnetic potentials

Instead of using vector fields $\vec{E}(\vec{x},t), \vec{B}(\vec{x},t)$, Maxwell's equations can be reformulated using electromagnetic potentials $\phi(\vec{x},t), \vec{A}(\vec{x},t)$, where ϕ is a scalar and $\vec{A}$ is a vector. There are several advantages with this approach: (1) scalar + vector are simpler than two vectors; (2) the resulting equations are simpler; (3) it is simpler to do relativistic formulation.

From $\nabla \cdot \vec{B}$, we see that we can introduce vector potential $\vec{A}$ such that

$$\vec{B} = \nabla \times \vec{A},$$

since $\nabla \cdot \vec{B} = \nabla \cdot (\nabla \times \vec{A}) = 0$.
From Faraday's induction law we get:

$$\nabla \times \vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t} = -\frac{1}{c}\nabla \times \frac{\partial \vec{A}}{\partial t}, \tag{4.1}$$

or

$$\nabla \times \left(\vec{E} + \frac{1}{c}\frac{\partial \vec{A}}{\partial t}\right) = 0. \tag{4.2}$$

Then the expression in brackets can be written as the gradient of a scalar field $-\nabla\phi$:

$$\vec{E} = -\nabla\phi - \frac{1}{c}\frac{\partial \vec{A}}{\partial t}. \tag{4.3}$$

Potentials are not unique since the following changes (called *gauge transformation*)

$$\vec{A}' = \vec{A} + \nabla\psi, \qquad \phi' = \phi - \frac{1}{c}\frac{\partial\psi}{\partial t} \tag{4.4}$$

do not change the physical quantities $\vec{E}$ and $\vec{B}$:

$$\vec{B}' = \nabla\times\vec{A}' = \nabla\times\vec{A} + \nabla\times\nabla\psi = \nabla\times\vec{A} = \vec{B}, \tag{4.5}$$

$$\vec{E}' = -\nabla\phi' - \frac{1}{c}\frac{\partial\vec{A}'}{\partial t} = -\nabla\phi + \frac{1}{c}\frac{\partial}{\partial t}\nabla\psi - \frac{1}{c}\frac{\partial\vec{A}}{\partial t} - \frac{1}{c}\frac{\partial}{\partial t}\nabla\psi = \vec{E}. \tag{4.6}$$

Here ψ is an arbitrary function.

One can choose ψ to make equations for $\vec{A}, \phi$ simpler. The Lorentz gauge for radiation problems means that we choose ψ so that

$$\nabla\cdot\vec{A} + \frac{1}{c}\frac{\partial\phi}{\partial t} = 0, \qquad \text{Lorentz condition.} \tag{4.7}$$

It makes this formalism Lorentz invariant: $[\vec{A}, i\phi]$ - four-vector; $[\nabla, ic\frac{\partial}{\partial t}]$ - four-dimensional gradient.

4.2 Maxwell's equations with electromagnetic potentials

Our aim is to solve Maxwell's equations with the given source terms. Let us rewrite the two Maxwell's equations containing source terms.

Coulomb law: $\nabla\cdot\vec{E} = 4\pi\rho_e$

$$\nabla\cdot\nabla\phi + \frac{1}{c}\frac{\partial\nabla\cdot\vec{A}}{\partial t} = -4\pi\rho_e. \tag{4.8}$$

Using Lorentz gauge $\nabla\cdot\vec{A} = -\frac{1}{c}\frac{\partial\phi}{\partial t}$, we get

$$\nabla^2\phi - \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} = -4\pi\rho_e. \tag{4.9}$$

Ampere's law: $\nabla\times\vec{B} = \frac{4\pi\vec{j}_e}{c} + \frac{1}{c}\frac{\partial\vec{E}}{\partial t}$

$$\nabla\times(\nabla\times\vec{A}) = \frac{4\pi\vec{j}_e}{c} - \frac{1}{c}\nabla\left(\frac{\partial\phi}{\partial t}\right) - \frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2}. \tag{4.10}$$

The lhs is $\nabla(\nabla\cdot\vec{A})-\nabla^2\vec{A}$. In the 2nd term in the rhs change $\frac{1}{c}\frac{\partial\phi}{\partial t}$ to $-\nabla\cdot\vec{A}$ (Lorentz gauge). We get then

$$\nabla(\nabla\cdot\vec{A})-\nabla^2\vec{A}=\frac{4\pi\vec{j}_e}{c}+\nabla(\nabla\cdot\vec{A})-\frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2}. \tag{4.11}$$

Finally,

$$\nabla^2\vec{A}-\frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2}=-\frac{4\pi\vec{j}_e}{c}. \tag{4.12}$$

Inhomogeneous wave equations for ϕ and $\vec{A}$:

$$\Box^2\begin{pmatrix}\phi\\ \vec{A}\end{pmatrix}=-4\pi\begin{pmatrix}\rho\\ \vec{j}_e/c\end{pmatrix}, \tag{4.13}$$

where $\Box^2$ is the d'Alembertian operator $=\nabla^2-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}$, i.e. 4-dimensional Laplasian = wave operator. In vacuum, ϕ and $\vec{A}$ also have wave solutions propagating with velocity c.

Our strategy is to determine $\vec{E},\vec{B}$ from given $\rho,\vec{j}_e$:
1) solve the inhomogeneous wave equation for $\phi,\vec{A}$;
2) then compute $\vec{B}=\nabla\times\vec{A}$ and $\vec{E}=-\nabla\phi-\frac{1}{c}\frac{\partial\vec{A}}{\partial t}$

Ways of solving these equations:
i) Rybicki & Lightman: see Jackson 2nd ed. (1975);
ii) Shu: (1) see Jackson 1st ed. (1962), ch 6.6; (2) 'physical derivation' from Landau & Lifshitz (1951) + verification of solution;
iii) most stringent: Jackson 1st ed., ch 6.6 - Fourier methods + complex integration using residue calculus;
iv) Jackson 2nd ed. ch 6.6: Fourier transforms + physical derivation in Fourier space.

4.3 Green's function method

Equations (4.13) have the general form

$$\Box^2\psi(\vec{x},t)=-4\pi S(\vec{x},t), \tag{4.14}$$

where S is the charge ρ or current density $\vec{j}_e/c$ and ψ is either ϕ or $\vec{A}$.

Determine the effect of a source (e.g. a charge) at $\vec{x}', t'$ given by $\delta(\vec{x}-\vec{x}', t-t')$. Instead of $\psi(\vec{x}, t)$, determine Green's function $G(\vec{x}, t; \vec{x}', t')$ satisfying

$$\Box^2 G(\vec{x}, t; \vec{x}', t') = -4\pi\delta(\vec{x} - \vec{x}')\delta(t - t'). \tag{4.15}$$

Once you have the particular solution for a source at $(\vec{x}', t')$, you can get the solution for the source distribution $S(\vec{x}', t')$ by integrating over S:

$$\psi(\vec{x}, t) = \int G(\vec{x}, t; \vec{x}', t')S(\vec{x}', t')\mathrm{d}^3\vec{x}'\mathrm{d}t'. \tag{4.16}$$

Check:

$$\begin{aligned}\Box^2\psi(\vec{x}, t) &= \int \Box^2 G(\vec{x}, t; \vec{x}', t')S(\vec{x}', t')\mathrm{d}^3\vec{x}'\mathrm{d}t' = \qquad (4.17)\\ &= -4\pi \int \delta(\vec{x} - \vec{x}')\delta(t - t')S(\vec{x}', t')\mathrm{d}^3\vec{x}'\mathrm{d}t' = -4\pi S(\vec{x}, t). \quad \text{QED}\end{aligned}$$

Below we describe the heuristic solution of equation (4.15). Let us consider a charge source $Q(t)$ located at $\vec{x}' = 0$. The equation we wish to solve is

$$\Box^2\phi = -4\pi Q(t)\delta(\vec{r}). \tag{4.18}$$

The problem is then spherically symmetric. The effects are the same in all directions. We rewrite d'Alembertian operator in spherical coordinates and get the equation outside the origin ($r \neq 0$)

$$\Box^2\phi(r, t) = \frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial\phi}{\partial r} - \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} = \frac{1}{r}\frac{\partial^2(r\phi)}{\partial r^2} - \frac{1}{r}\frac{1}{c^2}\frac{\partial^2(r\phi)}{\partial t^2} = 0. \tag{4.19}$$

This equation has a spherical wave solution:

$$r\phi(r, t) = f_+(t - r/c) + f_-(t + r/c). \tag{4.20}$$

We first solution represent outgoing wave, the second - ingoing wave. We skip the second solution due to causality. The solution is thus

$$\phi(r, t) = \frac{1}{r} f_+(t - r/c), \quad r \neq 0. \tag{4.21}$$

What is then f_+? We find it by normalizing the solution at the origin.

Near the origin, the function ϕ varies rapidly $\sim 1/r$ and the radial derivatives are much larger than the time derivatives, which we then can neglect. We arrive at the Poisson equation for electrostatic (Coulomb) field

$$\nabla^2\phi = -4\pi Q(t)\delta(\vec{r}). \tag{4.22}$$

$\nabla\times\vec{E} = 0 \quad \rightarrow \quad E = -\nabla\phi$ combined with $\nabla\cdot\vec{E} = 4\pi\rho$ gives $\nabla^2\phi = -4\pi\rho$. Solve instead

$$\nabla^2 G = -4\pi\delta(\vec{r}). \tag{4.23}$$

The solution is $G(r) = 1/r$. At $r \neq 0$, both $\delta(\vec{r})$ and $\nabla^2(1/r)$ are zeros. The correctness of normalization can be checked by integrating both sides of the equation over the volume. Instead of $1/r$ substitute $G(r) = 1/\sqrt{r^2+a^2}$:

$$\int_0^\infty \nabla^2\left(\frac{1}{\sqrt{r^2+a^2}}\right) 4\pi r^2 \mathrm{d}r = -4\pi \tag{4.24}$$

for any a. Considering a limit $a \rightarrow 0$, we prove equation (4.23). The solution of (4.22) is then

$$\phi(r,t) = Q(t)/r \tag{4.25}$$

close to the origin, i.e. $r/c \ll t$. Identifying $f_+(t)$ with $Q(t)$ we get the general solution of the wave equation

$$\phi(r,t) = \frac{1}{r}Q(t - r/c). \tag{4.26}$$

For the source at $\vec{x}', t'$ the retarded Green function is

$$G_{\mathrm{ret}}(\vec{x},t;\vec{x}',t') = \frac{\delta(t - t' - r/c)}{|\vec{x}-\vec{x}'|} = \frac{\delta(t' - (t - |\vec{x}-\vec{x}'|/c))}{|\vec{x}-\vec{x}'|}, \tag{4.27}$$

where we used $\delta(x) = \delta(-x)$ in the last equality.

The general solution is given by expression (4.16).

4.4 Retarded potentials

The solutions for the scalar and vector potentials are thus given by so called retarded potentials:

$$\begin{aligned}\begin{pmatrix}\phi(\vec{x},t)\\ \vec{A}(\vec{x},t)\end{pmatrix} &= \int \frac{\delta(t' - (t - |\vec{x}-\vec{x}'|/c))}{|\vec{x}-\vec{x}'|}\begin{pmatrix}\rho(\vec{x}',t')\\ \vec{j}_e(\vec{x}',t')/c\end{pmatrix}\mathrm{d}^3x'\,\mathrm{d}t' \\ &= \int_{Vol}\begin{pmatrix}\rho(\vec{x}',t_{\mathrm{ret}})\\ \vec{j}_e(\vec{x}',t_{\mathrm{ret}})/c\end{pmatrix}\frac{\mathrm{d}^3x'}{|\vec{x}-\vec{x}'|},\end{aligned} \tag{4.28}$$

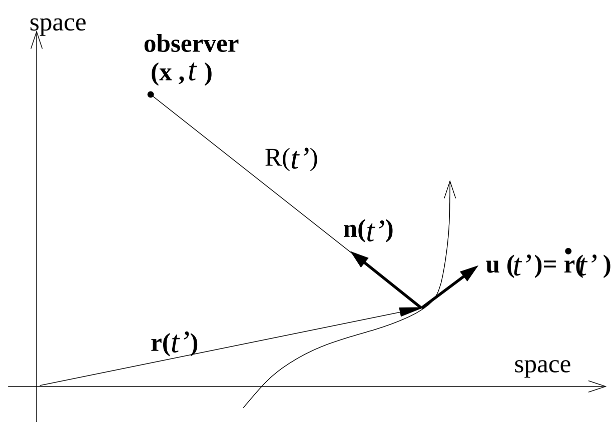
where retarded time $t_{\rm ret} = t - |\vec{x} - \vec{x}'|/c$. Potentials at position $\vec{x}$ and time t contain contributions from the past light cone.

4.5 Lienard–Wiechert potentials

Consider the potential from a single charge moving along the path $\vec{x} = \vec{r}(t)$. The corresponding charge and current densities are

$$\rho(\vec{x}, t) = q\delta(\vec{x} - \vec{r}(t))$$

$$\vec{j}(\vec{x}, t) = q\vec{u}(t)\delta(\vec{x} - \vec{r}(t))$$



Move back one step (to before the integration over t')

$$\phi(\vec{x}, t) = \int \frac{\delta(t' - t + |\vec{x} - \vec{x}'|/c)}{|\vec{x} - \vec{x}'|}\rho(\vec{x}', t')\mathrm{d}^3x'\mathrm{d}t'.$$

Insert now $\rho(\vec{x}, t)$:

$$\phi(\vec{x}, t) = q\int \frac{\delta(t' - t + |\vec{x} - \vec{x}'|/c)}{|\vec{x} - \vec{x}'|}\delta(\vec{x}' - \vec{r}(t'))\mathrm{d}^3x'\mathrm{d}t'$$

and integrate over the volume instead of time:

$$\phi(\vec{x}, t) = q\int \frac{\delta(t' - t + R(t')/c)}{R(t')}\mathrm{d}t',$$

where we defined $R(t') = |\vec{x} - \vec{r}(t')|$.

Using the rule for δ-function (see Arfken):

$$\int a(t)\delta(f(t))\mathrm{d}t = \frac{a(t_0)}{|f'(t_0)|}, \quad t_0 : f(t_0) = 0,$$

we get

$$\phi(\vec{x}, t) = \frac{q}{R(t_{\rm ret})\left|\frac{\mathrm{d}}{\mathrm{d}t'}(t' - t + R(t')/c)\right|_{t_{\rm ret}}} = \frac{q}{R(t_{\rm ret})\left|1 + \frac{\mathrm{d}R(t')}{c\mathrm{d}t'}\right|_{t_{\rm ret}}},$$

where $t_{\rm ret} = t - R(t_{\rm ret})/c$.

Let's do some mathematical exercises to compute $\frac{\mathrm{d}R(t')}{\mathrm{d}t'}$. Since $\vec{R}(t') = \vec{x} - \vec{r}(t')$, we get $\dot{\vec{R}} = -\dot{\vec{r}}(t') = -\vec{u}(t')$. Notice that $R^2 = \vec{R}\cdot\vec{R}$. Therefore, $R\dot{R} = \vec{R}\cdot\dot{\vec{R}} = -\vec{R}\cdot\vec{u}$. Thus one gets $\dot{R}(t') = -\frac{\vec{R}(t')}{R(t')}\cdot\vec{u}(t') = -\vec{n}(t')\cdot\vec{u}(t')$.

Finally, we get the Lienard - Wiechert potentials

$$\phi(\vec{x}, t) = \frac{q}{(1 - \vec{n}(t_{\rm ret})\cdot\vec{u}(t_{\rm ret})/c)\, R(t_{\rm ret})} = \left.\frac{q}{\kappa R}\right|_{t_{\rm ret}}, \tag{4.29}$$

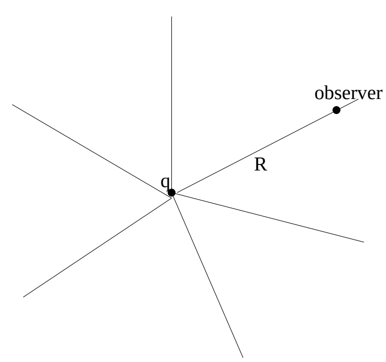
and similarly for $\vec{A}$:

$$\vec{A}(\vec{x}, t) = \frac{q\vec{u}(t_{\rm ret})}{c(1 - \vec{n}(t_{\rm ret})\cdot\vec{u}(t_{\rm ret})/c)\, R(t_{\rm ret})} = \left.\frac{q\vec{u}/c}{\kappa R}\right|_{t_{\rm ret}}, \tag{4.30}$$

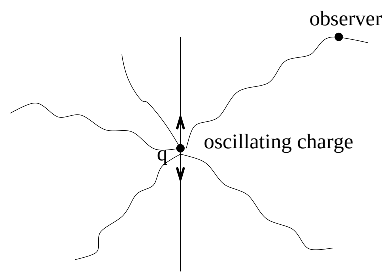
where we defined the Doppler factor $\kappa = 1 - \vec{n}(t_{\rm ret})\cdot\vec{u}(t_{\rm ret})/c$.

<u>Comments</u>

(1) if $\vec{u} = 0$, $\phi = q/R$, $\vec{A} = 0$, i.e. the static Coulomb potentials.

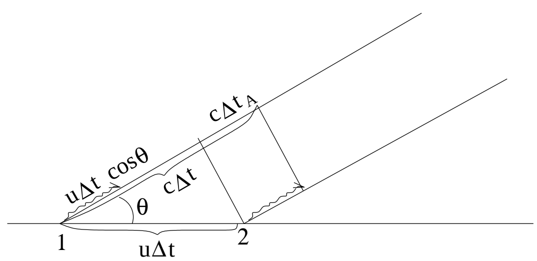


(2) $\vec{u} \neq 0$. The Lienard-Wiechert potentials show the effect of charge motion on the Coulomb potentials. E.g. if the charge is oscillating, the observer sees oscillating potentials reflecting what the charge did $\Delta t = R/c$ time ago. The observer sees an oscillating E-field superimposed on the static field. One also sees an oscillating B-field, since from the induction law $\nabla \times B = -\frac{1}{c}\frac{\partial E}{\partial t}$. The moving charge is current. One gets an electromagnetic wave.



(3) The theory is relativistically correct. No assumptions about smallness of the velocities, $u \ll c$.

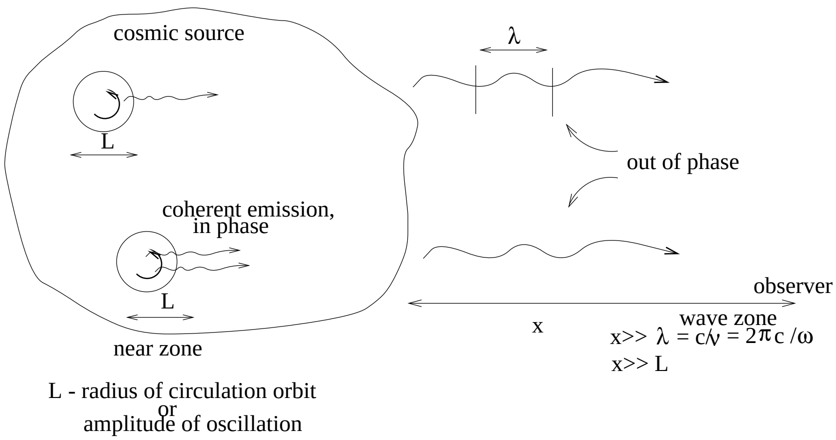
(4) $1 - \vec{n} \cdot \vec{u}/c$ factor accounts for the fact that the time intervals between two signals are not equal for the emitting observer and the receiving observer (the Doppler effect). When $\vec{n} \| \vec{u}$ and $|\vec{u}| \sim c$ the Doppler factor is very small and potentials become large. This gives rise to the so called Doppler beaming: radiation is beamed toward the direction of the charge motion.



(5) $\vec{n}, \vec{u}, R$ are all evaluated at retarded time $t_{\rm ret}$.

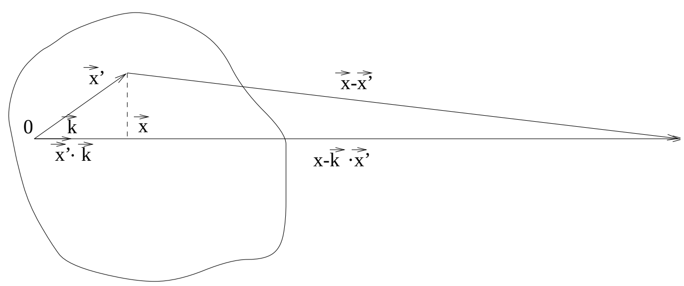
(6) $\vec{B} = \nabla \times \vec{A}$ and $\vec{E} = -\nabla\phi - \frac{1}{c}\frac{\partial \vec{A}}{\partial t}$: Lienard-Wiechert potentials seem to behave as $A \propto 1/R$ and $\phi \propto 1/R$ are large R. This would mean that $B \propto 1/R^2$, $E \propto 1/R^2$ and the energy crossing the sphere at R is $4\pi R^2 S \propto R^2 BE \propto 1/R^2 \to 0$ as for static fields. However, one should also include the R-dependence of the retardation time $t_{\rm ret} = t - R(t_{\rm ret})/c$. This will give terms behaving as $B \propto 1/R$, $E \propto 1/R$ and then the energy reaching a sphere at R is $\propto R^2 EB \propto R^2/R^2 = {\rm const}$. This is radiation.

4.6 Different zones



Space and time variations of fields at the observer are caused by
1) variations in x due to particle motion
2) time retardation effects

In the wave zone, one can use the small angle approximation



Distance to the particle in the wave-zone approximation

$$|\vec{x}-\vec{x}'| = (x^2+x'^2-2\vec{x}\cdot\vec{x}')^{1/2} \approx x\left(1-\frac{2}{2}\frac{\vec{x}\cdot\vec{x}'}{x^2}...\right) = x-\vec{k}\cdot\vec{x}' \approx x.$$

Here we used the fact that $x'^2 \approx L^2 \ll x^2$. The last approximation is valid if just the distance is important.

Retarted time in the wave-zone approximation:

$$t_{\rm ret} \equiv t' = t - \frac{|\vec{x}-\vec{x}'|}{c} \approx t - \frac{1}{c}(x-\vec{k}\cdot\vec{x}') = t - \frac{1}{c}(\vec{k}\cdot\vec{x}-\vec{k}\cdot\vec{x}').$$

We keep $\vec{k}\cdot\vec{x}'$, which represent time retardation effects within the source, when considering rapidly varying fields ($t_{\rm var} \ll x/c$) which is the case for observed photons. For example, $t_{\rm var} \approx 1/\nu \approx 10^{-10} - 10^{-20}$ s for radio-γ-rays, while $x/c \approx 1 - 10^9$ years for astronomical sources. Wave oscillates faster that the light travel time to the source.

4.7 Electromagnetic potentials in the wave zone

$$\begin{pmatrix} \phi(\vec{x},t) \\ \vec{A}(\vec{x},t) \end{pmatrix} = \int_{Vol} \begin{pmatrix} \rho(\vec{x}',t_{\rm ret}) \\ \frac{\vec{j}_e(\vec{x}',t_{\rm ret})}{c} \end{pmatrix} \frac{\mathrm{d}^3x'}{|\vec{x}-\vec{x}'|} \approx \frac{1}{x}\int_{Vol} \begin{pmatrix} \rho(\vec{x}',t_{\rm ret}) \\ \frac{\vec{j}_e(\vec{x}',t_{\rm ret})}{c} \end{pmatrix} \mathrm{d}^3x'. \quad (4.31)$$

Now consider space variations of, e.g., scalar potential at the observer (there is x dependence in $1/x$ and $t_{\rm ret}$).

$$\begin{aligned} \nabla\phi &= \nabla(\frac{1}{x})\int \rho \mathrm{d}^3x' + \frac{1}{x}\int \nabla t_{\rm ret}\frac{\partial\rho}{\partial t_{\rm ret}}(x',t_{\rm ret})\mathrm{d}^3x' \\ &= -\frac{\nabla x}{x^2}\int\rho\mathrm{d}^3x' + \frac{1}{x}\int\nabla t_{\rm ret}\frac{\partial t}{\partial t_{\rm ret}}\frac{\partial\rho}{\partial t}(x',t_{\rm ret})\mathrm{d}^3x', \end{aligned} \quad (4.32)$$

where we used the chain rule for partial derivatives.

Let us note that $\nabla x = \vec{k} \equiv \vec{x}/x$ (since $\nabla f(r) = \frac{\vec{r}}{r}\frac{\partial f(r)}{\partial r}$, see Arfken). One can show (see exercises) that

$$\nabla t_{\rm ret} = -\frac{\vec{n}}{c(1-\vec{n}\cdot\frac{\vec{u}(t_{\rm ret})}{c})}, \quad (4.33)$$

where $\vec{n} \equiv \vec{R}/R$ (remember that $\vec{R} = \vec{x}-\vec{x}'$). Also

$$\frac{\partial t}{\partial t_{\rm ret}} = \frac{\partial}{\partial t_{\rm ret}}(t_{\rm ret}+R/c) = 1-\vec{n}\cdot\frac{\vec{u}(t_{\rm ret})}{c}. \quad (4.34)$$

Thus we see that $\nabla t_{\rm ret}\frac{\partial t}{\partial t_{\rm ret}} = -\frac{\vec{n}}{c} \approx -\frac{\vec{k}}{c}$. One gets

$$\begin{aligned} \nabla\phi &= -\frac{\vec{k}}{x}\phi - \frac{\vec{k}}{c}\frac{1}{x}\int\frac{\partial\rho}{\partial t}\mathrm{d}^3x' = -\frac{\vec{k}}{x}\phi - \frac{\vec{k}}{c}\frac{\partial}{\partial t}\phi \\ &= -\vec{k}\left(\frac{\phi}{x}+\frac{1}{c}\frac{\partial\phi}{\partial t}\right) = -\vec{k}\frac{1}{c}\frac{\partial\phi}{\partial t} \end{aligned} \quad (4.35)$$

since in the wave zone $x \gg \lambda = c/\nu$ and $\phi/x \ll \phi/\lambda = \frac{\nu}{c}\phi \sim \frac{1}{c}\frac{\partial\phi}{\partial t}$ (if one considers Fourier components in wave packet $\phi_\nu \sim e^{i2\pi\nu t}$). This is equivalent to $t_{\rm var} \ll x/c$ or oscillation time $\ll$ light travel time.

In the wave zone, thus we can substitute the ∇ operator by

$$\nabla \to -\frac{\vec{k}}{c}\frac{\partial}{\partial t}.$$

Thus $\nabla^2 \to \frac{1}{c^2}\frac{\partial^2}{\partial t^2}$, and therefore $\vec{E}, \vec{B}, \phi, \vec{A}$ all satisfy wave equation in vacuum, whose solutions is radiation.

4.8 Electromagnetic fields in the wave zone

Now let us compute the electric and magnetic fields in the wave zone:

$$\begin{aligned} \vec{B} &= \nabla\times\vec{A} = -\frac{\vec{k}}{c}\times\frac{\partial\vec{A}}{\partial t}, \qquad (4.36)\\ \vec{E} &= -\nabla\phi - \frac{1}{c}\frac{\partial\vec{A}}{\partial t} = \text{(wave zone)} \quad \frac{\vec{k}}{c}\frac{\partial\phi}{\partial t} - \frac{1}{c}\frac{\partial\vec{A}}{\partial t}\\ &= \text{(Lorentz gauge)} \quad -\vec{k}(\nabla\cdot\vec{A}) - \frac{1}{c}\frac{\partial\vec{A}}{\partial t}\\ &= \text{(wave zone)} \quad \vec{k}\left(\frac{\vec{k}}{c}\cdot\frac{\partial\vec{A}}{\partial t}\right) - \frac{1}{c}\frac{\partial\vec{A}}{\partial t}(\vec{k}\cdot\vec{k})\\ &= \left(\frac{1}{c}\frac{\partial\vec{A}}{\partial t}\times\vec{k}\right)\times\vec{k} = \vec{B}\times\vec{k}. \qquad (4.37) \end{aligned}$$

We used identity $\vec{c}(\vec{a}\cdot\vec{b}) - \vec{b}(\vec{a}\cdot\vec{c}) = (\vec{b}\times\vec{c})\times\vec{a}$ getting to the last line. So we get

$$\vec{E} = \vec{B}\times\vec{k}$$

just like for radiation in vacuum.

Furthermore,

$$\vec{A} \propto \frac{1}{x}\int\frac{\vec{j}}{c}d^3x' \propto \frac{1}{x}.$$

Magnetic field $\vec{B} \propto \frac{\partial\vec{A}}{\partial t} \propto \frac{1}{x}$, and $|\vec{E}| = |\vec{B}| \propto \frac{1}{x}$, which is different from static fields $|\vec{E}| \propto 1/x^2$.

Poynting flux

$$\vec{S} = \frac{c}{4\pi}(\vec{E} \times \vec{B}) = \frac{c}{4\pi} B^2 \vec{k}.$$

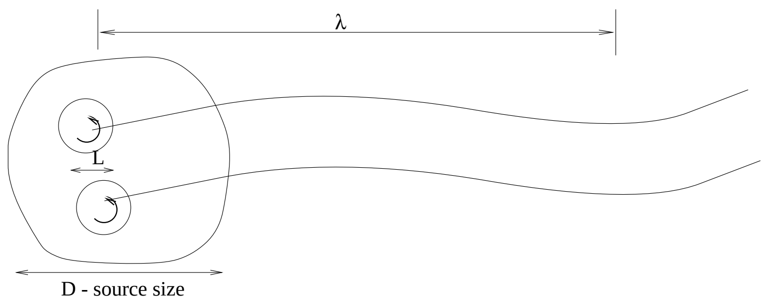
The energy radiated in solid angle dΩ per unit time is then

$$\mathrm{d}P = \frac{\mathrm{d}W}{\mathrm{d}t} = \vec{S} \cdot (\vec{k} x^2 \mathrm{d}\Omega) = B^2 x^2 \vec{k} \cdot \vec{k} \frac{c \mathrm{d}\Omega}{4\pi},$$

which is independent of x since $B \propto 1/x$. Energy thus can be transported to infinity. This is radiation.

4.9 Dipole Radiation

Consider several charged particles in a source. Different particles have different retarded times. Normally, one may need to keep track of phase differences. But if $\lambda \gg D$, then one can forget about phase differences.



L is the distance over which the particle changes its motion. Then the typical frequency is $\nu \sim 1/(\text{time to change motion}) \sim u/L$, where u is particle velocity. Thus,

$$\lambda \gg D \Rightarrow \frac{c}{\nu} \gg D \Rightarrow \frac{cL}{u} \gg D \Rightarrow \frac{u}{c} \ll \frac{L}{D} < 1 \Rightarrow$$

the motion is non-relativistic.

Forgetting the phase difference implies that $t' = t - \frac{1}{c}(x - \vec{k} \cdot \vec{x}') \approx t - \frac{x}{c}$, i.e. the 'same' distance to all particles. One neglects that the source is extended in space and time. This is 'dipole approximation'.

The retarded potentials become

$$\begin{aligned} \vec{A}(\vec{x}, t) &= \int_{vol} \frac{\vec{j}(\vec{x}', t_{\mathrm{ret}})}{c} \frac{\mathrm{d}^3 x'}{|\vec{x} - \vec{x}'|} \approx (\text{wave zone}) \quad \frac{1}{x} \int \frac{\vec{j}(\vec{x}', t_{\mathrm{ret}})}{c} \mathrm{d}^3 x' \\ &\approx (\text{dipole approx.}) \quad \frac{1}{cx} \int \vec{j}(\vec{x}', t - x/c) \mathrm{d}^3 x'. \end{aligned} \tag{4.38}$$

Non-relativistic charges are described by

$$\vec{j}(\vec{x}', t_{\rm ret}) = \sum_a q_a \vec{u}_a(t_{\rm ret})\delta(\vec{x}' - \vec{r}(t_{\rm ret})), \quad (4.39)$$

$$\Rightarrow \vec{A}(\vec{x}, t) = \frac{1}{cx}\sum_a q_a \vec{u}_a(t - x/c). \quad (4.40)$$

Define the dipole moment of a charge distribution as

$$\vec{d} \equiv \sum_a q_a \vec{x}_a \Rightarrow \dot{\vec{d}} = \sum_a q_a \dot{\vec{x}}_a = \sum_a q_a \vec{u}_a, \quad (4.41)$$

then

$$\vec{A}(\vec{x}, t) = \frac{\dot{\vec{d}}}{cx}. \quad (4.42)$$

We want to determine $\vec{E}$ and $\vec{B}$ fields and the radiated power. The magnetic field is

$$\vec{B} = \nabla \times \vec{A} = \frac{1}{c}\left(\frac{\partial \vec{A}}{\partial t} \times \vec{k}\right) = \frac{1}{c^2 x}\ddot{\vec{d}} \times \vec{k}. \quad (4.43)$$

The electric field vector is then

$$\vec{E} = \vec{B} \times \vec{k} = \frac{1}{c^2 x}\vec{k} \times (\vec{k} \times \ddot{\vec{d}}), \quad (4.44)$$

and its strength is

$$E(t) = \ddot{d}(t)\frac{\sin\theta}{c^2 x}, \quad (4.45)$$

where θ is the angle between $\ddot{\vec{d}}$ and $\vec{k}$. The Poynting flux in the direction $\vec{k}$ is

$$S = \frac{c}{4\pi}B^2. \quad (4.46)$$

The radiated power per unit solid angle can be obtained by multiplying S by the area $x^2 \mathrm{d}\Omega$ through which radiation passes at distance x and dividing by $\mathrm{d}\Omega$:

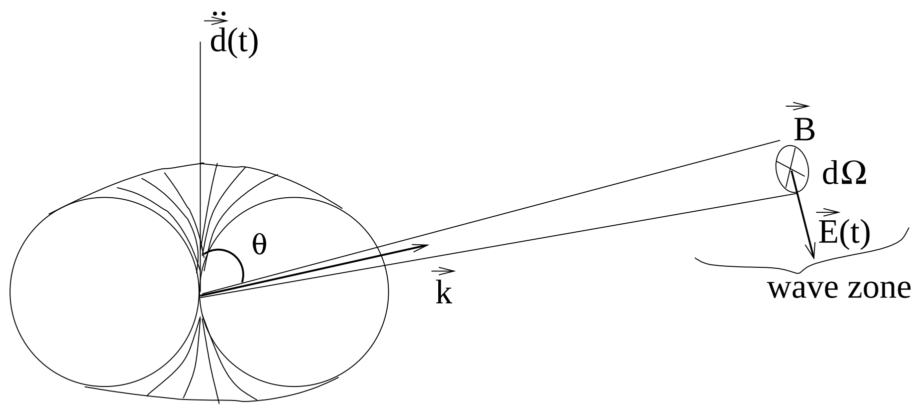
$$\frac{\mathrm{d}P}{\mathrm{d}\Omega} = \frac{c}{4\pi}(Bx)^2 = \frac{1}{4\pi c^3}\left|\ddot{\vec{d}} \times \vec{k}\right|^2 = \frac{\left|\ddot{\vec{d}}\right|^2}{4\pi c^3}\sin^2\theta. \quad (4.47)$$

The total radiated power (Larmor formula)

$$P = \int_{4\pi} \frac{\mathrm{d}P}{\mathrm{d}\Omega}\mathrm{d}\Omega = \frac{\left|\ddot{\vec{d}}\right|^2}{c^3} \int \sin^2\theta \frac{\mathrm{d}\Omega}{4\pi} = \frac{\left|\ddot{\vec{d}}\right|^2}{c^3} \int_{-1}^{1} (1-\mu^2)\frac{\mathrm{d}\mu}{2} = \frac{2}{3}\frac{\left|\ddot{\vec{d}}\right|^2}{c^3}, \tag{4.48}$$

where $\mu = \cos\theta$.

Radiation pattern is a dipole pattern



In the wave zone $\vec{E}$ lies in plane defined by $\vec{k}$ and $\ddot{\vec{d}}$.
$\vec{B}$ is $\perp$ to that plane: $\vec{E} = \vec{B} \times \vec{k} \propto \vec{k} \times (\vec{k} \times \dot{\vec{u}})$.

Note: for a single charge, we have $\vec{d} = e\vec{x}$ and
1) $P \propto e^2$ and $P \propto \dot{\vec{u}}^2$, i.e. (acceleration)^2;
2) if acceleration is along the straight line, then the radiation is 100% linearly polarized in the $\vec{k} - \dot{\vec{u}}$ plane;
3) no radiation in the $\dot{\vec{u}}$-direction.

Spectrum of radiation. Let us consider a Fourier transform of the dipole moment:

$$d(t) = \int e^{-i\omega t}\hat{d}(\omega)\mathrm{d}\omega. \tag{4.49}$$

Its second time derivative is

$$\ddot{d}(t) = -\int \omega^2 e^{-i\omega t}\hat{d}(\omega)\mathrm{d}\omega. \tag{4.50}$$

Using equation (4.45), we get

$$\hat{E}(\omega) = -\frac{1}{c^2 x}\omega^2\hat{d}(\omega)\sin\theta. \tag{4.51}$$

The energy per unit solid angle per unit frequency is then (see eq.[3.52] and note that $dA = x^2 d\Omega$)

$$\frac{dW}{d\omega d\Omega} = cx^2 \left|\hat{E}(\omega)\right|^2 = \frac{\omega^4}{c^3} \left|\hat{d}(\omega)\right|^2 \sin^2\theta. \tag{4.52}$$

The total energy per frequency is

$$\frac{dW}{d\omega} = \frac{8\pi}{3}\frac{\omega^4}{c^3} \left|\hat{d}(\omega)\right|^2. \tag{4.53}$$

We see that the spectrum of the emitted radiation is related to the spectrum of oscillations of the dipole moment.

4.10 Dipole Radiation: Examples

4.10.1 Electron-electron collision

Consider a collision between electrons or in general particles with the same q/m ratio:

$$\dot{\vec{d}} = \sum_a q_a \vec{u}_a = -e \sum_e \vec{u}_e = -\frac{e}{m_e} \sum m_e \vec{u}_e = -\frac{e}{m_e}\vec{P}, \tag{4.54}$$

where $\vec{P}$ is the total momentum of all electrons. The total momentum is conserved $\frac{d\vec{P}}{dt} = 0$, thus

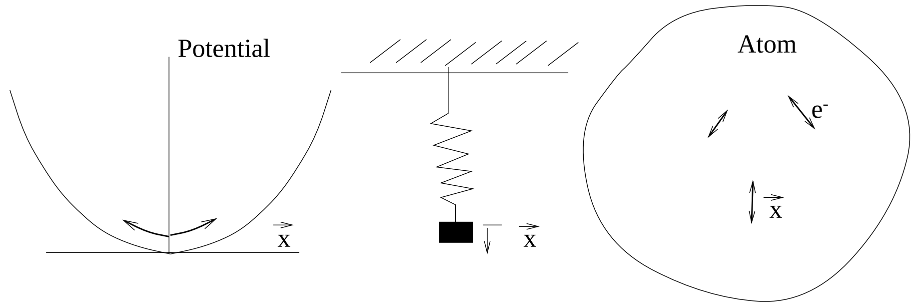
$$\ddot{\vec{d}} = -\frac{e}{m_e}\dot{\vec{P}} = 0. \tag{4.55}$$

Thus, no bremsstrahlung in $e - e$ (non-relativistic) collisions in the dipole approximation. (In the next order, quadrupole, there is radiation.)

Bremsstrahlung between charged particles in the dipole approximation requires particles with different q/m, e.g. ions and electrons. In the case of relativistic particles, they “forget” about their masses, and radiation exists even in dipole approximation.

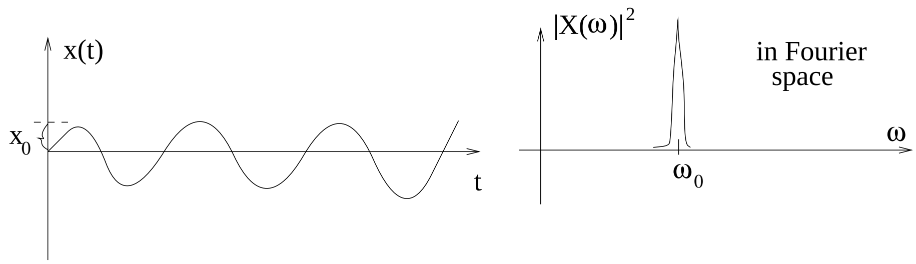
4.10.2 Thomson classical model of an atom

Consider harmonically bound charges that perform "free" oscillations around their equilibrium positions.



Central force $\vec{F} = -k\vec{x} \equiv -m\omega_0^2\vec{x}$. Here k - spring constant, ω_0 - oscillation frequency, x - displacement. Newton's 2nd law states:

$$m\ddot{\vec{x}} = \vec{F} = -m\omega_0^2\vec{x}, \Rightarrow \vec{x}(t) = \vec{x}_0 \cos(\omega_0 t). \tag{4.56}$$



Dipole radiation from oscillating charge:

dipole moment $\vec{d} = -e\vec{x}$,

radiated power $P = \frac{2}{3c^3}|\ddot{\vec{d}}|^2 = \frac{2}{3}\frac{e^2}{c^3}|\ddot{\vec{x}}|^2 = \frac{2}{3}\frac{e^2}{c^3}x_0^2\omega_0^4 \cos^2 \omega_0 t$,

mean power $P = \frac{2}{3}\frac{e^2}{c^3}x_0^2\omega_0^4 \langle\cos^2 \omega_0 t\rangle = \frac{e^2}{3}\frac{\omega_0^4}{c^3}x_0^2$,

since $1/(2\pi)\int_0^{2\pi} \cos^2\phi \, d\phi = 1/2$.

As the particle radiates, it should lose energy and the oscillations should damp. This is radiation reaction.

Equation of motion

$$m_e\ddot{\vec{x}} = -m_e\omega_0^2\vec{x} - m_e\gamma\dot{\vec{x}} = \vec{F} + \vec{F}_{Rad.React}. \tag{4.57}$$

What is γ? We know that radiation losses are given by Larmor formula

$$\frac{\mathrm{d}W}{\mathrm{d}t} = -P = -\frac{2}{3}\frac{e^2}{c^3}|\ddot{\vec{x}}|^2. \tag{4.58}$$

The time averaged radiation loss is then proportional to

$$\frac{1}{\tau}\int_t^{t+\tau}\ddot{\vec{x}}\cdot\ddot{\vec{x}}\mathrm{d}t = -\frac{1}{\tau}\int_t^{t+\tau}\dddot{\vec{x}}\cdot\dot{\vec{x}}\mathrm{d}t, \tag{4.59}$$

where the integration in parts was used. The average emitted power is equal to the average work done by reaction force per unit time:

$$\langle\frac{\mathrm{d}W}{\mathrm{d}t}\rangle = \langle\vec{F}_{Rad.React}\cdot\dot{\vec{x}}\rangle, \tag{4.60}$$

and we can introduce the radiation reaction force

$$\vec{F}_{Rad.React} = \frac{2}{3}\frac{e^2}{c^3}\dddot{\vec{x}}. \tag{4.61}$$

For simplicity, for the oscillating solution we can rewrite $\dddot{\vec{x}} = -\omega^2\dot{\vec{x}}$, so that

$$\vec{F}_{R.R.} = -m_e\gamma\dot{\vec{x}}, \tag{4.62}$$

with $\gamma = \frac{2e^2\omega^2}{3m_ec^3} = \frac{2}{3}\frac{r_e}{c}\omega^2$, where $r_e \equiv \frac{e^2}{m_ec^2} = 2.8\cdot 10^{-13}$ cm is the classical electron radius.

Limits of Validity

Damping term should be small ($\vec{x}\propto e^{i\omega t}$):

$$\gamma\dot{\vec{x}} \ll \ddot{\vec{x}} \Rightarrow \gamma\omega x_0 \ll \omega^2 x_0 \Rightarrow \gamma \ll \omega \Rightarrow \frac{r_e}{c}\omega^2 \ll \omega \Rightarrow \omega \ll \frac{c}{r_e} = 10^{23}\text{rad/sec}, \tag{4.63}$$

i.e. photon energy $\hbar\omega \ll m_ec^2/\alpha_f = 70$ MeV (here the fine structure constant $\alpha_f = e^2/(\hbar c) = 1/137$).

$\omega \ll \frac{c}{r_e} \Leftrightarrow r_e \ll \frac{c}{\omega} \sim \lambda$, i.e. theory is valid only for wavelengths larger than the electron radius.

Solution of equation (4.57). Make a guess $x(t) = x_0e^{\alpha t}$. Substitute this into equation and get

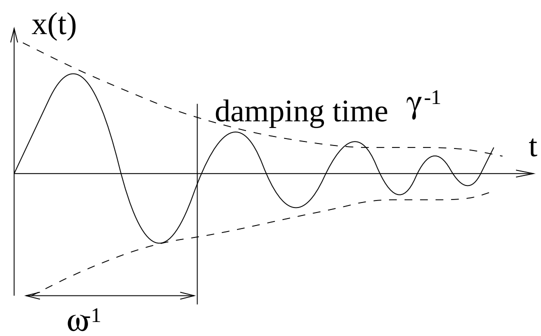
$$\alpha^2 + \gamma\alpha + \omega_0^2 = 0, \tag{4.64}$$

which has two solutions $\alpha = -\gamma/2 \pm \sqrt{-\omega_0^2 + (\gamma/2)^2}$. For $\gamma \ll \omega_0$ they can be approximated as

$$\alpha = -\gamma/2 \pm i\omega_0. \tag{4.65}$$

Thus the general solution is the linear combination of the two possible solutions, with arbitrary coefficients. Taking initial conditions $x(0) = x_0, \dot{x}(0) = 0$, we get the solution

$$x(t) = x_0 e^{-\frac{\gamma}{2}t} \cos\omega_0 t = \frac{x_0}{2}\left(e^{-\frac{\gamma}{2}t - i\omega_0 t} + e^{-\frac{\gamma}{2}t + i\omega_0 t}\right). \qquad (4.66)$$



Making Fourier transform we get

$$\hat{x}(\omega) = \frac{1}{2\pi}\int_0^\infty x(t) e^{i\omega t} \mathrm{d}t = \frac{x_0}{4\pi}\left[\frac{1}{\frac{\gamma}{2} - i(\omega - \omega_0)} + \frac{1}{\frac{\gamma}{2} - i(\omega + \omega_0)}\right]. \qquad (4.67)$$

The first term in the brackets dominates, because for ω close to ω_0 the denominator there becomes small. Thus we can approximate

$$\hat{x}(\omega) = \frac{x_0}{4\pi}\left[\frac{1}{\frac{\gamma}{2} - i(\omega - \omega_0)}\right], \qquad (4.68)$$

and

$$|\hat{x}(\omega)|^2 = \left(\frac{x_0}{4\pi}\right)^2 \frac{1}{(\omega - \omega_0)^2 + (\gamma/2)^2}. \qquad (4.69)$$

Using eq. (4.53), we can get the radiation spectrum

$$\frac{\mathrm{d}W}{\mathrm{d}\omega} = \frac{8\pi}{3}\frac{\omega^4 e^2}{c^3}\left(\frac{x_0}{4\pi}\right)^2 \frac{1}{(\omega - \omega_0)^2 + (\gamma/2)^2} = \frac{1}{2}kx_0^2 \frac{\gamma/2\pi}{(\omega - \omega_0)^2 + (\gamma/2)^2}, \qquad (4.70)$$

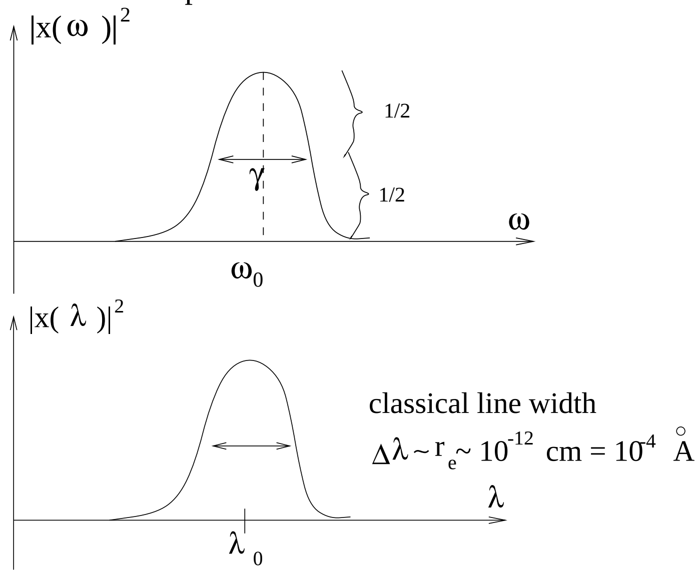
where $k = m_e\omega_0^2$ is the spring constant. The factor $kx_0^2/2$ is the potential energy of the oscillator. Integrating over frequencies, we get the total emitted energy

$$W = \int_{-\infty}^{\infty} \frac{\mathrm{d}W}{\mathrm{d}\omega}\mathrm{d}\omega = \frac{1}{2}kx_0^2, \qquad (4.71)$$

as it should by conservation of energy (here we can extend integral to $-\infty$ because the function is strongly peaked at $\omega \approx \omega_0$). The function

$$\frac{\gamma/2\pi}{(\omega-\omega_0)^2+(\gamma/2)^2}. \tag{4.72}$$

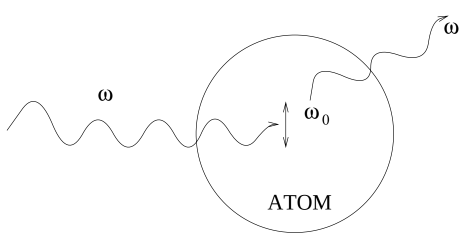
is called Lorentz profile.



The line width in units of λ is a physical constant:

$$\Delta\lambda = \Delta(c/\nu) = 2\pi c\Delta(1/\omega) = 2\pi c\frac{\Delta\omega}{\omega_0^2} = 2\pi c\frac{\gamma}{\omega_0^2} = 2\pi c\frac{2}{3}\frac{r_e}{c}\frac{\omega_0^2}{\omega_0^2} = \frac{4\pi}{3}r_e. \tag{4.73}$$

4.10.3 Externally driven oscillations of the bound atomic oscillators (charges) = the classical theory of absorption and scattering



Consider incoming fields

$$\vec{E} = \vec{E}_0 e^{i(\vec{k}\cdot\vec{x}-\omega t)}, \quad \vec{B} = \vec{B}_0 e^{i(\vec{k}\cdot\vec{x}-\omega t)}, \tag{4.74}$$

where the electron field is the real part of $\vec{E}$. At charge, $\vec{x} = 0$,

$$\vec{E} = \vec{E}_0 e^{-i\omega t}.$$

Consider particle to be nonrelativistic

$$\vec{F}_{\text{external}} = -e\vec{E} - e\frac{\vec{u}}{c} \times \vec{B} \approx -e\vec{E}. \tag{4.75}$$

The equation of motion can be written as

$$\begin{aligned} m\ddot{\vec{x}} &= \vec{F}_{\text{central}} + \vec{F}_{\text{RR}} + \vec{F}_{\text{external}}, \\ m\ddot{\vec{x}} &= -m\omega_0^2\vec{x} - m\gamma\dot{\vec{x}} - e\vec{E}_0 e^{-i\omega t}. \end{aligned} \tag{4.76}$$

<u>Ansatz</u> (guess) $\vec{x} = \vec{x}_0 e^{-i\omega t}$. Thus

$$\vec{x}_0(i\omega)^2 e^{-i\omega t} = \left(-\omega_0^2\vec{x}_0 + i\omega\gamma\vec{x}_0 - \frac{e\vec{E}_0}{m_e}\right)e^{-i\omega t}, \tag{4.77}$$

$$(\omega^2 - \omega_0^2 + i\omega\gamma)\vec{x}_0 = \frac{e\vec{E}_0}{m_e}, \quad \Rightarrow \quad \vec{x}_0 = \frac{e\vec{E}_0/m_e}{(\omega^2 - \omega_0^2) + i\omega\gamma}. \tag{4.78}$$

Response $\vec{x}$ is out of phase with the E-field (note imaginary $i\omega\gamma$ in the dominator). Radiated energy per unit time = power (averaged over period)

$$\begin{aligned} \langle P \rangle &= \frac{2}{3}\frac{\langle|\ddot{\vec{d}}|^2\rangle}{c^3} = \frac{2e^2}{3c^3}\langle|\ddot{\vec{x}}|^2\rangle = \frac{2e^2}{3c^3}\langle(\omega^2\Re\vec{x}_0 e^{-i\omega t})^2\rangle \\ &= \frac{2e^2}{3c^3}\omega^4\frac{1}{2}\vec{x}_0\cdot\vec{x}_0^* = \frac{e^2\omega^4}{3c^3}\frac{(e^2/m_e^2)E_0^2}{[(\omega^2-\omega_0^2)+i\omega\gamma][(\omega^2-\omega_0^2)-i\omega\gamma]} \\ &= \frac{e^2\omega^4}{3c^3}\frac{(e^2/m_e^2)E_0^2}{(\omega^2-\omega_0^2)^2+\omega^2\gamma^2}. \end{aligned} \tag{4.79}$$

The time averaging of the $\langle(\Re\vec{x}_0 e^{-i\omega t})^2\rangle$ can be done in the following way. Express $\Re\vec{x}_0 e^{-i\omega t} = (\vec{x}_0 e^{-i\omega t} + \vec{x}_0^* e^{i\omega t})/2$, then $(\Re\vec{x}_0 e^{-i\omega t})^2 = (\vec{x}_0^2 e^{-i2\omega t} + (\vec{x}_0^*)^2 e^{i2\omega t} + 2\vec{x}_0\vec{x}_0^*)/4$. Averaging the first two terms over time gives zero, while the third term does not depend on time.

Incoming Poynting flux $\langle S \rangle$ (erg/s/cm^2) is:

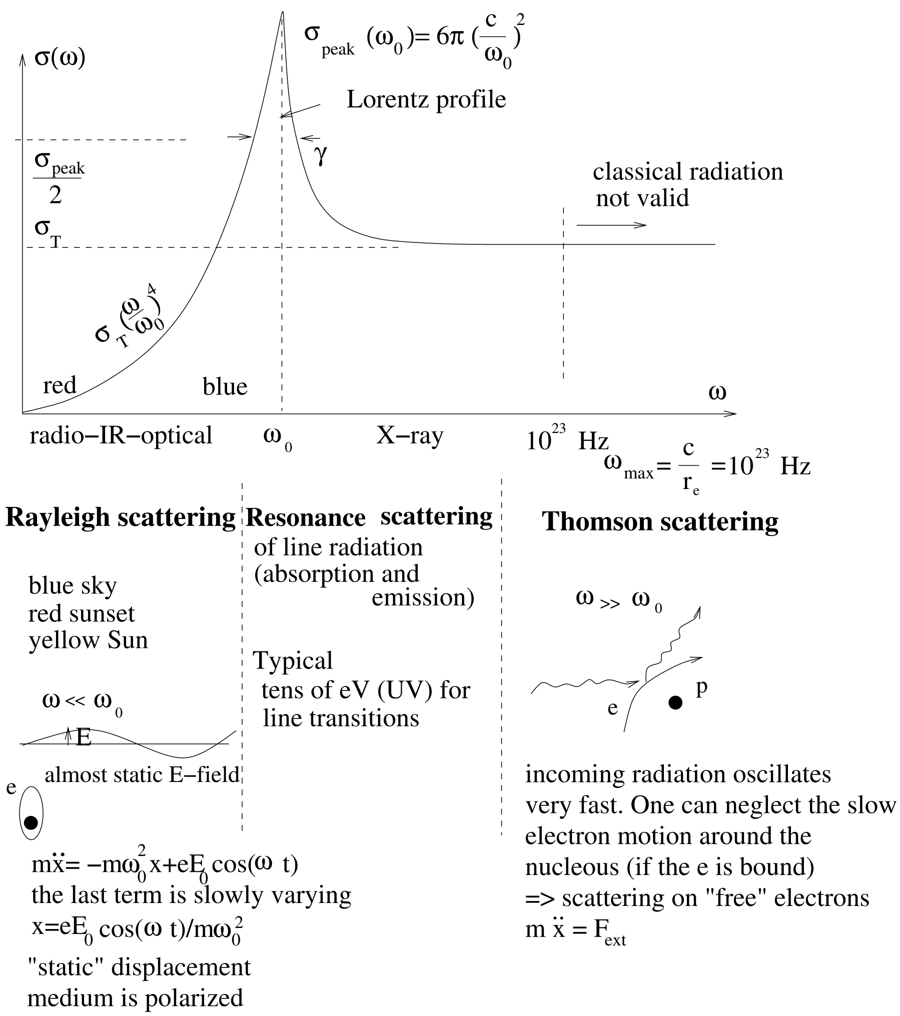
$$\langle S \rangle = \frac{c}{4\pi}\langle(\Re\vec{E}_0 e^{-i\omega t})^2\rangle = \frac{c}{8\pi}E_0^2. \tag{4.80}$$

We can define the cross-section as the ratio of the scattered energy per unit time P (erg/s) to the incoming Poynting flux (erg/s/cm^2):

$$\sigma_{\rm scat}(\omega) = \frac{\langle P\rangle}{\langle S\rangle} = \frac{8\pi e^4}{3m_e^2c^4}\frac{\omega^4}{(\omega^2-\omega_0^2)^2+\omega^2\gamma^2} = $$

$$= \frac{8\pi r_e^2}{3}\frac{\omega^4}{(\omega^2-\omega_0^2)^2+\omega^2\gamma^2} = \sigma_T\frac{\omega^4}{(\omega^2-\omega_0^2)^2+\omega^2\gamma^2}, \quad (4.81)$$

where we introduced the Thomson cross-section $\sigma_T = \frac{8\pi}{3}r_e^2 \approx 0.665\times10^{-24}$ cm^2. The frequency dependence of the cross-section $\sigma(\omega)$ is shown below.

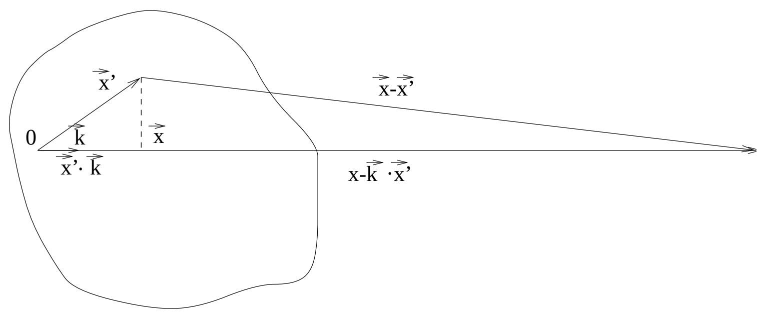


4.11 Multipole Radiation

Higher orders beyond electric dipole are important, when electric dipole radiation is zero. Here we only consider next two terms in the multipole expansion: magnetic dipole and electric quadrupole.

The retarded potentials once again

$$\vec{A}(\vec{x},t) = \int_V \frac{\vec{j}(\vec{x}', t_{\text{ret}})}{c} \frac{d^3x'}{|\vec{x}-\vec{x}'|}. \tag{4.82}$$



In the wave zone

$$|\vec{x}-\vec{x}'| \approx x - \vec{k}\cdot\vec{x}' \approx x,$$

$$t_{\text{ret}} = t - \frac{|\vec{x}-\vec{x}'|}{c} \approx (t - \frac{x}{c}) + \frac{\vec{k}\cdot\vec{x}'}{c} = t_0 + \Delta t.$$

The first term $t_0 = t - x/c$ for t_{ret} was kept in the dipole approximation, while the second (the difference in retardation time due to different distances to different e^- in the source) was neglected.

We can Taylor expand the current density for small Δt:

$$\vec{j}(\vec{x}', t_{\text{ret}}) = \vec{j}(\vec{x}', t_0) + \Delta t \left.\frac{\partial \vec{j}(\vec{x}', t_{\text{ret}})}{\partial t_{\text{ret}}}\right|_{t_{\text{ret}}=t_0} + O((\Delta t)^2). \tag{4.83}$$

The first term in the rhs gives dipole approximation, the next terms give next higher orders:

$$\begin{aligned} \vec{A}(\vec{x}', t) &= \vec{A}_0(\vec{x},t) + \vec{A}_1(\vec{x},t) + ... \\ &= \frac{1}{cx}\int_V \vec{j}(\vec{x}', t - x/c)\mathrm{d}^3x' + \frac{1}{cx}\int_V \Delta t \left.\frac{\partial \vec{j}(\vec{x}', t_{\text{ret}})}{\partial t_{\text{ret}}}\right|_{t_{\text{ret}}=t_0} \mathrm{d}^3x'. \end{aligned} \tag{4.84}$$

Consider the term A_1:

$$\vec{A}_1(\vec{x}', t) = \frac{1}{c^2 x} \int_V \vec{k} \cdot \vec{x}' \left. \frac{\partial \vec{j}(\vec{x}', t_{\mathrm{ret}})}{\partial t_{\mathrm{ret}}} \right|_{t_{\mathrm{ret}} = t_0} \mathrm{d}^3 x' = \frac{\vec{k}}{c^2 x} \cdot \frac{\partial}{\partial t} \int_V \vec{x}' \vec{j}(\vec{x}', t_0) \mathrm{d}^3 x', \quad (4.85)$$

where we used $\frac{\partial}{\partial t_{\mathrm{ret}}} = \frac{\partial}{\partial t} \frac{\partial t}{\partial t_{\mathrm{ret}}} = \frac{\partial}{\partial t}$.

For non-relativistic charges

$$\vec{j}(\vec{x}', t - x/c) = \sum_a q_a \vec{u}_a (t - x/c) \delta(\vec{x}' - \vec{x}_a(t - x/c)),$$

we obtain

$$\vec{A}_1(\vec{x}, t) = \frac{\vec{k}}{c^2 x} \cdot \frac{d}{dt} \sum_a q_a [\vec{x}_a \vec{u}_a]_{t_{\mathrm{ret}} = t - x/c}. \quad (4.86)$$

Note that there is *no* scalar product within the brackets [], but there is in $(\vec{k} \cdot \vec{x}_a)\vec{u}_a$. Representing $\vec{x}_a \vec{u}_a = \frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t}(\vec{x}_a \vec{x}_a) + \vec{x}_a \vec{u}_a / 2 - \vec{u}_a \vec{x}_a / 2$ we get:

$$\begin{aligned} \vec{A}_1(\vec{x}, t) &= \frac{\vec{k}}{6c^2 x} \cdot \frac{\mathrm{d}^2}{\mathrm{d}t^2} \sum_a q_a [3\vec{x}_a \vec{x}_a - |\vec{x}_a|^2 \mathbf{I}] + \frac{1}{cx} \frac{\mathrm{d}}{\mathrm{d}t} \sum_a \frac{q_a}{2cm_a} (\vec{x}_a \times m_a \vec{u}_a) \times \vec{k} \\ &= \frac{1}{6c^2 x} \vec{k} \cdot \ddot{\mathbf{Q}} + \frac{1}{cx} \dot{\vec{M}} \times \vec{k}. \end{aligned} \quad (4.87)$$

Here $\mathbf{Q} = \sum_a q_a [3\vec{x}_a \vec{x}_a - |\vec{x}_a|^2 \mathbf{I}]$ is electric quadrupole moment (which is symmetric traceless tensor), $\mathbf{I}$ - unit tensor, $\vec{M} = \sum_a \frac{q_a}{2cm_a} (\vec{x}_a \times m_a \vec{u}_a)$ - magnetic dipole moment, and $\vec{x}_a \times m_a \vec{u}_a$ - orbital angular momentum of ath charge. We added a vector proportional to $\vec{k} \cdot \mathbf{I} = \vec{k}$ in the first term. This is possible since such addition does not change the value of physical magnetic (and electric) field ($\vec{B} = \frac{1}{c} \frac{\partial \vec{A}}{\partial t} \times \vec{k}$). Thus, we have in the wave zone

$$\vec{A} = \vec{A}_0 + \vec{A}_1 = \frac{\dot{\vec{d}}}{cx} + \frac{1}{6c^2 x} \vec{k} \cdot \ddot{\mathbf{Q}} + \frac{1}{cx} \dot{\vec{M}} \times \vec{k}. \quad (4.88)$$

electric dipole	electric quadrupole	magnetic dipole

The magnetic and electric fields are given by

$$\begin{aligned} \vec{B} &= \nabla \times \vec{A} = \frac{1}{c} \frac{\partial \vec{A}}{\partial t} \times \vec{k} = \frac{1}{c^2 x} \left[\ddot{\vec{d}} \times \vec{k} + \frac{1}{6c} (\vec{k} \cdot \dddot{\mathbf{Q}}) \times \vec{k} + (\ddot{\vec{M}} \times \vec{k}) \times \vec{k} \right], \\ \vec{E} &= \vec{B} \times \vec{k}. \end{aligned} \quad (4.89)$$

The total radiated power integrated over all solid angles is given by

$$P = \int \vec{S} \cdot \vec{k} x^2 \mathrm{d}\Omega = c \int (Bx)^2 \frac{\mathrm{d}\Omega}{4\pi} [\mathrm{erg/s}].$$

Now we square the expression for $\vec{B}$ and average it over angles. One should note that all the cross terms give zero and only squares of individual terms are left. Noting that $\int \sin^2\theta \frac{\mathrm{d}\Omega}{4\pi} = \frac{2}{3}$ and doing some messy calculations for the middle term (see Landau & Lifshitz, vol 2) one gets

$$P = \frac{2}{3}\frac{|\ddot{\vec{d}}|^2}{c^3} + \frac{1}{180}\frac{|\dddot{\mathbf{Q}}|^2}{c^5} + \frac{2}{3}\frac{|\ddot{\vec{M}}|^2}{c^3}. \tag{4.90}$$

Futher comments:
1) The multipole expansion is valid for $L \ll \lambda$, $u/c \ll 1$ and cannot be used for relativistic particles.
2)

$$\frac{\text{quadrupole power}}{\text{dipole power}} = \frac{|\dddot{\mathbf{Q}}|^2/c^5}{|\ddot{\vec{d}}|^2/c^3} = \frac{|\frac{\mathrm{d}^3}{\mathrm{d}t^3}\vec{x}\vec{x}|^2/c^5}{|\frac{\mathrm{d}^2}{\mathrm{d}t^2}\vec{x}|^2/c^3} \approx \frac{|\frac{\mathrm{d}\vec{x}}{\mathrm{d}t}|^2}{c^2} \approx \left(\frac{u}{c}\right)^2.$$

3) In the near zone $L \ll x \ll \lambda$, one must keep further terms. For example: pulsars (rotating magnetic dipoles).

Chapter 5

Relativistic covariance and kinematics

5.1 Lorentz transformations

The special theory of relativity is based on two postulates:

1. The laws of nature are the same in two reference frames in uniform relative motion with no rotation.

2. The speed of light is constant c in all such frames.

Consider two coordinate systems K and K' that moves with relative velocity V along the x-axis. From the postulates one can show that the coordinates in the two systems are related through Lorentz transformations (LT):

$$\begin{aligned} x' &= \gamma(x - \beta ct), \\ y' &= y, \\ z' &= z, \\ ct' &= \gamma(ct - \beta x), \end{aligned} \tag{5.1}$$

where $\gamma = \frac{1}{\sqrt{1-\beta^2}}$, $\beta = V/c$. This also can be written using the Lorentz transformation tensor

$$\Lambda = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{5.2}$$

as

$$\begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = \Lambda \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}. \tag{5.3}$$

The inverse transform can be obtained changing $\beta \rightarrow -\beta$:

$$\begin{aligned} x &= \gamma(x' + \beta ct'), \\ y &= y', \\ z &= z', \\ ct &= \gamma(ct' + \beta x'). \end{aligned} \tag{5.4}$$

The Lorentz transformation with arbitrary velocity $\vec{\beta}$ of a 4-vector $\underline{a} = \{a_0, \vec{a}\}$ is given by:

$$\begin{aligned} a'_0 &= \gamma(a_0 - \vec{\beta} \cdot \vec{a}), \\ \vec{a}' &= \vec{a} - a_0\gamma\vec{\beta} + (\gamma - 1)\vec{\beta}(\vec{\beta} \cdot \vec{a})/\beta^2. \end{aligned} \tag{5.5}$$

5.1.1 Proper time

Some quantities are Lorentz invariants, i.e. they have the same value in all Lorentz-frames. Proper time, $\mathrm{d}\tau$, between events with time- and spatial distances $\mathrm{d}t$, $\mathrm{d}x$, $\mathrm{d}y$, $\mathrm{d}z$, is defined as

$$c^2\mathrm{d}\tau^2 \equiv c^2\mathrm{d}t^2 - (\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2). \tag{5.6}$$

One can prove that $\mathrm{d}\tau$ is the Lorentz invariant by using the Lorentz transformation, $\mathrm{d}\tau = \mathrm{d}\tau'$. Prove this at home!

Proper time τ is the time shown by the clocks that observers carry along, i.e. τ is the time in the rest frame of the observer, where $\mathrm{d}x = \mathrm{d}y = \mathrm{d}z = 0$. In this system we have $\mathrm{d}\tau = \mathrm{d}t$.

5.1.2 Lorentz-Fitzgerald length contraction

The K'-observer carries a rod of length $L' = x'_2 - x'_1$. In K-system the rod's length is measured by determining the coordinates of the ends of the rod at the same time t. Therefore

$$L' = x'_2 - x'_1 = \gamma(x_2 - Vt - x_1 + Vt) = \gamma(x_2 - x_1) = \gamma L, \tag{5.7}$$

where we used LT. Here L is the length in K-system. Thus,

$$L = (1-\beta^2)^{1/2}L' = L'/\gamma. \tag{5.8}$$

The rod is shorter for the K-observer.

If the K-observer carries the rod, then the K'-observer finds it to be shorter. The effect is symmetric. The reason for differences is that the measurements are not simultaneous between the two frames.

5.1.3 Time dilation

Consider a clock at rest in K' (a comoving clock, $x' = 0$) that measures a time interval $T' = t'_2 - t'_1$. In the K-system (using clocks in the K-system), one measures

$$T = t_2 - t_1 = \gamma(t'_2 - t'_1) = \gamma T'. \tag{5.9}$$

Here we used LT with $x' = 0$. For K, the clock in K' seems to be slower. On the contrary, for K' the clocks in K seem slower.

5.1.4 Velocity transformation

LT can be written in differential form:

$$\begin{aligned} \mathrm{d}x &= \gamma(\mathrm{d}x' + \beta\, c\mathrm{d}t'), \\ \mathrm{d}y &= \mathrm{d}y', \\ \mathrm{d}z &= \mathrm{d}z', \\ c\mathrm{d}t &= \gamma(c\mathrm{d}t' + \beta \mathrm{d}x'), \end{aligned} \tag{5.10}$$

We get then for the velocities

$$\begin{aligned} u_x &= \frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\gamma(\mathrm{d}x' + \beta\, c\mathrm{d}t')}{\gamma(\mathrm{d}t' + \beta \mathrm{d}x'/c)} = \frac{u'_x + V}{1 + \beta u'_x/c}, \\ u_y &= \frac{\mathrm{d}y}{\mathrm{d}t} = \frac{u'_y}{\gamma(1 + \beta u'_x/c)}, \\ u_z &= \frac{\mathrm{d}z}{\mathrm{d}t} = \frac{u'_z}{\gamma(1 + \beta u'_x/c)}, \end{aligned} \tag{5.11}$$

or rewriting this for velocities parallel and perpendicular to V:

$$u_{\|} = \frac{u'_{\|} + V}{1 + \beta u'_{\|}/c}, \quad (5.12)$$

$$u_{\perp} = \frac{u'_{\perp}}{\gamma(1 + \beta u'_{\|}/c)}. \quad (5.13)$$

5.1.5 Transformation of velocity directions = aberration

The angle θ the velocity makes to some direction can be defined via the ratio of the projections of the velocity to the direction and perpendicular to it:

$$\tan\theta = \frac{u_{\perp}}{u_{\|}} = \frac{u'_{\perp}}{\gamma(u'_{\|} + V)} = \frac{u' \sin\theta'}{\gamma(u' \cos\theta' + V)}. \quad (5.14)$$

Note that the azimuth angle does not change.

To determine aberration of light, we need to substitute $u' = c$:

$$\tan\theta = \frac{\sin\theta'}{\gamma(\cos\theta' + \beta)}. \quad (5.15)$$

At home: use $\cos^2\theta = \frac{1}{1+\tan^2\theta}$ to show that

$$\cos\theta = \frac{\cos\theta' + \beta}{1 + \beta\cos\theta'}, \quad (5.16)$$

$$\sin\theta = \frac{\sin\theta'}{\gamma(1 + \beta\cos\theta')}. \quad (5.17)$$

Example: How does $\theta' = \pi/2$ transforms?

$$\theta' = \pi/2 \Rightarrow \tan\theta = 1/(\gamma\beta) \Rightarrow \cos\theta = \beta \Rightarrow \sin\theta = 1/\gamma. \quad (5.18)$$

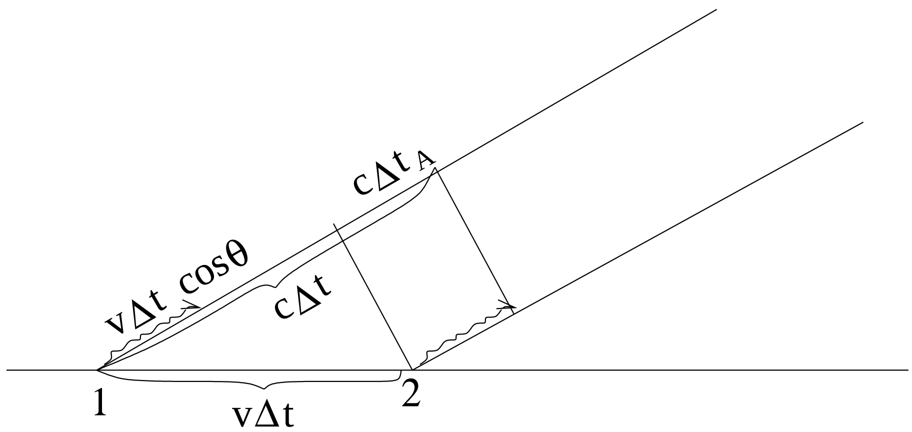
If $\gamma \gg 1 \Rightarrow \theta = 1/\gamma$.

Isotropic emission in K' becomes ”beamed emission” in K-frame, where the particle moves with Lorentz factor γ.

5.1.6 Doppler effect

There are three different intervals to keep track of:
1) the time interval $\Delta t'$ in the moving particle frame K' (e.g. related to the frequency of emitted radiation $\Delta t' = 2\pi/\omega'$);

2) the time interval Δt in the observer's system K: $\Delta t = \gamma\Delta t'$ (time dilation);
3) the time interval $\Delta t_{\rm Arrival}$, during which a pulse is received by the observer.



A particle of velocity V emits photons at 1 and 2 with time interval Δt, towards the observer. When the 2nd photon is emitted, the 1st one has travelled a distance $c\Delta t$. From the figure we get

$$c\Delta t_A = c\Delta t - V\Delta t\cos\theta \Rightarrow \Delta t_A = \Delta t(1-\beta\cos\theta). \tag{5.19}$$

If Δt_A is the time interval for receiving one wavelength, the observed frequency becomes:

$$\omega = \frac{2\pi}{\Delta t_A} = \frac{2\pi}{\Delta t(1-\beta\cos\theta)} = \frac{2\pi}{\gamma\Delta t'(1-\beta\cos\theta)} = \frac{\omega'}{\gamma(1-\beta\cos\theta)}. \tag{5.20}$$

<u>Relativistic Doppler formula</u>:

$$\omega' = \omega\gamma(1-\beta\cos\theta), \qquad \omega = \omega'\gamma(1+\beta\cos\theta'). \tag{5.21}$$

Show that $\gamma(1-\beta\cos\theta) = \frac{1}{\gamma(1+\beta\cos\theta')}$ using the $\cos\theta$-$\cos\theta'$ formula.

For small θ and large γ (i.e. $\beta = \sqrt{1-1/\gamma^2} \sim 1 - 1/2\gamma^2$),

$$\omega = \frac{\omega'}{\gamma[1-(1-1/2\gamma^2)(1-\theta^2/2)]} = \frac{\omega' 2\gamma}{1+\theta^2\gamma^2}. \tag{5.22}$$

5.1.7 4-vectors

Examples:

position $(ct, \vec{x})$; 4-velocity $\gamma(c, \vec{u})$; 4-momentum for a photon $k = \frac{\hbar\omega}{c}(1, \vec{n})$ where $\vec{n}$ is the unit vector in the direction of the photon propagation; 4-current density $(\rho c, \vec{j})$; 4-potential $(\phi, \vec{A})$; 4-momentum for particle $p = (E/c, \vec{p})$

All Lorentz transformed in the same way as the ”position” vector, e.g. for the 0th component:

time

$$ct' = \gamma(ct - \beta x), \tag{5.23}$$

particle energy

$$E'/c = \gamma(E/c - \beta p_x), \tag{5.24}$$

photon energy

$$\frac{\hbar\omega'}{c} = \gamma\frac{\hbar\omega}{c}(1 - \beta n_x) = \gamma\frac{\hbar\omega}{c}(1 - \beta\cos\theta) \Rightarrow \tag{5.25}$$

$\omega' = \omega\gamma(1 - \beta\cos\theta)$, i.e. we obtained the relativistic Doppler effect formula directly from the LT.

5.1.8 Lorentz invariants

Scalar products of 4-vectors are Lorentz invariants. For example:

a)

$$(ct, \vec{x})\begin{pmatrix} ct \\ \vec{x} \end{pmatrix} = -c^2t^2 + x^2 = \text{const.} \tag{5.26}$$

Minus sign appears by rule in space-time metric. One can instead use standard definition of the scalar product, but introduce imaginary i infront of the 0th element of the 4-vector, i.e.

$$(ict, \vec{x})\begin{pmatrix} ict \\ \vec{x} \end{pmatrix} = (ict)^2 + x^2 = -c^2t^2 + x^2. \tag{5.27}$$

b)

$$(E/c, \vec{p})\begin{pmatrix} E/c \\ \vec{p} \end{pmatrix} = p^2 - E^2/c^2 = \text{const.} \tag{5.28}$$

What constant? Consider its value in the rest frame where $\vec{p} = 0$, then $0 - \frac{(mc^2)^2}{c^2} = -(mc)^2$. This means that $E^2 - (pc)^2 = (mc^2)^2$. If one introduces energy $E = \gamma mc^2$ and momentum $\vec{p} = \gamma m\vec{u}$, one gets $\gamma^2 - \beta^2\gamma^2 = 1$, i.e. $\gamma = 1/\sqrt{1 - \beta^2}$.

The dependence between E and pc (hyperbola $E^2 = (pc)^2 + (mc^2)^2$) is the same as we had when discussing proper time. Here the constant is the square of rest mass energy $(mc^2)^2$.

5.1.9 Electro-magnetic field transformation

The electric or magnetic fields cannot be represented as 4-vectors. Instead one can introduce the electro-magnetic field tensor:

$$F = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & B_x & -B_y \\ -E_y & -B_z & 0 & B_x \\ -E_z & B_y & -B_x & 0 \end{pmatrix}. \tag{5.29}$$

The LT to the system K' moving with velocity $V = c\beta$ along the x-axis can be written using the LT tensor as

$$F' = \Lambda^{\mathrm{T}} F \Lambda. \tag{5.30}$$

This can be rewritten in the form

$$\vec{E}'_{\parallel} = \vec{E}_{\parallel}, \qquad \vec{B}'_{\parallel} = \vec{B}_{\parallel}, \tag{5.31}$$

$$\vec{E}'_{\perp} = \gamma(\vec{E}_{\perp} + \beta \times \vec{B}), \quad \vec{B}'_{\perp} = \gamma(\vec{B}_{\perp} - \beta \times \vec{E}). \tag{5.32}$$

The immediate consequence is that the concept of pure electric or magnetic field is not Lorentz invariant. If in one frame the field is purely electric ($\vec{B} = 0$), in some other frame it will be, in general, a mixed electric and magnetic field. Thus the general term *electro-magnetic field*.

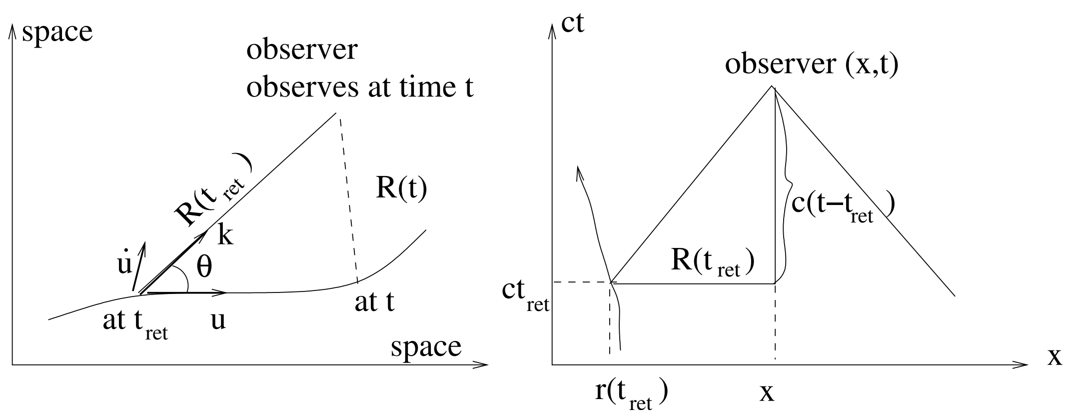
Note that $\vec{B}^2 - \vec{E}^2$ and $\vec{E} \cdot \vec{B}$ are Lorentz invariants.

5.2 Radiation from relativistic charges

Multipole expansion is an expansion in u/c, which cannot be used for $u \approx c$. For $u/c \leq 1$, start from the exact potentials for one particle, the Lienard-Wiechert potentials:

$$\begin{pmatrix} \phi(\vec{x},t) \\ \vec{A}(\vec{x},t) \end{pmatrix} = \left[\frac{q}{R - \vec{R}\cdot\frac{\vec{u}}{c}} \begin{pmatrix} 1 \\ \vec{u}/c \end{pmatrix} \right]_{t_{\text{ret}}}, \tag{5.33}$$

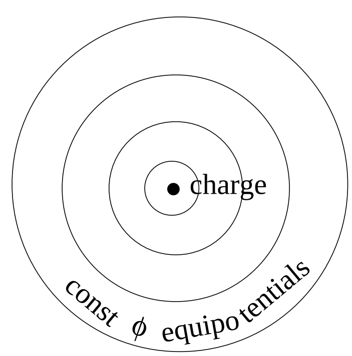
where $t_{\text{ret}} = t - R(t_{\text{ret}})/c$, $\vec{R}(t_{\text{ret}}) = \vec{x} - \vec{r}(t_{\text{ret}})$, and $\vec{r}(t_{\text{ret}})$ is the position of particle at t_{ret}.



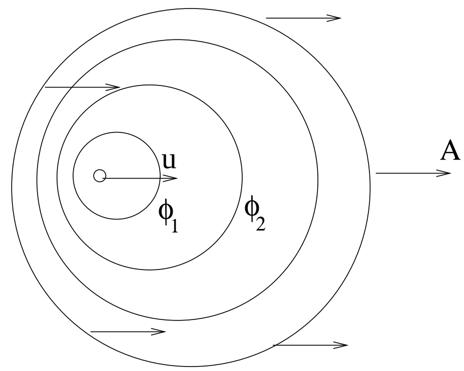
All is valid in both near and wave zone. Just to remind the notations:
$\vec{k} = \vec{R}/R$ - direction to the observer; $\vec{\beta} = \vec{u}/c$ velocity; $\dot{\vec{\beta}} = \dot{\vec{u}}/c$ acceleration; R distance to the observer; $\gamma = 1/(1-\beta^2)^{1/2}$ Lorentz factor

Lienard-Wiechert Potentials

1) if velocity $\vec{u}/c = 0$ (or $\vec{u}/c \ll 1$), then $\phi(\vec{x},t) = q/R$, $\quad \vec{A}(\vec{x},t) \approx 0$.



2) if velocity $\vec{u} \neq 0$ and a reasonable fraction of c, then $R - \vec{R} \cdot \vec{u}/c = R(1 - \beta \cos\theta)$ causes "beaming" in the $\vec{u}$ direction. Here $\vec{A}$ is parallel to $\vec{u}$ everywhere and is largest in forward direction due to Doppler factor.



$\vec{E}$ and $\vec{B}$ fields

As usual, $\vec{E}$ and $\vec{B}$ are obtained through

$$\begin{aligned} \vec{B} &= \nabla \times \vec{A}, \\ \vec{E} &= -\nabla\phi - \frac{1}{c}\frac{\partial \vec{A}}{\partial t}. \end{aligned} \tag{5.34}$$

In home exercise 3.2 we have shown that

$$\begin{aligned} \vec{E}(\vec{x}, t) &= \frac{q}{(R - \vec{R}\cdot\vec{\beta})^3} \left\{ (1-\beta^2)(\vec{R} - R\vec{\beta}) + \frac{\vec{R}}{c} \times [(\vec{R} - R\vec{\beta}) \times \dot{\vec{\beta}}] \right\}_{t_{\rm ret}}, \\ \vec{B}(\vec{x}, t) &= \frac{\vec{R}}{R} \times \vec{E}. \end{aligned} \tag{5.35}$$

The first term in curved brackets goes as $\propto R/R^3 \propto R^{-2}$ which is as in Coulomb field. The second term $\propto R^2/R^3 \propto R^{-1}$ makes transverse, radiation field. $\vec{B} \perp \vec{E}$ and $\vec{B} \perp \vec{k}$ in both near and wave zone.

Expression $(R - \vec{R}\cdot\vec{\beta}) = R(1 - \vec{k}\cdot\vec{\beta}) = R(1 - \beta\cos\theta)$ contains the Doppler factor $\kappa = 1 - \beta\cos\theta$. It appears in $\vec{E}$ and $\vec{B}$ partly due to

$$\frac{\partial t}{\partial t_{\rm ret}} = 1 - \beta\cos\theta, \tag{5.36}$$

which is identical to

$$\frac{\Delta t_A}{\Delta t} = 1 - \beta\cos\theta \tag{5.37}$$

discussed in the section 5.1.6 with different notations $\Delta t_A \leftrightarrow \Delta t$ and $\Delta t \leftrightarrow \Delta t_{\rm ret}$.

5.2.1 Electromagnetic field from charge with contant velocity

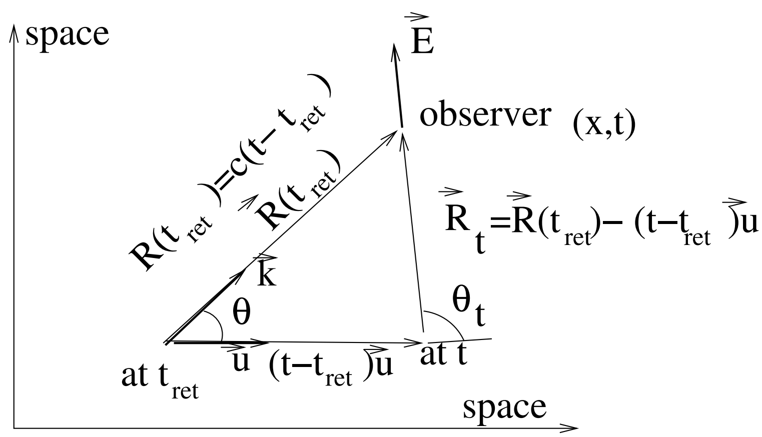
1) If velocity $\vec{\beta} = 0$, then one recovers the usual static Coulomb field

$$\vec{E} = \frac{q}{R^3}\vec{R} = \frac{q}{R^2}\vec{k}, \quad \vec{B} = \vec{k} \times \vec{E} = 0, \quad \text{Coulomb field.} \tag{5.38}$$

Uniform motion, $\dot{\vec{\beta}} = 0 \Rightarrow$ only the Coulomb term left.

$$\vec{E}(\vec{x}, t) = q \left[\frac{(1-\beta^2)(\vec{R} - R\vec{\beta})}{(R - \vec{R} \cdot \vec{\beta})^3} \right]_{t_{\rm ret}} . \tag{5.39}$$

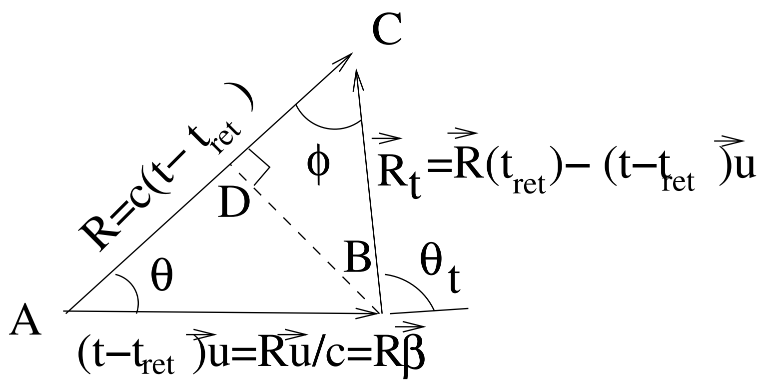
One can ask a question where does $\vec{E}$-vector points?



The vector $\vec{R}_t \equiv \vec{R}(t_{\rm ret}) - (t - t_{\rm ret})\vec{u} = \vec{R}(t_{\rm ret}) - \frac{R(t_{\rm ret})}{c}\vec{u}$ points towards the observer from the present position of the charge. Thus, $\vec{E} \propto (\vec{R} - R\vec{\beta}) = \vec{R}_t$ points away from the charge's present position!, although the field is caused by what the charge did at time $t_{\rm ret}$!

The denominator (i.e. Doppler factor) can be written in terms of present angle θ_t and distance R_t:

$$R - \vec{R} \cdot \vec{\beta} = R_t (1 - \beta^2 \sin^2 \theta_t)^{1/2}. \tag{5.40}$$



Proof: Note that the length AD is $R\beta\cos\theta$. Then DC is $R - R\beta\cos\theta$, which is just the factor we want to express in terms of θ_t and R_t. Now by Pythagoras' theorem for triangle BCD we get

$$(R - \vec{R}\cdot\vec{\beta})^2 = R_t^2 - R_t^2 \sin^2\phi. \tag{5.41}$$

Considering triangle ABC, and using the fact that the sine for an angle divided by the length of the opposite side is a constant, gives $\sin\phi/R\beta = \sin\theta_t/R$, or

$$\sin\phi = \beta\sin\theta_t. \tag{5.42}$$

Substituting $\sin\phi$ taking a square root of both sides one gets the needed equality. QED.

Now we have

$$\vec{E}(\vec{x},t) = \frac{q}{\gamma^2}\frac{\vec{R}_t}{R_t^3(1-\beta^2\sin^2\theta_t)^{3/2}}, \tag{5.43}$$

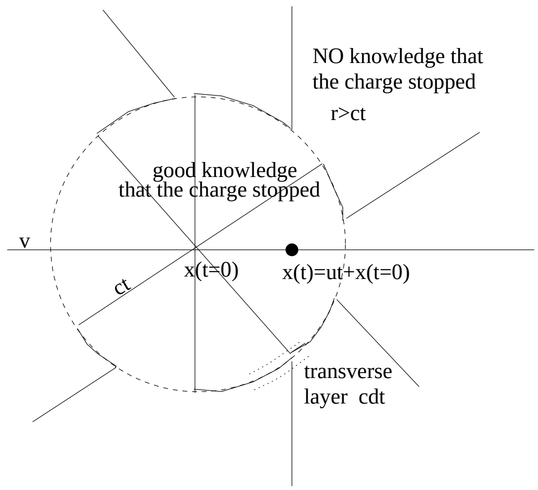
$$\vec{B}(\vec{x},t) = \frac{\vec{R}}{R}\times\vec{E} = (\vec{\beta} + \frac{\vec{R}_t}{R})\times\vec{E} = \vec{\beta}\times\vec{E}. \tag{5.44}$$

a) If $\vec{\beta} = 0$,

$$\vec{E} = q\frac{\vec{R}_t}{R_t^3}, \quad \vec{B} = 0, \quad \text{i.e. Coulomb field.} \tag{5.45}$$

b) If $\vec{\beta} \neq 0$, $\vec{R} - R\vec{\beta}$ gives rise to a beaming effect.

c) Sudden deceleration (Bremsstrahlung): Consider a charge with a constant velocity that rapidly stops during time dt at time $t = 0$. This gives rise to a spherical transverse pulse that propagates outwards with speed of light.

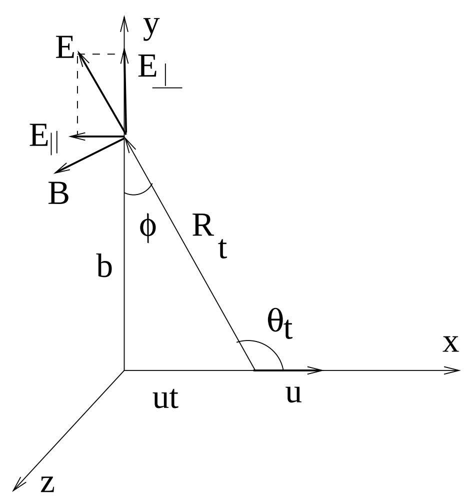


The thickness of the transverse layer cdt. Number of flux lines through a ring is constant. The area is $2\pi R \mathrm{d}R$, the thickness $\mathrm{d}R = c\mathrm{d}t$. Field strength = (number of flux lines)/area = constant/ R, i.e. radiation! Average static field strength = (number of flux lines)/area = const/$4\pi R^2$.

d) $\vec{E}$ and $\vec{B}$ fields from *relativistic* particle in uniform motion;

$$\vec{E} = \frac{q(1-\beta^2)\vec{R}_t}{R_t^3(1-\beta^2\sin^2\theta_t)^{3/2}}, \quad \vec{B} = \vec{\beta}\times\vec{E}. \tag{5.46}$$

When $\beta \to 1$, we have $|\vec{B}| \approx |\vec{E}|$. Furthermore, $\vec{B}\perp\vec{E}$ always. This is similar to radiation! Consider a field at a point located at distance b from the track of the charge. The charge passes the origin at $t = 0$.



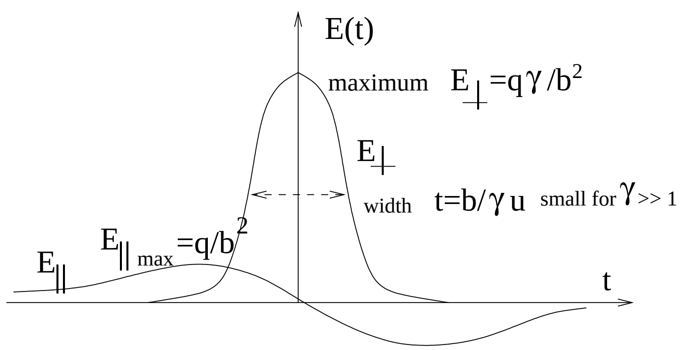
Here we define $R_t = b/\sin\theta_t = b/\cos\phi$. The field can be decomposited into a $\parallel$ and a $\perp$ fields:

$$E_\perp = \frac{q(1-\beta^2)\sin^3\theta_t}{b^2(1-\beta^2\sin^2\theta_t)^{3/2}} = \frac{q(1-\beta^2)\cos^3\phi}{b^2(1-\beta^2\cos^2\phi)^{3/2}}, \quad (5.47)$$

$$E_\parallel = \frac{q(1-\beta^2)\sin^2\theta_t\cos\theta_t}{b^2(1-\beta^2\sin^2\theta_t)^{3/2}} = \frac{q(1-\beta^2)\cos^2\phi\sin\phi}{b^2(1-\beta^2\cos^2\phi)^{3/2}}. \quad (5.48)$$

For $\gamma \gg 1$, the denominator becomes $1-\beta^2\cos^2\phi = 1-(1-\frac{1}{\gamma^2})(1-\sin^2\phi) \approx \frac{1}{\gamma^2} + \sin^2\phi = \frac{1+\gamma^2\sin^2\phi}{\gamma^2}$. The denominator is small for $\gamma^2\sin^2\phi \ll 1$, i.e. $\sin\phi \ll 1/\gamma \ll 1$, i.e. $\phi \ll 1/\gamma$ or $\frac{ut}{b} \ll 1/\gamma$, i.e. for times $t \ll \frac{b}{u\gamma}$.

The observer sees a pulse $\vec{E}(t)$:



$E_\perp$ has same sign and has maximum at $\phi = 0$:

$$E_\perp = \frac{q(1-\beta^2)}{b^2(1-\beta^2)^{3/2}} = \frac{q\gamma}{b^2} \approx \gamma \times \text{static field}. \quad (5.49)$$

$E_\parallel$ changes sign and has smaller amplitude $\sim q/b^2 \approx$ static field:

$$E_\parallel \approx \frac{q}{b^2}\frac{\gamma^3\sin\phi}{\gamma^2(1+\gamma^2\sin^2\phi)^{3/2}}, \quad (5.50)$$

and maximum $\approx \frac{q}{b^2}$ occurs at $\phi \approx \frac{1}{\gamma}$. The field lines are thus concentrated within an angle $1/\gamma$ relative to the transverse direction. The observer thus sees a transverse field with $|E| \approx |B|$ and $\vec{E}\perp\vec{B}$.

This Coulomb field can be Fourier-decomposed and be considered as a field consisting of virtual photons. (This is used in semi-classical calculations, e.g. Jackson ch 15.4, Weizsächer-Williams method). When charge is accelerated, one can consider the emitted photons to be virtual photons that have been shaken off.

5.2.2 Electromagnetic field from accelerated charge in the wave zone, i.e. radiation field

In the wave zone $\vec{R} \approx \vec{k}x$ and $R = x$, where x is mean distance to charge. Acceleration field (radiation field) is given by

$$\begin{aligned}
\vec{E}(\vec{x}, t) &\approx \frac{q}{(R - \vec{R}\cdot\vec{\beta})^3}\left\{\frac{\vec{R}}{c}\times[(\vec{R} - R\vec{\beta})\times\dot{\vec{\beta}}]\right\}_{\rm ret} \\
&\approx \frac{q}{c^2 x \kappa^3}\left\{\vec{k}\times[(\vec{k} - \vec{\beta})\times\dot{\vec{u}}]\right\}_{\rm ret} \qquad (5.51)\\
\vec{B}(\vec{x}, t) &= \frac{\vec{R}}{R}\times\vec{E} = \vec{k}\times\vec{E},
\end{aligned}$$

where $\kappa = 1 - \vec{k}\cdot\vec{\beta}$. We see that $\vec{E} \propto \vec{k}\times[\ ] \Rightarrow \vec{E}\perp\vec{k}$. Also $\vec{B}\perp\vec{k}$ and $|\vec{E}| = |\vec{B}| \propto \frac{1}{x}$, i.e. radiation.

Let us consider a few special cases.

1) <u>Non-relativistic</u> motion $\beta \ll 1$, $\kappa \approx 1$:

$$\begin{aligned}
\vec{B} &= \vec{k}\times\vec{E} = \frac{q}{c^2 x}\{\vec{k}\times[\vec{k}\times(\vec{k}\times\dot{\vec{u}})]\} = \frac{q}{c^2 x}\{\vec{k}\times[\vec{k}(\vec{k}\cdot\dot{\vec{u}}) - \dot{\vec{u}}(\vec{k}\cdot\vec{k})]\} \\
&= -\frac{q}{c^2 x}(\vec{k}\times\dot{\vec{u}}) = \frac{\ddot{\vec{d}}\times\vec{k}}{c^2 x}, \qquad (5.52)
\end{aligned}$$

i.e. the classical Larmour formula for "accelerating" displacement. Radiation pattern $\frac{dP}{d\Omega} = \frac{c}{4\pi}(Bx)^2 = \frac{|\ddot{\vec{d}}|^2}{4\pi c^3}\sin^2\theta$.

2) <u>Relativistic motion</u> $\beta \to 1$, $\gamma \gg 1$:

Note two things.

a) The Doppler factor $\kappa = 1 - \beta\cos\theta$ can be very small when $\beta \sim 1$ and for certain angles $\theta \sim 0$. Then $1/\kappa$ is very large. For $\gamma \gg 1$, $\beta = 1 - \frac{1}{2\gamma^2}$ and

$$\kappa = 1 - \beta\cos\theta \approx 1 - (1 - \frac{1}{2\gamma^2})(1 - \frac{\theta^2}{2}) \approx \frac{1 + \gamma^2\theta^2}{2\gamma^2} \qquad (5.53)$$

and

$$\frac{1}{\kappa} = \frac{2\gamma^2}{1 + \gamma^2\theta^2}. \qquad (5.54)$$

This is large when $\theta \ll 1/\gamma \ll 1$. Due to the $1/\kappa^3$ factor the radiation field is beamed, i.e. concentrated towards the $\theta = 0$ direction.

b) For observers with $\vec{k}$, such that $(\vec{k} - \vec{\beta}) \parallel \dot{\vec{u}}$, $\vec{E} = 0$, i.e. no radiation. Note that $(\vec{k} - \vec{\beta}) \parallel (\vec{R} - R\vec{\beta}) = \vec{R}_t$, thus $\vec{E} = 0$ if $\dot{\vec{u}}(t_{\rm ret})$ parallel to the direction from the observer to the location where the charge would have been at time t if it had been in uniform motion.

3) The general case:

Angular distribution of radiated power. As before, the radiated power passing on area $x^2 d\Omega$ in direction $\vec{k}$, becomes

$$\mathrm{d}P = \frac{c}{4\pi}|\vec{E} \times \vec{B}| x^2 \mathrm{d}\Omega = \frac{cB^2}{4\pi}(x^2 \mathrm{d}\Omega), \tag{5.55}$$

i.e. the received power per unit solid angle becomes

$$\frac{\mathrm{d}P}{\mathrm{d}\Omega}_{\rm received} = \frac{c(xE)^2}{4\pi} = \frac{q^2}{4\pi c^3}\left\{\frac{\vec{k} \times [(\vec{k} - \vec{\beta}) \times \dot{\vec{u}}]}{\kappa^3}\right\}^2 . \tag{5.56}$$

Denote the expression in {} as $\vec{g}$:

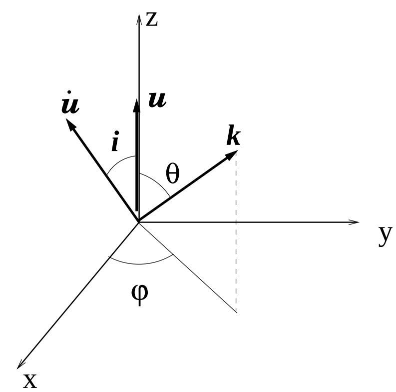
$$\vec{g} = \frac{1}{\kappa^3}[(\vec{k} \cdot \dot{\vec{u}})(\vec{k} - \vec{\beta}) - \vec{k} \cdot (\vec{k} - \vec{\beta})\dot{\vec{u}}]. \tag{5.57}$$

Then

$$\begin{aligned} g^2 &= \frac{1}{\kappa^6}[(\vec{k} \cdot \dot{\vec{u}})^2(\vec{k} - \vec{\beta})^2 + \kappa^2|\dot{\vec{u}}|^2 - 2\kappa(\vec{k} \cdot \dot{\vec{u}})(\vec{k} - \vec{\beta}) \cdot \dot{\vec{u}}] \\ &= \frac{1}{\kappa^6}\{(\vec{k} \cdot \dot{\vec{u}})^2(1 + \beta^2 - 2\vec{k} \cdot \vec{\beta}) + \kappa^2|\dot{\vec{u}}|^2 - 2\kappa[(\vec{k} \cdot \dot{\vec{u}})^2 - (\vec{k} \cdot \dot{\vec{u}})(\vec{\beta} \cdot \dot{\vec{u}})]\} \\ &= \frac{1}{\kappa^4}|\dot{\vec{u}}|^2 + \frac{2}{\kappa^5}(\vec{k} \cdot \dot{\vec{u}})(\vec{\beta} \cdot \dot{\vec{u}}) - \frac{1}{\kappa^6}(\vec{k} \cdot \dot{\vec{u}})^2(1 - \beta^2), \end{aligned} \tag{5.58}$$

where we used $\kappa = 1 - \vec{k} \cdot \vec{\beta}$.

Define a coordinate system: $\vec{u} = (0, 0, u)$, $\dot{\vec{u}} = |\dot{\vec{u}}|(\sin i, 0, \cos i)$, $\vec{k} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$.



Then $\vec{k} \cdot \dot{\vec{u}} = |\dot{\vec{u}}|(\sin\theta\cos\phi\sin i + \cos\theta\cos i)$, and $\dot{\vec{u}} \cdot \vec{\beta} = |\dot{\vec{u}}|\beta\cos i$.

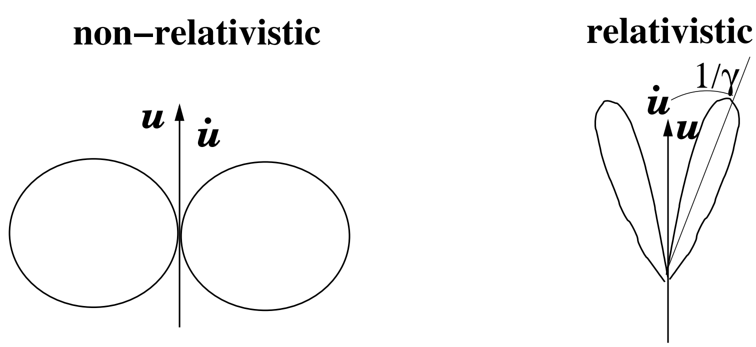
Special case $\dot{\vec{u}}||\vec{u}$, i.e. $i = 0$, then $\vec{k} \cdot \dot{\vec{u}} = |\dot{\vec{u}}|\cos\theta$ and $\dot{\vec{u}} \cdot \vec{\beta} = |\dot{\vec{u}}|\beta$. Then (show this!)

$$g^2 = |\dot{\vec{u}}|^2 \frac{\sin^2\theta}{(1-\beta\cos\theta)^6}. \tag{5.59}$$

Then the received power is

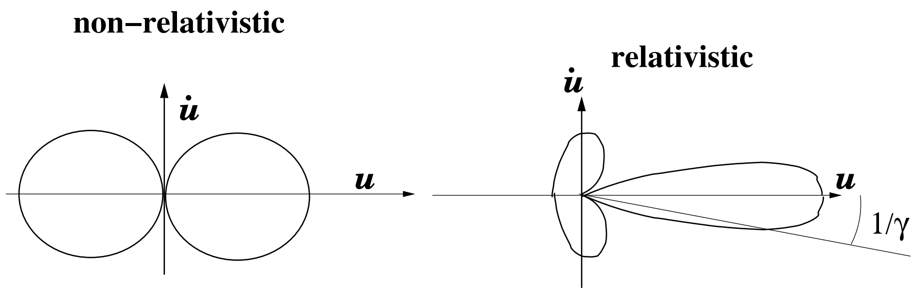
$$\frac{\mathrm{d}P}{\mathrm{d}\Omega}_{\text{received}} = \frac{q^2}{4\pi c^3} g^2 \approx \frac{16 q^2 |\dot{\vec{u}}|^2}{\pi c^3} \gamma^{10} \frac{\gamma^2\theta^2}{(1+\gamma^2\theta^2)^6}. \tag{5.60}$$

Radiation pattern (angular distribution of $\mathrm{d}P/\mathrm{d}\Omega$). If $\beta \sim 1$, $\gamma \gg 1$, and $\ddot{\vec{d}}||\vec{u}$, the torus becomes very elongated. Maximum at $\theta \sim 1/\gamma$, no radiation at $\theta = 0$. It is simply the non-relavistic torus (applicable in the instantaneous rest frame of the charge) that has been Lorentz transformed (see Rybicki & Lightman pp. 140-143 for details). The relativistic result can be obtained by Lorentz transforming $\mathrm{d}P/\mathrm{d}\Omega$, $\sin\theta$, and $|\vec{u}_{||}|$.



Special case $\dot{\vec{u}} \perp \vec{u}$, i.e. $i = \pi/2$, then $\vec{k} \cdot \dot{\vec{u}} = |\dot{\vec{u}}| \sin\theta \cos\phi$ and $\dot{\vec{u}} \cdot \vec{\beta} = 0$. Then

$$\frac{g^2}{|\dot{\vec{u}}|^2} = \frac{1}{\kappa^4} - \frac{(1-\beta^2)}{\kappa^6} \sin^2\theta \cos^2\phi. \tag{5.61}$$



As previously discussed, the radiation observer's time interval, dt (denoted Δt_A previous lecture), is not equal to the particle observer's time interval $\mathrm{d}t_{\mathrm{ret}}$ (denoted Δt before). We have $\frac{\mathrm{d}t}{\mathrm{d}t_{\mathrm{ret}}} = 1 - \beta\cos\theta = \kappa$.

The emitted power per unit solid angle

$$\frac{\mathrm{d}P}{\mathrm{d}\Omega}_{\mathrm{emitted}} = \frac{\mathrm{d}W}{\mathrm{d}t_{\mathrm{ret}}\mathrm{d}\Omega} = \left(\frac{\mathrm{d}t}{\mathrm{d}t_{\mathrm{ret}}}\right)\frac{\mathrm{d}W}{\mathrm{d}t\mathrm{d}\Omega} = \left(\frac{\mathrm{d}t}{\mathrm{d}t_{\mathrm{ret}}}\right)\frac{\mathrm{d}P}{\mathrm{d}\Omega}_{\mathrm{received}} = \kappa\frac{\mathrm{d}P}{\mathrm{d}\Omega}_{\mathrm{received}}, \tag{5.62}$$

i.e. emitted power is not equal to the received power as the same amount of energy $\mathrm{d}W$ is emitted and received during different time intervals.

Radiated power P

Integrate $\mathrm{d}P/\mathrm{d}\Omega$ over $\mathrm{d}\Omega$. One must choose if it the received or emitted power that is of interest. To compute local energy losses in the gas, requires a knowledge of the emitted power.

$$\frac{\mathrm{d}P}{\mathrm{d}\Omega}_{\mathrm{emitted}} = \kappa\frac{\mathrm{d}P}{\mathrm{d}\Omega}_{\mathrm{received}} = \frac{q^2\kappa g^2}{4\pi c^3}, \tag{5.63}$$

$$P_{\mathrm{emitted}} = \frac{q^2}{4\pi c^3}\int \kappa g^2 \mathrm{d}\Omega = \frac{2e^2}{3c^3}\gamma^6[|\dot{\vec{u}}|^2 - (\dot{\vec{u}}\times\vec{\beta})^2] = \frac{2e^2}{3c^3}\gamma^6|\dot{\vec{u}}|^2(1-\beta^2\sin^2 i). \tag{5.64}$$

Parallel acceleration $a_{\parallel} \equiv \dot{\vec{u}}_{\parallel}$, $i = 0$,

$$P_{\mathrm{emitted},\parallel} = \frac{2e^2}{3c^3}\gamma^6 a_{\parallel}^2. \tag{5.65}$$

Perpendicular acceleration $a_\perp \equiv \dot{\vec{u}}_\perp$, $i = \pi/2$,

$$P_{\text{emitted},\perp} = \frac{2e^2}{3c^3}\gamma^6(1-\beta^2)a_\perp^2 = \frac{2e^2}{3c^3}\gamma^4 a_\perp^2. \tag{5.66}$$

In a general case, $\vec{a} = \vec{a}_\parallel + \vec{a}_\perp$:

$$P_{\text{emitted}} = \frac{2e^2}{3c^3}\gamma^4(a_\perp^2 + \gamma^2 a_\parallel^2). \tag{5.67}$$

This is relativistic Larmor formula. For a given acceleration, a relativistic particle radiates a factor γ^4 or γ^6 more than a non-relativistic.

5.2.3 Lorentz invariance of the radiated power P

The expression for P_{emitted} can be derived very elegantly using Lorentz invariance of P. Lienard who derived P_{emitted} in 1898 did not have access to special relativity.

K' is the instantaneous rest frame. During a short moment electron is at rest in this system, and in a short time interval before and after the velocity of electron is non-relativistic in K'.

For a non-relativistic charge one can use Larmor formula (dipole radiation). Consider the energy $\mathrm{d}W'$ that is radiated during $\mathrm{d}t'$. Corresponding total momentum change is $\mathrm{d}p' = 0$ due to the symmetry of the torus.

Now Lorentz transform to K (energy transforms as time):

$$\mathrm{d}W = \gamma(\mathrm{d}W' + \beta c\,\mathrm{d}p') = \gamma\,\mathrm{d}W', \tag{5.68}$$

since $\mathrm{d}p' = 0$ and the time interval

$$\mathrm{d}t = \gamma(\mathrm{d}t' + \beta\,\mathrm{d}x'/c) = \gamma\,\mathrm{d}t', \tag{5.69}$$

since $\mathrm{d}x' = 0$ in K'. The power

$$P = \frac{\mathrm{d}W}{\mathrm{d}t} = \frac{\gamma\,\mathrm{d}W'}{\gamma\,\mathrm{d}t'} = P', \tag{5.70}$$

i.e. total power is Lorentz invariant for processes with symmetry in the rest frame.

We have

$$P' = \frac{2q^2}{3c^3}|a'|^2 = \frac{2q^2}{3c^3}({a'}_\perp^2 + {a'}_\parallel^2) \tag{5.71}$$

in the instantaneous rest frame. In K we have

$$P = \frac{2q^2}{3c^3}\gamma^4(a_\perp^2 + \gamma^2 a_\parallel^2). \tag{5.72}$$

Since $P = P'$, the acceleration must Lorentz transform as $a'_\perp = \gamma^2 a_\perp$ and $a'_\parallel = \gamma^3 a_\parallel$.

Chapter 6

Bremsstrahlung (free-free radiation)

6.1 Free-free emission

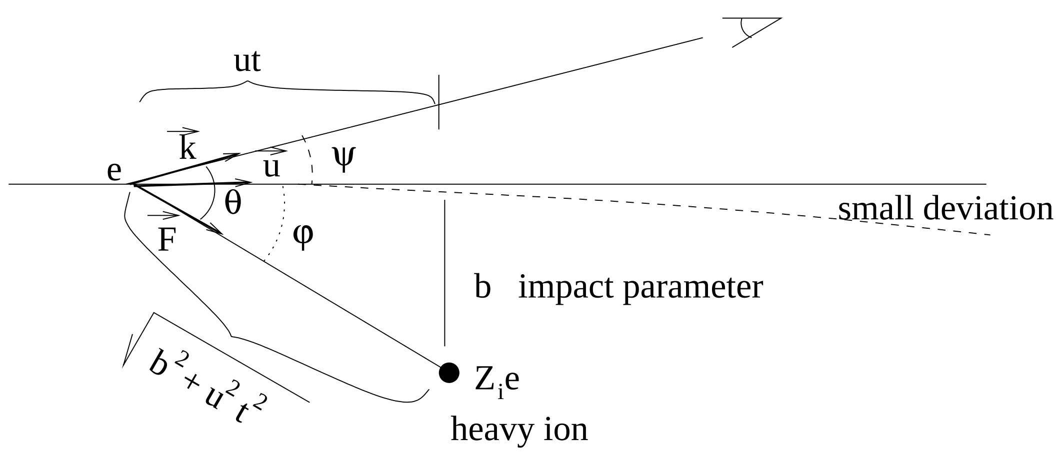
Consider bremsstrahlung radiated from a plasma of temperature T and densities n_e cm^{-3} electrons with charge $-e$ and n_i cm^{-3} ions with charge $Z_i e$.

Important ratio:

$$\frac{\text{Coulomb potential energy}}{\text{thermal kinetic energy}} \approx \frac{Z_i e^2/\langle r\rangle}{kT} \approx \frac{Z_i e^2 n_e^{1/3}}{kT} \ll 1 \tag{6.1}$$

for typical n_e and $T \sim 10^4 - 10^8$ K. Here we used the mean distance between particles $\langle r\rangle \sim n_e^{-1/3}$. Coulomb interaction is only a perturbation on the thermal motions of the electrons.

Consider one electron of velocity $\vec{u}$. Approximate the orbit as a straight line:



The electron is accelerated in the Coulomb field by the force

$$|\vec{F}| = \frac{Z_i e^2}{b^2 + u^2 t^2}. \tag{6.2}$$

Newtons 2nd law states: $m_e \ddot{\vec{x}} = \vec{F} \Rightarrow$, therefore the acceleration is

$$|\ddot{\vec{x}}| = \frac{Z_i e^2}{m_e (b^2 + u^2 t^2)}. \tag{6.3}$$

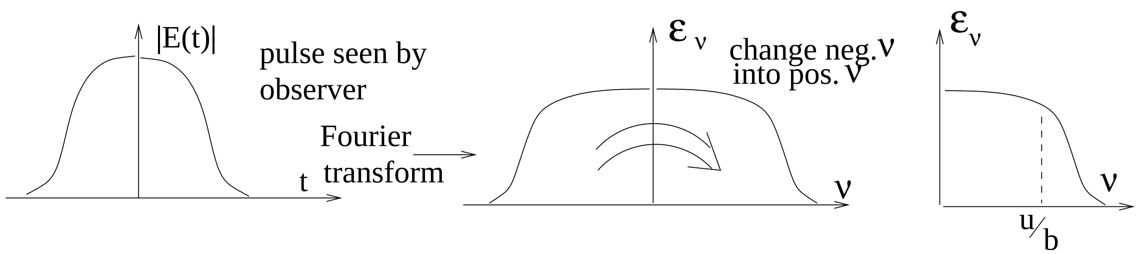
The electric field in the wave zone in dipole approximation is given by

$$\vec{E} = \vec{B} \times \vec{k} = \frac{1}{c^2 r} (\ddot{\vec{d}} \times \vec{k}) \times \vec{k}, \tag{6.4}$$

where r is the distance to the observer. Thus

$$|\vec{E}| = \frac{|\ddot{\vec{d}}|}{c^2 r} \sin\theta(t) = \frac{e|\ddot{\vec{x}}|}{c^2 r} \sin\theta(t) = \frac{Z_i e^3 \sin\theta(t)}{m_e c^2 r (b^2 + u^2 t^2)}. \tag{6.5}$$

Shu considers $\sin\theta$ to be constant (which leads to somewhat wrong normalization). For now, we follow Shu. Fourier transform to get the frequency spectrum of the time-varying $\vec{E}$-field at the observer (see Rybicki & Lightman, ch 2.3).



Fourier transform:

$$\hat{E}(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} |\vec{E}| e^{i\omega t} \mathrm{d}t = \frac{Z_i e^3 \sin\theta}{2\pi m_e c^2 r} \int_{-\infty}^{\infty} \frac{e^{i\omega t}}{b^2 + u^2 t^2} \mathrm{d}t = \frac{Z_i e^3 \sin\theta}{2\pi m_e c^2 r} \left(\frac{\pi}{bu}\right) e^{-|\omega| b/u}, \tag{6.6}$$

i.e. the spectrum cuts off at $\omega > u/b$. We used here contour integration in complex plane to find the integral (see Arfken). Total time-integrated Poynting flux (fluence) at the observer from the full pulse is

$$\int S(t) \mathrm{d}t = \frac{c}{4\pi} \int_{-\infty}^{\infty} |E|^2 \mathrm{d}t = c \int_0^{\infty} |\hat{E}(\omega)|^2 \mathrm{d}\omega, \tag{6.7}$$

where we used Parseval's theorem transforming integral over time to the integral over frequency (see Fourier analysis in Arfken or Rybicki & Lightman ch 2.3).

Time integrated flux the observer sees at frequency ν is thus (remember $\omega = 2\pi\nu$)

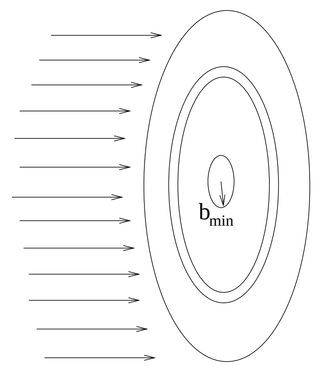
$$c2\pi|\hat{E}(\omega)|^2 = \frac{\sin^2\theta}{2\pi r^2}\left(\frac{Z_i^2 e^6}{m_e^2 c^3}\right)\frac{\pi^2}{(bu)^2}e^{-4\pi\nu b/u} \quad [\text{erg cm}^{-2}\ \text{Hz}^{-1}]. \tag{6.8}$$

The electron radiates a total energy (sum over all solid angles, i.e. a sphere at radius r)

$$P_\nu(b) \equiv \frac{\mathrm{d}W}{\mathrm{d}\nu} = \int c2\pi|\hat{E}(\omega)|^2\, r^2 \mathrm{d}\Omega = \left(\frac{Z_i^2 e^6}{m_e^2 c^3}\right)\frac{\pi^2}{(bu)^2}e^{-4\pi\nu b/u}\frac{4}{3} \quad [\text{erg Hz}^{-1}], \tag{6.9}$$

where we accounted for the fact that $\int \sin^2\theta\, r^2 \mathrm{d}\Omega/(2\pi r^2) = 4/3$.

So far, we considered only one electron interacting with the ion. Now consider a flux of $n_e\vec{u}$ electrons approaching the ion.



Flux $n_e\vec{u}$ [electrons/cm^2/s]. There is a minimum impact parameter $b_{\min}$ that we consider. The existance of the minimum caused by (1) for b not small, deviations or orbit are not small anymore, (2) if $b = \Delta x$ is small, the uncertainty principle ($\Delta x \Delta p > \hbar$) not fulfilled.

Emitted spectrum from flux of electrons

$$P_\nu = \frac{\mathrm{d}W}{\mathrm{d}t\mathrm{d}\nu} = \int_{b_{\min}}^{\infty} P_\nu(b)\, n_e u\, 2\pi b\, \mathrm{d}b \quad [\text{erg s}^{-1}\text{Hz}^{-1}]. \tag{6.10}$$

If electrons have a Maxwellian velocity distribution, then we must integrate over the normalized distribution function:

$$f_e(u) = \left(\frac{m_e}{2\pi kT}\right)^{3/2} e^{-m_e u^2/2kT}, \tag{6.11}$$

$$P_\nu = \int_{u_{\min}}^{\infty} n_e f_e(u)\, 4\pi u^2 \mathrm{d}u \int_{b_{\min}}^{\infty} P_\nu(b)\, u\, 2\pi b\, \mathrm{d}b, \tag{6.12}$$

here $u_{\min}$ is the smallest velocity needed to emit photon of frequency ν, i.e. $mu^2/2 > h\nu \Rightarrow u_{\min} = \sqrt{2h\nu/m_e}$. Let's make the variable transformation

$$\xi = \frac{4\pi\nu b}{u}, \quad x = \frac{m_e u^2}{2kT} \Rightarrow$$

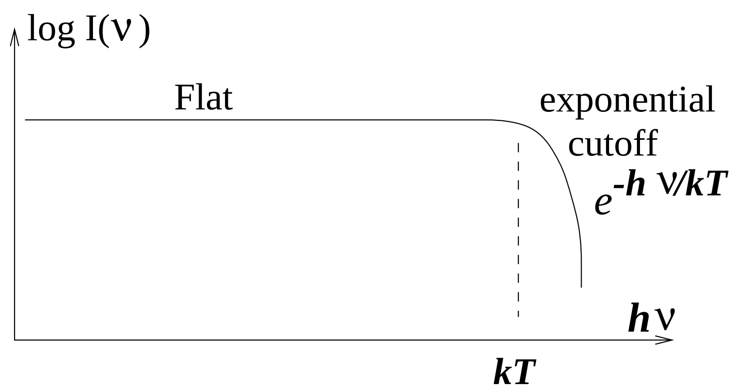
$$P_\nu = n_e \left(\frac{2m_e}{\pi kT}\right)^{1/2} \left(\frac{8\pi^3 Z_i^2 e^6}{3m_e^2 c^3}\right) I(\nu), \tag{6.13}$$

where

$$I(\nu) = \int_{x_{\min}}^{\infty} \mathrm{d}x e^{-x} \int_{\xi_{\min}}^{\infty} \frac{e^{-\xi}}{\xi} \mathrm{d}\xi \tag{6.14}$$

contains all frequency dependence.

In classical deviation (i.e. here) $I(\nu) \propto e^{-h\nu/kT}$.



Exponential ”cut off” appears becauses there are only exponentially few electrons ($e^{-mu^2/2kT}$) at electron energies $\gg kT$ and that these are the only electrons that can radiate at $h\nu \gg kT$. The spectrum if flat since the encounter is short (δ-function).

To get exact $I(\nu)$, one must use quantum mechanical deviation

$$I(\nu) = \frac{4}{\pi\sqrt{3}} g_{ff}(\nu) e^{-h\nu/kT}, \tag{6.15}$$

where g_{ff} is called (free-free) Gaunt factor ≈ 1 and weakly depends on ν.

Summing over all ion species gives the emission coefficient j_ν^{ff} [erg//cm^3/str/sec/Hz]

$$\begin{aligned}\frac{\mathrm{d}W}{\mathrm{d}t\mathrm{d}\nu\mathrm{d}V} &= 4\pi\ j_\nu^{ff} = \sum_i n(Z_i)n_e\left(\frac{2m_e}{3\pi kT}\right)^{1/2}\left(\frac{32\pi^2 Z_i^2 e^6}{3m_e^2c^3}\right)g_{ff}(\nu)e^{-h\nu/kT}\\ &= 6.8\cdot 10^{-38}\sum_i Z_i^2 n(Z_i)n_e T^{-1/2}e^{-h\nu/kT}g_{ff}(\nu)\ \mathrm{erg/cm^3/s/Hz.}\end{aligned} \tag{6.16}$$

A more intelligent way is to rewrite this as:

$$\frac{\mathrm{d}W}{\mathrm{d}t\mathrm{d}V\mathrm{d}(h\nu/m_ec^2)} = \frac{16}{3}\left(\frac{2\pi}{3}\right)^{1/2} g_{ff}(\nu)\alpha r_e^2 c m_e c^2\left(\frac{m_ec^2}{kT}\right)^{1/2}\sum_i Z_i^2 n(Z_i)n_e e^{-h\nu/kT}\ , \tag{6.17}$$

where $r_e = e^2/m_ec^2 = 2.82\ 10^{-13}$ cm is the classical electron radius, $\alpha = e^2/\hbar c = 1/137$ is the fine structure constant, and αr_e^2 is the typical cross-section.

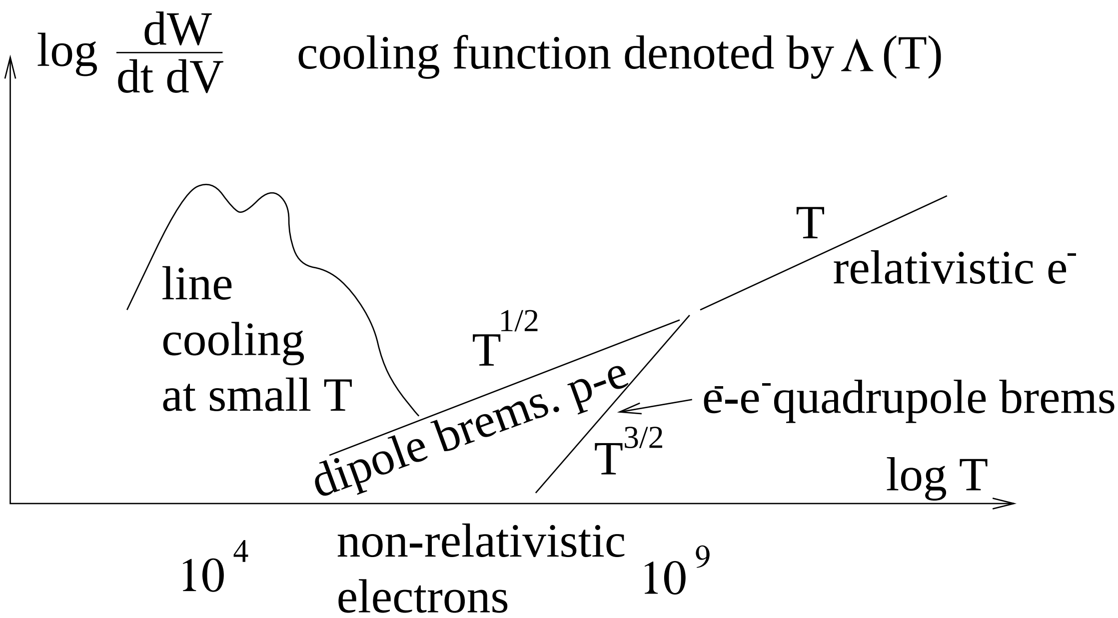
Total radiated power/unit volume= cooling function [erg/s/cm^3] is obtained by integrating over frequency spectrum:

$$\begin{aligned}\frac{\mathrm{d}W}{\mathrm{d}t\mathrm{d}V} &= \frac{16}{3}\left(\frac{2\pi}{3}\right)^{1/2} g_B\alpha r_e^2 c m_e c^2\left(\frac{kT}{m_ec^2}\right)^{1/2}\sum_i Z_i^2 n(Z_i)n_e\\ &= 1.4\ 10^{-27}T^{1/2}\sum_i Z_i^2 n(Z_i)n_e g_B\ ,\end{aligned} \tag{6.18}$$

where $g_B \sim 1.1-1.5$ is the new Gaunt factor.

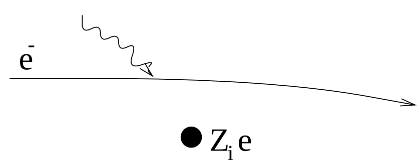
For $e-e$ bremsstrahlung (=0 in dipole approximation) we have

$$\begin{aligned}\frac{\mathrm{d}W}{\mathrm{d}t\mathrm{d}V}\Big|_{ee,quadrupole} &\approx \left(\frac{u}{c}\right)^2\frac{\mathrm{d}W}{\mathrm{d}t\mathrm{d}V}\Big|_{ion-e,dipole} \approx \left(\frac{kT}{m_ec^2}\right)\frac{\mathrm{d}W}{\mathrm{d}t\mathrm{d}V}\Big|_{ion-e,dipole}\\ &\propto TT^{1/2}\propto T^{3/2}.\end{aligned} \tag{6.19}$$



6.1.1 Free-Free Absorption

This is a 3-body interaction



A useful trick to compute the absorption coefficient when you know the emission coefficient is to use the fact that in complete thermodynamic equilibrium we have emission=absorption at each ν:

$$j_\nu^{ff} = \alpha_\nu^{ff} B_\nu(T), \tag{6.20}$$

where the lhs is the emission coefficient [erg/g/sec/ster/Hz], α is the absorption coefficient [cm^{-1}], and B_ν is the Planck function [erg/cm^2/sec/ster/Hz].

Planck function:

$$B_\nu(T) = 2\left(\frac{\nu}{c}\right)^2 \frac{h\nu}{e^{h\nu/kT} - 1}. \tag{6.21}$$

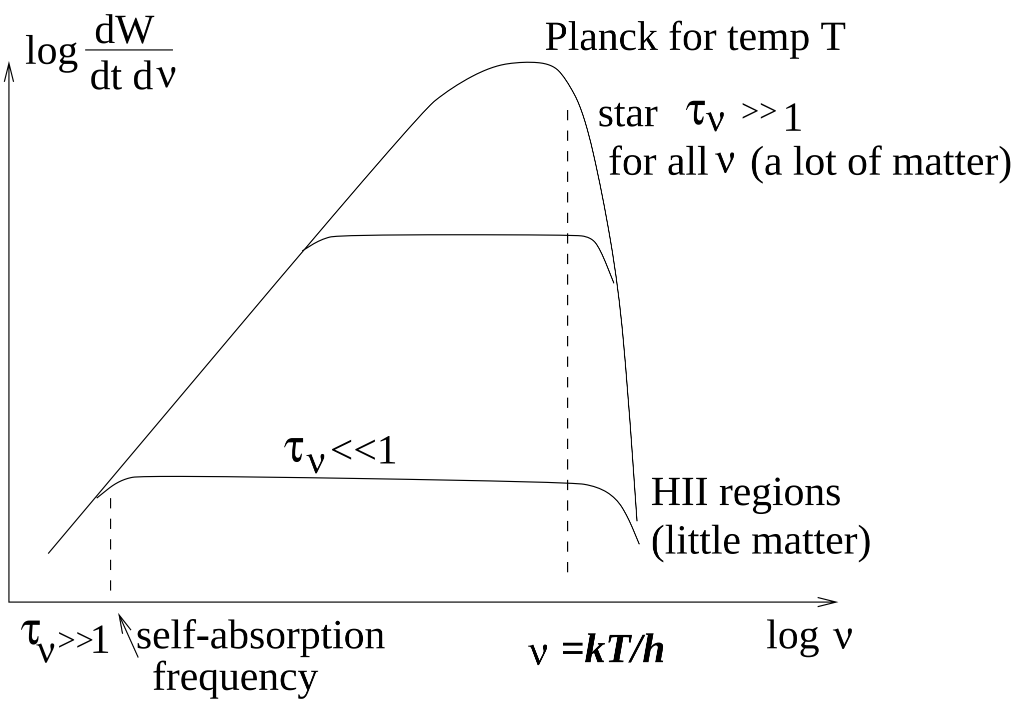
We get

$$\alpha_\nu^{ff} = \frac{2}{3}\left(\frac{2}{3\pi}\right)^{1/2}\left(\frac{m_e c^2}{kT}\right)^{1/2} \alpha r_e^2 n_e \sum_i Z_i^2 n(Z_i) g_{ff}(\nu) h \left(\frac{c}{h\nu}\right)^3 (1 - e^{-h\nu/kT})$$
$$= 3.7\ 10^8 T^{-1/2} n_e \sum_i Z_i^2 n(Z_i) \nu^{-3} g_{ff}(\nu)(1 - e^{-h\nu/kT}) \quad \mathrm{cm}^{-1} \qquad (6.22)$$

At $h\nu \gg kT$, the exponential is negligible and $\alpha_\nu^{ff} \propto \nu^{-3}$. For $h\nu \ll kT$, we get

$$\alpha_\nu^{ff} = 0.018\ T^{-3/2} n_e \nu^{-2} \sum_i Z_i^2 n(Z_i) g_{ff}(\nu). \qquad (6.23)$$

The optical depth of a cosmic gas cloud to free-free absorption $\tau_\nu^{ff} = \alpha_\nu^{ff} R$, where R is the size of the source. Since $\tau_\nu^{ff} \propto \nu^{-2}$ at small ν, the source is always optically thick at sufficiently small frequency. It is optically thin at large frequencies. Let us fill a cloud of a fixed temperature T with more and more material. The evolution of the resulting spectrum is presented below.



In optically thick objects, e.g. stars, the photons have a distribution close to the Planck distribution. When computing stellar structure, one does not consider the absorption at every frequency, but makes an average over (a derivative) of the Planck function, obtaining a Rosseland (from Norway) mean. The result is that $h\nu$ in α_ν is replaced with the typical photon energy kT of the Planck distribution:

$$\alpha_R \propto n_e \sum_i n(Z_i) T^{-1/2} T^{-3}. \tag{6.24}$$

Thus we get Kramer's law:

$$\alpha_R = 1.7 \times 10^{-25} T^{-7/2} Z^2 n_e n_i g_R, \tag{6.25}$$

where g_R is a weighted average of g_ν^{ff} and is of the order unity.

Chapter 7

Synchrotron and cyclotron radiation

7.1 Conservation laws and particle orbit

Radiation produced by a charge moving in the magnetic field is called cyclo-synchrotron or magneto-bremsstahlung radiation. One can consider two limits: non-relativistic motion results in cyclotron radiation with $\omega \sim \omega_B$, while the relativistic particles produce synchrotron radiation at frequencies $\omega \gg \omega_B$. We neglect here (1) radiation reaction (i.e. one neglects radiation losses during one revolution); (2) the effect on the motion by the fields generated by the particle.

Consider a charge q moving in a homogeneous B-field and $\vec{E} = 0$.

Energy conservation $\frac{dE}{dt} = \vec{u} \cdot \vec{F}_{\rm Lorentz}$

$$\frac{d\gamma mc^2}{dt} = \vec{u} \cdot \left(q\vec{E} + \frac{\vec{u}}{c} \times \vec{B} \right) = q\vec{u} \cdot \vec{E} = 0, \tag{7.1}$$

i.e. γ=const and u=const.

Momentum conservation

$$\frac{d\vec{P}}{dt} = \vec{F}_{\rm Lorentz} = q\vec{E} + q\frac{\vec{u}}{c} \times \vec{B} = q\frac{\vec{u}}{c} \times \vec{B}, \tag{7.2}$$

$$\frac{d\gamma m\vec{u}}{dt} = q\frac{\vec{u}}{c} \times \vec{B} \Rightarrow \frac{d\vec{u}}{dt} = \frac{q}{\gamma mc}\vec{u} \times \vec{B}, \tag{7.3}$$

since γ=const.

Let $\vec{B}$ be $\parallel$ to the z-axis. Decompose now $\vec{u}$ in $\vec{u}_z + \vec{u}_{xy}$, parallel and $\perp$ to $\vec{B}$. We get:

$$\frac{d\vec{u}_z}{dt} = 0, \qquad \frac{d\vec{u}_{xy}}{dt} = \frac{q}{\gamma mc}\vec{u} \times \vec{B}. \tag{7.4}$$

We have u=const, u_z=const, therefore u_{xy}=const. The force $\frac{q}{\gamma mc} u_{xy} B$ is then also constant. Rewrite

$$\frac{\mathrm{d}\vec{u}_{xy}}{\mathrm{d}t} = \vec{u} \times (\vec{e}_z \frac{qB}{\gamma mc}) = \vec{u}_{xy} \times (\vec{e}_z \omega_B), \tag{7.5}$$

where $\omega_B \equiv \frac{qB}{(\gamma m)c}$ rad/s is the relativistic (angular) gyrofrequency and the frequency is $\nu = \omega/(2\pi)$ Hz. The electron performs a helical orbit and a circular orbit in $x - y$ plane (Show this!):

$$\vec{r}(t) = \vec{e}_z u_z t + \frac{u_{xy}}{\omega_B}(\vec{e}_x \cos \omega_B t + \vec{e}_y \sin \omega_B t). \tag{7.6}$$

One can consider γm as a relativistic mass with a larger inertia than a non-relativistic particle. It is harder to turn a relativistic particle therefore the gyroradius is larger and the frequency is smaller. Gyroradius is

$$R = \frac{u_{xy}}{\omega_B} = \frac{u_{xy}\gamma mc}{qB} = \frac{u\gamma mc \sin \alpha}{qB}, \tag{7.7}$$

where α is the pitch angle and $\cos \alpha = \vec{u} \cdot \vec{B}/(uB)$.

7.2 Total radiated power

We showed before that

$$P_{\mathrm{emitted}} = \frac{2q^2}{3c^3}\gamma^4(a_\perp^2 + \gamma^2 a_\parallel^2) \;\; \text{erg/s}, \tag{7.8}$$

where $a_\perp$ - acceleration $\perp \vec{u}$ and $a_\parallel$ - acceleration $\|\vec{u}$. For an electron in a B-field, the acceleration is $\perp \vec{u}$:

$$a_\perp = \left|\frac{\mathrm{d}\vec{u}}{\mathrm{d}t}\right| = |\vec{u} \times \vec{e}_z \omega_B| = u\omega_B \sin \alpha, \qquad a_\parallel = 0. \tag{7.9}$$

Therefore

$$\begin{aligned} P_{\mathrm{emitted}} &= \frac{2e^2}{3c^3}\gamma^4 u^2 \omega_B^2 \sin^2 \alpha = \frac{2e^2}{3c^3}\gamma^4 u^2 \frac{e^2 B^2}{\gamma^2 m^2 c^2} \sin^2 \alpha \\ &= \frac{2}{3} r_e^2 c \gamma^2 \beta^2 B^2 \sin^2 \alpha = 2\sigma_T c(\gamma^2\beta^2) U_B \sin^2 \alpha \end{aligned} \tag{7.10}$$

where $r_e = e^2/mc^2$ - classical electron radius, $\sigma_T = \frac{8\pi}{3} r_e^2 = \frac{2}{3} 10^{-24}$ cm^2 - Thomson cross-section and $U_B = B^2/(8\pi)$ is the magnetic energy density. One can consider U_B as the energy density of virtual magnetic field photons of energy $\hbar\omega_B$ with number density $U_B/(\hbar\omega_B)$. The emitted power = cross-section × velocity × energy density.

One can view the process quantum-mechanically as if the electron collides (scatters) with virtual B-field photons and ”knocks” them free, this produces radiation.

If the electron velocity distribution is isotropic then one can average over the pitch angle ($\int \sin^2\alpha \frac{d\Omega}{4\pi} = \frac{2}{3}$):

$$P_{\rm emitted} = \frac{4}{3}\sigma_T c \beta^2 \gamma^2 U_B. \tag{7.11}$$

This formula is valid for any velocity β.

7.3 Cooling time or radiative lifetime

Consider how the electron loses energy. The energy equation becomes:

$$mc^2 \frac{d\gamma}{dt} = -P_{\rm emitted} = -2\sigma_T c(\gamma^2\beta^2) U_B \sin^2\alpha. \tag{7.12}$$

One can solve this ODE. (At home: assume $\beta = 1$ and solve this equation!)

The typical timescale for the electron to lose about half of its energy (i.e. cooling time) is approximately

$$t_{\rm cool} = \frac{\rm Energy}{\rm cooling\ rate} = \frac{\gamma mc^2}{-mc^2 \frac{d\gamma}{dt}} = \frac{\gamma mc^2}{P_{\rm emitted}} = \frac{4\pi mc^2}{\sigma_T c} \frac{1}{\gamma B^2 \sin^2\alpha} = \frac{15 \text{ years}}{\gamma B^2 \sin^2\alpha}, \tag{7.13}$$

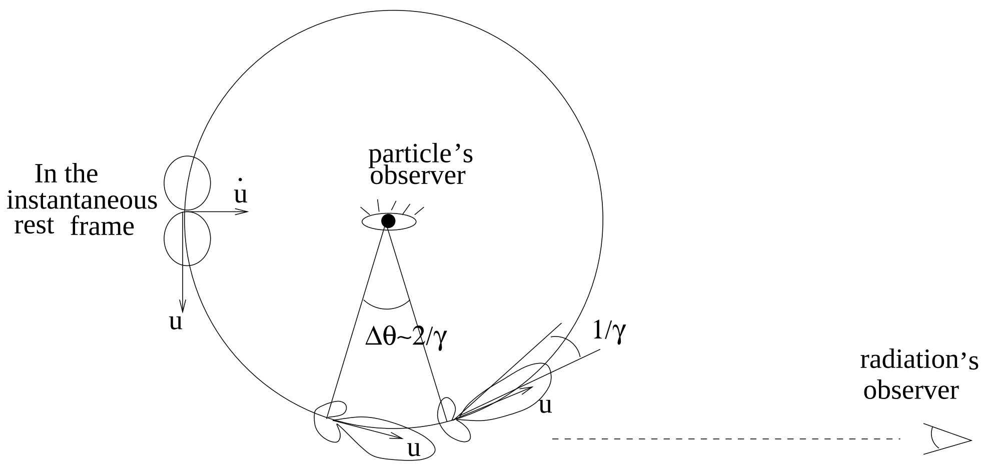
thus for $\gamma = 10^3$ this results in the following cooling times:

Location	Typical B	$t_{\rm cool}$	cooling length $\approx ct_{\rm cool}$	size of object
Interstellar medium	10^{-6} G	10^{10} years	10^{28} cm	10^{22} cm
Stellar atmosphere	1 G	5 days	10^{15} cm	10^{11} cm
Supermassive black hole	10^4 G	10^{-3} sec	$3\ 10^7$ cm	10^{14} cm
White dwarf	10^8 G	10^{-11} sec	3 mm	1000 km
Neutron star	10^{12} G	10^{-19} sec	$3\ 10^{-9}$ cm	10 km

In strong B-fields, the electron loses its energy before it can cross the source.

7.4 Spectrum emitted by a relativistic charge

Which frequency spectrum does the observer measure? This depends largerly on the width of the pulses measured by the observer. The narrower the pulse is, the broader is the frequency spectrum. Three relativistic effects determine the width.



The radiation's observer sees a pulse during a fraction $\frac{\Delta\theta}{2\pi} \sim \frac{1}{2\pi\gamma}$ of the orbital period $\frac{2\pi}{\omega_B}$. The pulse returns at intervals $\frac{2\pi}{\omega_B}$. Furthermore, the time interval Δt for length of pulse at radiation's observer is shorter that the time interval $\Delta t_{\rm ret} \approx \frac{2\pi}{\omega_B}\frac{\Delta\theta}{2\pi}$ measured by the particle's observer.

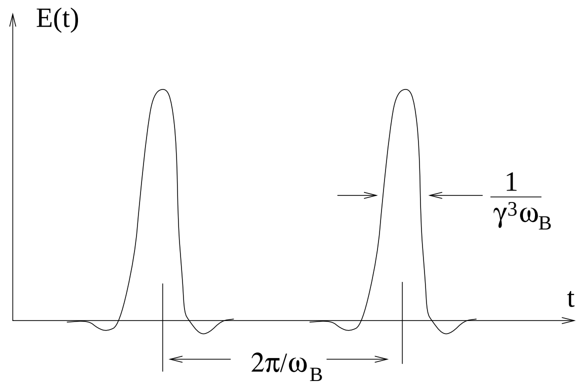
We have shown that

$$\frac{\Delta t}{\Delta t_{\rm ret}} = 1 - \beta\cos\theta \approx \frac{1+\gamma^2(\Delta\theta)^2}{2\gamma^2} \sim \frac{1}{\gamma^2}. \tag{7.14}$$

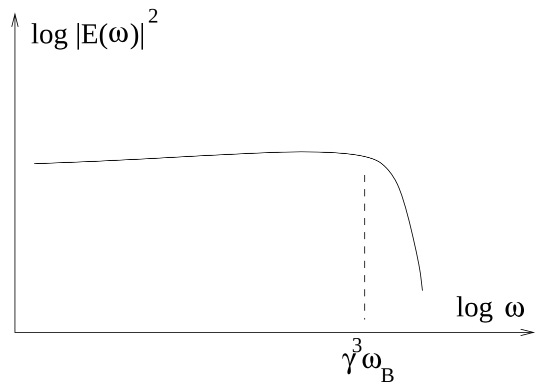
The pulse length at the radiation's observer becomes

$$\Delta t \approx \frac{1}{\gamma^2}\Delta t_{\rm ret} \approx \frac{1}{\gamma^2}\frac{\Delta\theta}{\omega_B} = \frac{1}{\gamma^3\omega_B}. \tag{7.15}$$

The radiation's observer sees



The frequency spectrum then extends up to very high harmonics $\omega \sim \gamma^3\omega_B$:



A typical frequency $\gamma^3\omega_B = \gamma^2\gamma\frac{eB}{\gamma mc} = \gamma^2\omega_L$, where $\omega_L \equiv \frac{eB}{mc}$ is Larmor frequency. Here we have three effects: γ^2 comes from Doppler factor, the next γ comes from the beaming of radiation, and the last γ in the denominator comes from the increase of the gyroradius (relativistic mass). Two effects cancel each other: (a) gyroradius increases with γ ($\omega_B \propto 1/\gamma$); (b) the beam becomes narrower with γ. The only remaining effect is time compression due to Doppler factor $\propto 1/\gamma^2$.

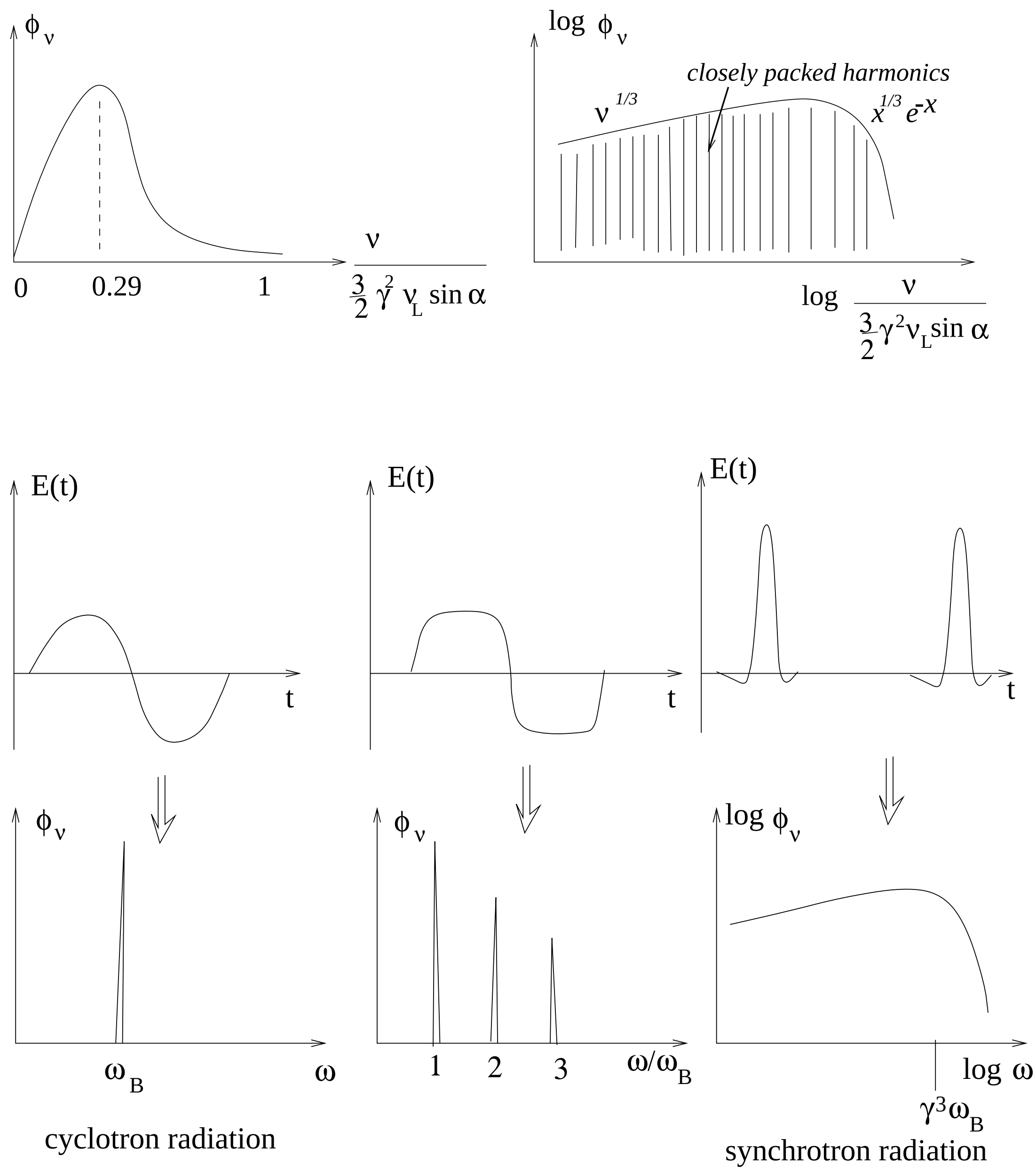
The emitted spectrum can be represented as

$$P_\nu(\gamma) = P_{\rm emitted}\ \phi_\nu(\gamma) \quad [\text{erg/s/Hz}],$$

where ϕ is normalized frequency distribution: $\int \phi_\nu(\gamma)d\nu = 1$. Detailed calculations give that

$$\phi_\nu(\gamma) = \frac{9\sqrt{3}}{8\pi}F\left(\frac{\nu}{\frac{3}{2}\gamma^2\nu_L\sin\alpha}\right)\frac{1}{\frac{3}{2}\gamma^2\nu_L\sin\alpha} \quad [\text{Hz}^{-1}], \tag{7.16}$$

where $F(x) = x\int_x^\infty K_{5/3}(y)dy \approx 1.8x^{1/3}e^{-x}$. Here $K_{5/3}$ is the modified Bessel function. The spectrum consists of closely packed harmonics. See Rybicki & Lightman ch 6.6.



7.5 Spectrum from a power-law electron distribution

Cosmic rays (protons and ions) that hit Earth have a power-law energy distribution. It is reasonable to expect that relativistic electrons also have power-law distributions due to acceleration processes in the Universe. A power-law distribution

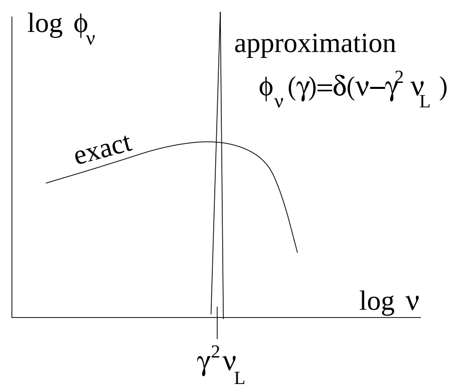
$$n(\gamma)\mathrm{d}\gamma = n_0\gamma^{-p}\mathrm{d}\gamma, \quad \gamma_{\min} < \gamma < \gamma_{\max}$$

has typically $p = 2 - 3$.

The spectrum from an electron distribution with $n(\gamma)\mathrm{d}\gamma$ cm^{-3} electrons between $\gamma m_e c^2$ and $(\gamma + \mathrm{d}\gamma)m_e c^2$ is given by

$$4\pi\ j_\nu = \int_1^\infty P_\nu(\gamma)n(\gamma)\ \mathrm{d}\gamma \quad \mathrm{erg/s/Hz/cm^3}. \tag{7.17}$$

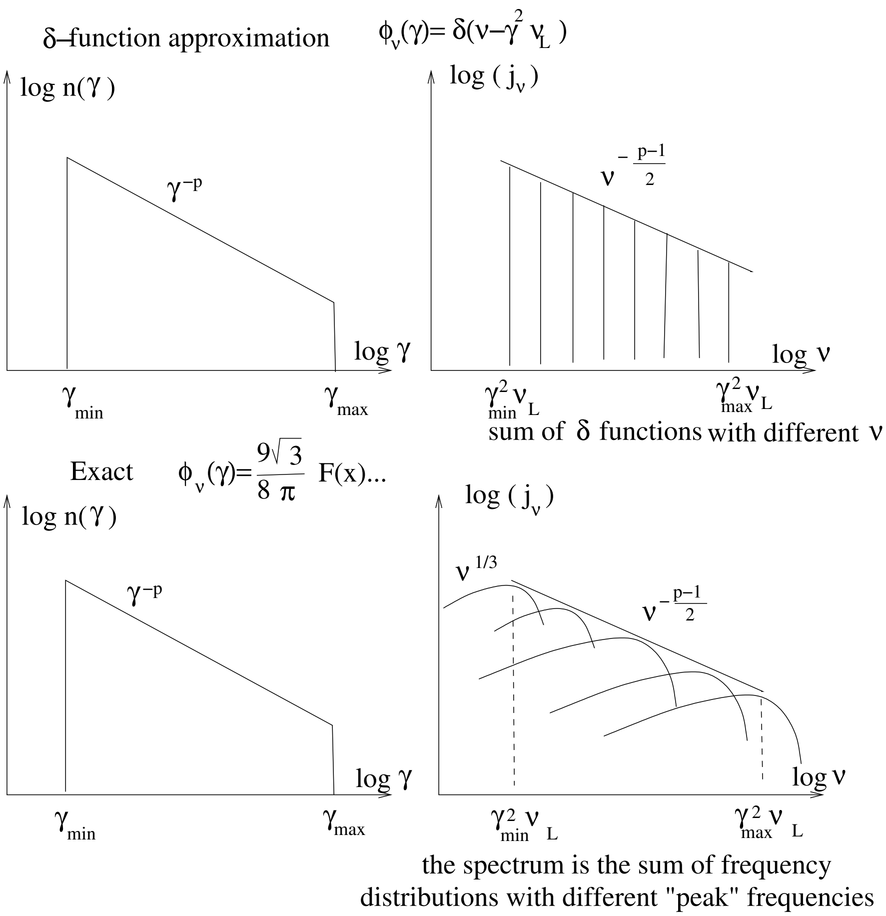
It is hard to do analytical integration using exact $P_\nu(\gamma)$. It is simpler to approximate $\phi_\nu(\gamma)$, e.g. by assuming that all emission occurs at $\nu \approx \gamma^2\nu_L$, i.e. $\phi_\nu(\gamma) = \delta(\nu - \gamma^2\nu_L)$. One can do this if $n(\gamma)$ is a broad distribution.

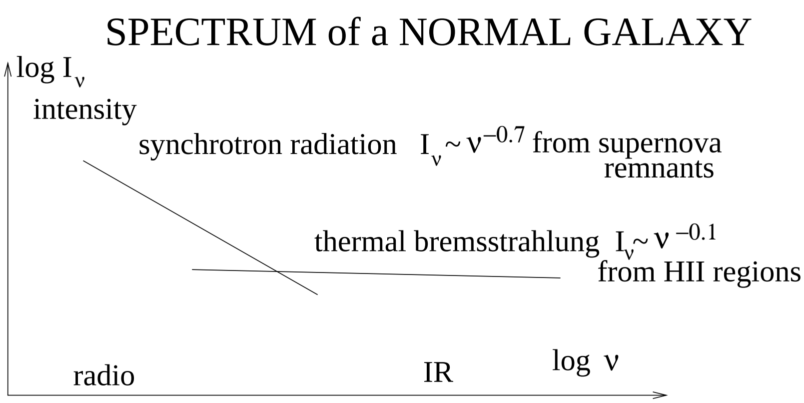


Then

$$\begin{aligned} 4\pi\ j_\nu &= \frac{4}{3}c\sigma_T U_B n_0 \int_{\gamma_{\min}}^{\gamma_{\max}} \mathrm{d}\gamma\ \gamma^{-p}\beta^2\gamma^2\delta(\nu - \gamma^2\nu_L) = \frac{4}{3}c\sigma_T U_B n_0 \left.\frac{(\gamma^{2-p})}{|\frac{\mathrm{d}}{\mathrm{d}\gamma}(\nu - \gamma^2\nu_L)|}\right|_{\gamma=\sqrt{\nu/\nu_L}} \\ &= \frac{2}{3}c\sigma_T U_B n_0 \frac{1}{\nu_L}(\gamma^{1-p})_{\gamma=\sqrt{\nu/\nu_L}} = \frac{2}{3}c\sigma_T U_B \frac{n_0}{\nu_L}\left(\frac{\nu}{\nu_L}\right)^{-\frac{p-1}{2}}, \end{aligned} \tag{7.18}$$

where $\gamma_{\min}^2 \nu_L < \nu < \gamma_{\max}^2 \nu_L$. The exponent $\frac{p-1}{2}$ is called the spectral index. From observed spectral indices one can determine the slope p of the electron power-law distribution.





7.6 Synchrotron self-absorption

Solution of the radiative transfer equation for a slab (the total observed intensity) is

$$I_\nu = I_\nu(0)e^{-\tau_\nu} + S_\nu(1 - e^{-\tau_\nu}) \tag{7.19}$$

We assume no background radiation $I_\nu(0) = 0$. For an optically thick / opaque source ($\tau_\nu \gg 1$), the intensity

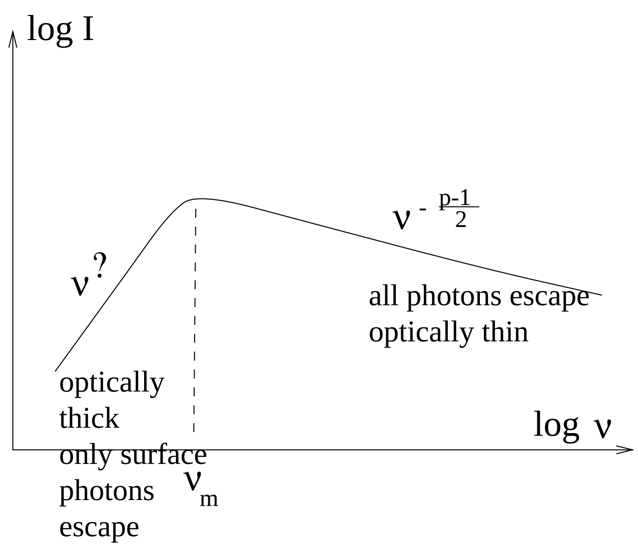
$$I_\nu = S_\nu = \frac{j_\nu}{\alpha_\nu} = \begin{cases} B_\nu(T), & \text{thermal gas, i.e. Maxwellian distribution} \\ \frac{j_\nu}{\alpha_\nu}, & \text{non-thermal gas, e.g. power-law electrons} \end{cases} \tag{7.20}$$

The thermal case (i.e. thermal bremsstrahlung). Planck function

$$B_\nu(T) = 2\left(\frac{\nu}{c}\right)^2 \frac{h\nu}{e^{h\nu/kT} - 1} \approx 2\left(\frac{\nu}{c}\right)^2 (kT). \tag{7.21}$$

Where $h\nu \ll kT$. Here 2 is the number of spins, $(\nu/c)^2$ is the phase space factor, and kT would be the typical energy of the electron doing the absorption.

In the nonthermal case



The typical electron energy is γmc^2 of the electron emitting at ν is determined from $\nu = \gamma^2\nu_L$, i.e. $\gamma mc^2 = (\nu/\nu_L)^{1/2}mc^2$. It is natural (and is indeed the case) that the electron that emits at ν also absorbs at ν, i.e. electron with energy γmc^2 absorbs mainly at $\nu = \gamma^2 \nu_L$. Then

$$\begin{aligned} I_\nu &= \frac{j_\nu}{\alpha_\nu} \approx 2\left(\frac{\nu}{c}\right)^2 \text{ [typical electron energy that absorbs at } \nu] \\ &= 2\left(\frac{\nu}{c}\right)^2 \left(\frac{\nu}{\nu_L}\right)^{1/2} mc^2, \end{aligned} \tag{7.22}$$

i.e.

$$I_\nu = 2m \frac{\nu^{5/2}}{\nu_L^{1/2}}. \tag{7.23}$$

Thus $I_\nu \propto \nu^{5/2}$ for $\tau_\nu \gg 1$.

7.7 Compact radio sources

In these sources, self-absorption frequency ν_m (i.e. frequency where $\tau_\nu = 1$) occurs in the radio or far IR. One can then observe ν_m and F_{ν_m}. We have earlier shown that for $\tau_\nu \gg 1$,

$$I_\nu = S_\nu \propto \frac{\nu^{5/2}}{\nu_L^{1/2}} \propto \frac{\nu^{5/2}}{B^{1/2}}. \tag{7.24}$$

The flux from the source at frequencies where $\tau_\nu \gg 1$ becomes

$$F_\nu = \pi S_\nu \theta_s^2 \propto \frac{\nu^{5/2}\theta_s^2}{B^{1/2}}. \tag{7.25}$$

This is also approximately valid at ν_m, where it is easy to measure. If one observes $\theta_s, F_{\nu_m}, \nu_m$, one can determine B. Typical values $B = 10^{-1} - 10^{-4}$ Gauss.

In compact radio sources the brightness temperature is typically $T_b \leq 10^{12}$ K. Which electrons emit at ν_m? Well, those with energy $\gamma m_e c^2 \approx kT_b$, i.e. $\gamma \approx \frac{kT_b}{m_e c^2} \sim \frac{10^{12}}{5\ 10^9} \approx 200$.

Chapter 8

Compton scattering

Compton scattering is scattering of photons against free (or if photon energies are large enough, against bound) electrons. Both photons and electrons change energies (incoherent scattering). In some cases the energy change is negligible (coherent or elastic scattering, also called Thomson scattering). The energy change is, however, never equal to zero.

8.1 Thomson scattering

8.1.1 Cross-section

We have already discussed as a special case of dipole radiation. We repeat it here. In this limit one can discard the photon picture and consider plane waves. Consider incoming plane against a free electron (or, bound electron if $\omega \gg \omega_0$, where ω_0 is the oscillating frequency of the bound system).

Incoming linear polarized electric field: $\vec{E}(t) = \vec{E}_0 e^{i(\vec{k}\cdot\vec{x}-\omega t)}$. The equation of motion for non-relativistic charge at $\vec{x} = 0$

$$m_e \ddot{\vec{x}} = \vec{F} = -e\vec{E} = -e\vec{E}_0 e^{-i\omega t}. \tag{8.1}$$

The charge is oscillating in the $\vec{E}$-field. Larmor formula gives

$$\begin{aligned} \frac{\mathrm{d}P}{\mathrm{d}\Omega} &= \frac{1}{4\pi c^3}|\ddot{\vec{d}} \times \vec{k}|^2 = \frac{|\ddot{\vec{d}}|^2}{4\pi c^3}\sin^2\Theta = \frac{e^2}{4\pi c^3}|\ddot{\vec{x}}|^2 \sin^2\Theta \\ &= \frac{e^4 E_0^2}{4\pi m_e^2 c^3}\sin^2\Theta\cos^2\omega t \quad [\mathrm{erg\ s^{-1}\ sr^{-1}}]. \end{aligned} \tag{8.2}$$

The time-average becomes

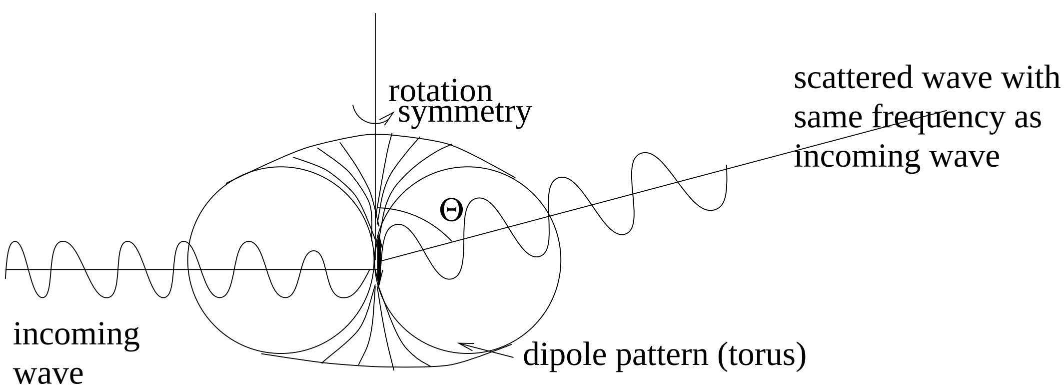
$$\langle\frac{\mathrm{d}P}{\mathrm{d}\Omega}\rangle = r_e^2\left[\frac{cE_0^2}{8\pi}\right]\sin^2\Theta. \qquad (8.3)$$

The expression in [ ] is the mean Poyting flux $\langle\vec{S}\rangle = \frac{c}{4\pi}\langle\vec{E}\times\vec{B}\rangle$.

Define the differential cross-section for linearly polarized radiation, that scatter into solid angle $d\Omega$ as

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{\langle \mathrm{d}P/\mathrm{d}\Omega\rangle}{\langle\vec{S}\rangle} = r_e^2\sin^2\Theta \quad \mathrm{cm}^2\mathrm{sr}^{-1}. \qquad (8.4)$$

Dipole pattern



The total cross-section

$$\sigma = \int \mathrm{d}\Omega\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = 2\pi r_e^2\int\sin^2\Theta\,\mathrm{d}\cos\Theta = \frac{8\pi r_e^2}{3} \equiv \sigma_T \approx \frac{2}{3}10^{-24} \quad \mathrm{cm}^2, \qquad (8.5)$$

which is called the Thomson cross-section.

Properties

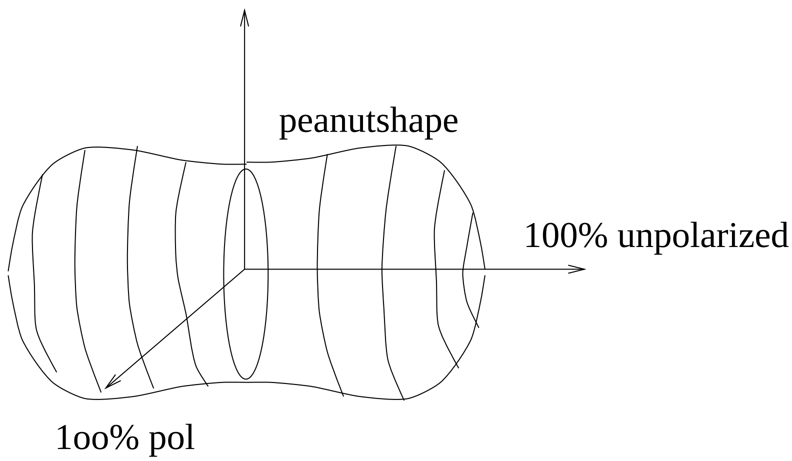
1) Frequency independent.
2) A classical cross-section, no energy exchange.
3) Valid for non-relativistic e^- with kinetic energy $\ll m_ec^2$ and photons with $h\nu \ll m_ec^2 \approx 511$ keV.

8.1.2 Thomson scattering of unpolarized radiation

Unpolarized radiation can be decomposed into 2 independent linearly polarized waves. One along $\vec{\epsilon}_1$ that scatters with angle Θ relative $\vec{\epsilon}_1$, and one along $\vec{\epsilon}_2$ that scatters with angle $\pi/2$ relative $\vec{\epsilon}_2$. The scattering angle $\theta = \pi/2 - \Theta$. The total differential cross-section becomes

$$\begin{aligned}\left(\frac{d\sigma}{d\Omega}\right)_{\rm unpol} &= \left[\frac{1}{2}\left(\frac{d\sigma}{d\Omega}\right)_{\rm pol}(\pi/2) + \frac{1}{2}\left(\frac{d\sigma}{d\Omega}\right)_{\rm pol}(\Theta)\right] \\ &= \frac{1}{2}r_e^2[1+\sin^2\Theta] = \frac{r_e^2}{2}(1+\cos^2\theta). \qquad (8.6)\end{aligned}$$

Radiation pattern (scattering pattern) is, in principle, the dipole pattern averaged over different directions. The zero along $\ddot{\vec{d}}$ causes the "waist" of the peanut.



Properties:
1) $\sigma_{\rm unpol} = \int d\Omega (\frac{d\sigma}{d\Omega})_{\rm unpol} = \frac{8\pi r_e^2}{3} = \sigma_T$,
2) axially and forward-back symmetric (peanut),
3) radiation becomes polarized. The intensity of the incident unpolarized radiation can be represented as a sum of two equal linearly polarized components $I = I_\perp + I_\parallel$, where $I_\perp = I_\parallel$ are the intensities of radiation polarized perpendicular and parallel to the scattering plane, respectively. The intensity of scattered radiation is then a sum of $I'_\perp + I'_\parallel$, where $I'_\perp \propto I_\perp \frac{d\sigma}{d\Omega}(\pi/2) \propto I_\perp r_e^2$ and $I'_\parallel \propto I_\parallel \frac{d\sigma}{d\Omega}(\Theta) \propto I_\parallel r_e^2 \cos^2\theta$. The polarization degree is

$$\Pi = \frac{Q'}{I'} = \frac{I'_\perp - I'_\parallel}{I'_\perp + I'_\parallel} = \frac{1-\cos^2\theta}{1+\cos^2\theta}. \qquad (8.7)$$

8.2 Compton effect

Quantum nature of Compton scattering appears in two ways: through the kinematics of the scattering process and through the change of the cross-section. Consider a scattering of a photon on an electron at rest. The initial photon has four-momentum $\underline{k} = h\nu/c(1, \vec{n})$ and the initial electron $\underline{p} = (m_e c, 0)$. After scattering they are $\underline{k}' = h\nu'/c(1, \vec{n}')$ and $\underline{p}' = (E'/c, \vec{p}')$. Here $\vec{n}$ and $\vec{n}'$ are the unit vectors in the direction of photon momentum, with $\vec{n} \cdot \vec{n}' = \cos\Theta$ and Θ being the scattering angle.

The conservation of four-momentum can be written as

$$\underline{k} + \underline{p} = \underline{k}' + \underline{p}'. \tag{8.8}$$

Expressing $\underline{p}' = \underline{k} + \underline{p} - \underline{k}'$ and computing the scalar product of the four-vector $\underline{p}'^2$, we get

$$\underline{p}'^2 = (m_e c)^2 = \underline{p}^2 + 2m_e c\frac{h\nu}{c} - 2m_e c\frac{h\nu'}{c} - 2\frac{h\nu}{c}\frac{h\nu'}{c}(1 - \vec{n} \cdot \vec{n}') \tag{8.9}$$

(here we used the fact that $\underline{k}^2 = \underline{k}'^2 = 0$). Thus we get

$$h\nu' = \frac{h\nu}{1 + \frac{h\nu}{m_e c^2}(1 - \cos\Theta)}. \tag{8.10}$$

We see that the scattering is no longer elastic, because of the recoil of the electron. In terms of wavelengths this can be rewritten as

$$\lambda' = \lambda + \lambda_c(1 - \cos\Theta), \tag{8.11}$$

where

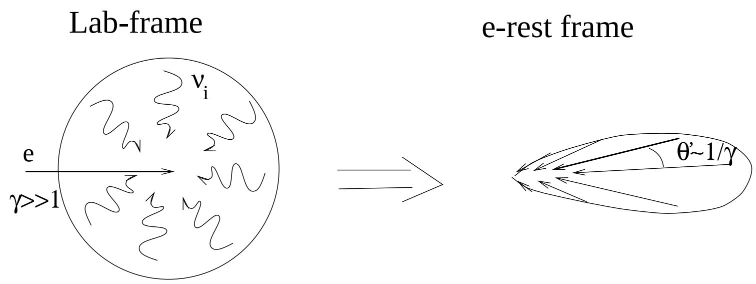
$$\lambda_c = \frac{h}{mc} \tag{8.12}$$

is the Compton wavelength.

8.3 Scattering by a relativistic electron

Consider now scattering of photons of frequency ν_i distributed isotropically by an electron moving with Lorentz factor $\gamma \gg 1$ through that photon field.

Before scattering



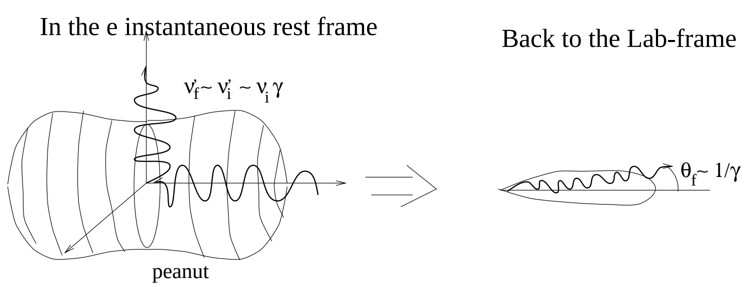
In the laboratory frame, photons are isotropic. Let us consider a photon with momentum making angle θ_i with the direction of electron. The frequency of that photon in the electron rest frame (marked by prime) is given by the Doppler formula:

$$\nu'_i = \nu_i \gamma (1 - \beta \cos\theta_i) \tag{8.13}$$

Let us consider such Lorentz factors and photon incident energies that the photon energy in the electron rest frame is still not very large, i.e. $h\nu'_i \sim h\nu_i\gamma \ll m_e c^2$. In that case, the scattering in the instantaneous rest frame, where the electron is at rest, can be considered as elastic (Thomson) and the frequency of the final scattered photon does not change upon scattering. So that

$$\nu'_f = \nu'_i. \tag{8.14}$$

After scattering



The frequency of the scattered photon in the laboratory frame is

$$\nu_f = \nu'_f \gamma (1 + \beta \cos\theta'_f), \tag{8.15}$$

where θ'_f is the angle the photon momentum makes to the electron propagation direction in the electron frame. Thus

$$\nu_f = \gamma^2 \nu_i (1 - \beta\cos\theta_i)(1 + \beta\cos\theta'_f) = \nu_i \frac{1 - \beta\cos\theta_i}{1 - \beta\cos\theta_f}, \tag{8.16}$$

where θ_f is the angle between photon and electron momenta in the lab frame.

For a typical incident angle $\theta_i \approx \pi/2$, we get $\nu'_i \approx \nu_i\gamma$. In the electron frame, the incoming photon moves at very small angle in the direction almost opposite to the direction of motion of the electron. The photons are scattered in all direction (according to ”peanut” pattern), i.e. typically at $\theta'_f \approx \pi/2$, so we have $\nu_f \approx \nu'_f\gamma \approx \nu'_i\gamma \approx \nu_i\gamma^2$.

8.4 Energy loss by Compton scattering

Consider an electron of energy $\gamma m_e c^2$ in an isotropic radiation field of energy density $U_{\rm rad}$ [erg cm^{-3}]. For simplicity assume that radiation consists of the photons of frequency ν_i. The energy density is proportional to the integral of the intensity over solid angles

$$U_{\rm rad} = \frac{1}{c}\int I\, d\Omega. \tag{8.17}$$

Let us define the number of photons per solid angle passing through unit area in unit time $dn/d\Omega = I/(h\nu_i)$, then for isotropic radiation

$$\frac{dn}{d\Omega} = \frac{c}{4\pi}\frac{U_{\rm rad}}{h\nu_i}. \tag{8.18}$$

The number of interactions of the electron with the photons per unit time is

$$\frac{dN}{dt} = \sigma_T \int (1-\beta\cos\theta_i)\,\frac{dn}{d\Omega}\,d\Omega. \tag{8.19}$$

The factor $1-\beta\cos\theta_i$ is related to the Doppler factor (the ratio $dt/dt_{\rm arr}=1-\beta\cos\theta$), which accounts for the difference in the intervals between the emission of the photons and their arrival to the moving electron. Each interaction produces more energetic photons of energy given by equation (8.16). Thus the emitted power is

$$\begin{aligned} P_{\rm Compton} &= \sigma_T\langle\int (1-\beta\cos\theta_i)(h\nu_f - h\nu_i)\,\frac{dn}{d\Omega}\,d\Omega\rangle \qquad (8.20)\\ &= c\sigma_T U_{\rm rad}\langle\frac{1}{4\pi}\int [\gamma^2(1-\beta\cos\theta_i)^2(1+\beta\cos\theta'_f) - (1-\beta\cos\theta_i)]\,d\Omega\rangle, \end{aligned}$$

where the angular brackets means averaging over directions in the electron rest frame. Because in the electron frame we assume the scattering to be elastic and it is forward-back symmetric, the term with θ'_f disappears after averaging. Computing the remaining integral we get

$$P_{\rm Compton} = c\sigma_T U_{\rm rad}[\gamma^2(1+\beta^2/3) - 1] = \frac{4}{3}c\sigma_T U_{\rm rad}\gamma^2\beta^2 \quad \text{erg s}^{-1}. \tag{8.21}$$

Compare this expression to the synchrotron power emitted by isotropic electrons moving in the magnetic field (consisting of virtual photons) with energy density U_B:

$$P_{\rm synchro} = \frac{4}{3}\beta^2\gamma^2 c\sigma_T U_B \quad \text{erg s}^{-1}. \tag{8.22}$$

The expressions are fully identical, although the processes seem so apparently different. The γ^2 factor in $P_{\rm synchro}$ comes from the observed photons having the frequency $\nu \approx \gamma^2\nu_L$. Similarly, for Compton scattering, the photons with initial energy ν_i are scattered to $\nu \approx \gamma^2\nu_i$.

The relative contribution of these processes in electron cooling can be estimated from the ratio

$$\frac{P_{\rm Compton}}{P_{\rm synchro}} = \frac{\frac{4}{3}\beta^2\gamma^2 c\sigma_T U_{\rm rad}}{\frac{4}{3}\beta^2\gamma^2 c\sigma_T U_B} = \frac{U_{\rm rad}}{U_B}. \tag{8.23}$$

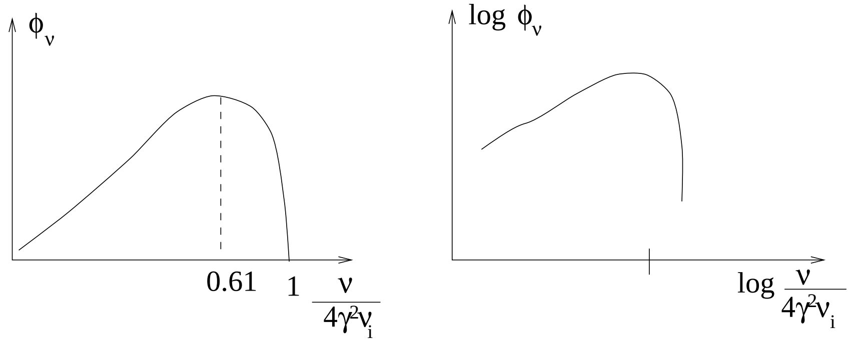
It is determined only by the ratio of the energy densities of radiation and magnetic field. By $U_{\rm rad}$ one should understand the energy density of the radiation field that can interact with the electron in the Thomson regime, i.e. $\gamma h\nu_i/m_e c^2 < 1$.

8.5 Spectrum from a single relativistic charge

In analogy with synchrotron radiation we write

$$P_\nu(\gamma) = P_{\rm Compton}\phi_\nu(\gamma), \tag{8.24}$$

where ϕ_ν is the normalized frequency distribution. It is not exactly the same as for synchrotron radiation but has similar properties. There are no photons with larger ν than $4\gamma^2\nu_i$ due to momentum and energy conservation. Most power emitted at $\nu \approx \nu_i\gamma^2$. The low-energy slope at $\nu \ll \nu_i\gamma^2$ is $P_\nu \propto \nu^1$.



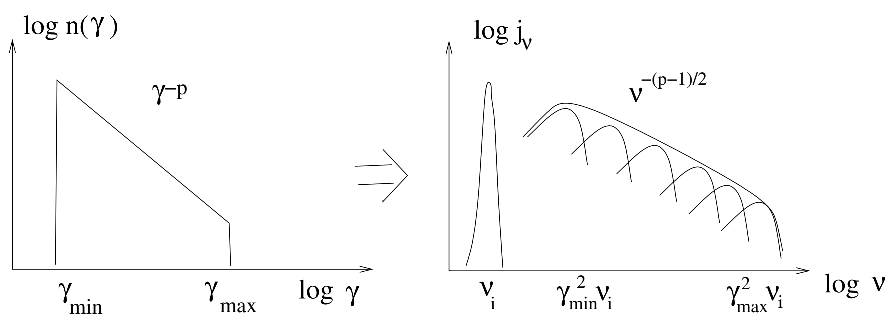
8.6 Spectrum from a power-law distribution

To estimate the emission coefficient for the power-law distribution, $n(\gamma)\mathrm{d}\gamma = n_0\gamma^{-p}\mathrm{d}\gamma$, of relativistic electrons, we can use the δ-function approximation similarly to the synchrotron case:

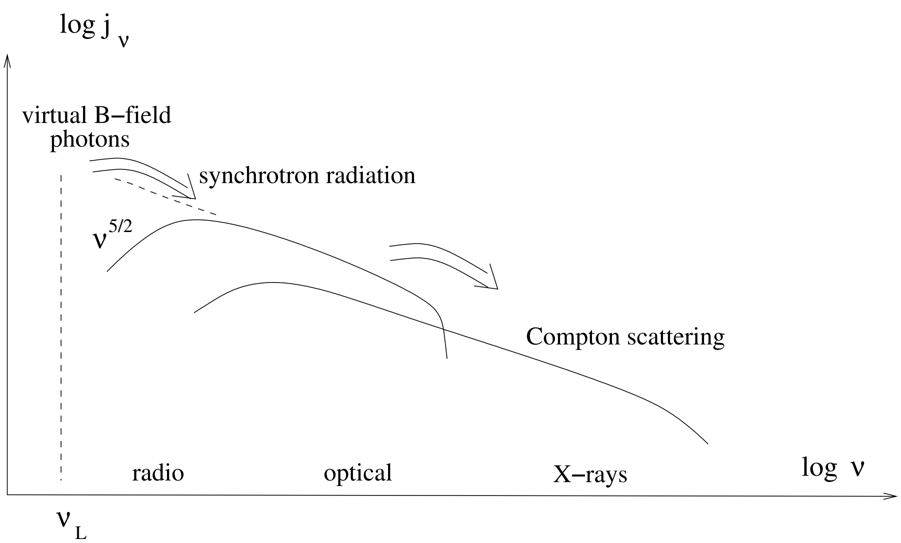
$$\phi_\nu(\gamma) \approx \delta(\nu - \gamma^2\nu_i). \tag{8.25}$$

A power-law electron distribution, scatters photons of frequency ν_i into a power-law:

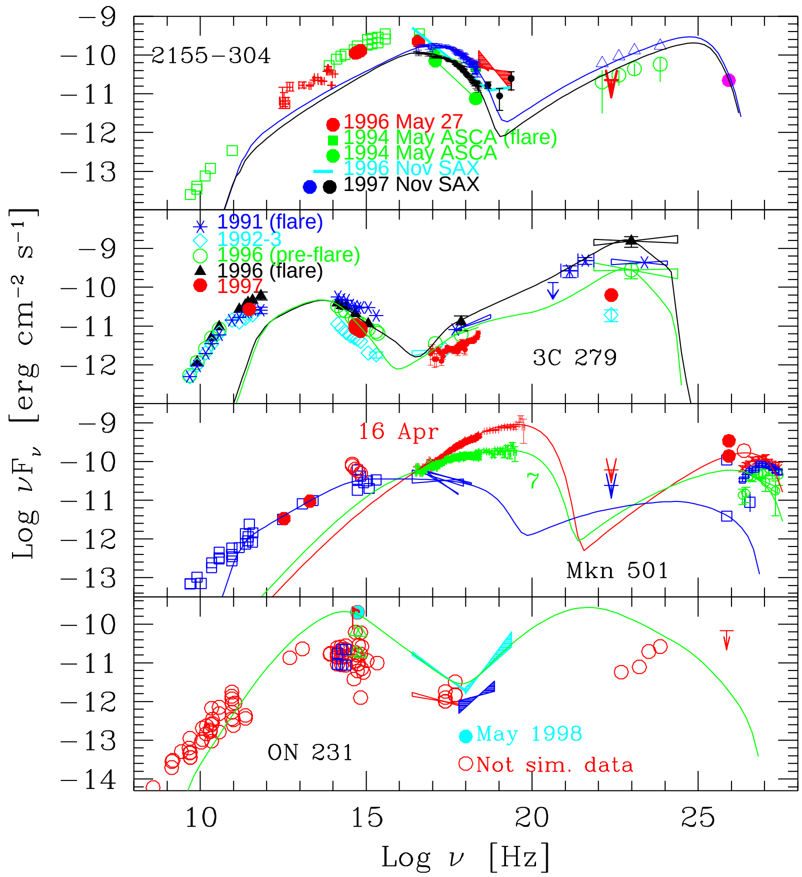
$$4\pi j_\nu = \frac{2}{3}c\sigma_T U_{\rm rad}\frac{n_0}{\nu_i}\left(\frac{\nu}{\nu_i}\right)^{-(p-1)/2}. \tag{8.26}$$



In radio-sources, the electrons that radiate synchrotron radiation in the radio to X-ray range, can scatter these photons up to gamma-ray energies. This mechanism is called Synchrotron self-Compton (SSC). The spectra of jets from blazars are believed to be produced this way (see pictures below).



log j ν
virtual B−field photons
synchrotron radiation
ν^5/2
Compton scattering
radio
optical
X−rays
log ν
ν L



2155−304
1996 May 27
1994 May ASCA (flare)
1994 May ASCA
1996 Nov SAX
1997 Nov SAX
1991 (flare)
1992-3
1996 (pre-flare)
1996 (flare)
1997
3C 279
16 Apr
7
Mkn 501
ON 231
May 1998
Not sim. data
Log νFν [erg cm−2 s−1]
Log ν [Hz]

8.7 Non-relativistic Compton scattering

When electrons are non-relativistic, i.e. when $\beta = u/c \ll 1$ or $kT \sim \langle m_e u^2/2 \rangle \ll m_e c^2$, then the energy exchange in a single scattering is very small.

This small energy exchange will be considered now in some detail. Consider the case when the electrons have more energy than the photons, $kT \gtrsim \epsilon_i$. Then the electrons lose the energy to photons. The energy loss per unit time for a nonrelativistic ($\beta \ll 1, \gamma \approx 1$) electron becomes:

$$\langle P_{\rm Compton} \rangle = \frac{4}{3}\beta^2 c\sigma_T U_{\rm rad} = \frac{4}{3}\beta^2 c\sigma_T n_{\rm photon}\epsilon_i \quad {\rm erg\ s^{-1}}, \tag{8.27}$$

where $\epsilon_i = h\nu_i/m_e c^2$ is the dimensionless photon energy, $n_{\rm photon}$ is the photon number density [cm^{-3}]. The number of collisions that the electron suffers per unit time is

$$\frac{{\rm d}N}{{\rm d}t} = c\sigma_T n_{\rm photon} \quad {\rm s^{-1}}. \tag{8.28}$$

The mean energy loss per collision, for the electron, i.e. the mean energy gain, $\langle \Delta\epsilon \rangle$, for the photon, becomes

$$\langle \Delta\epsilon \rangle = \frac{\langle P_{Compton} \rangle}{\frac{{\rm d}N}{{\rm d}t}} = \frac{4}{3}\beta^2\epsilon_i. \tag{8.29}$$

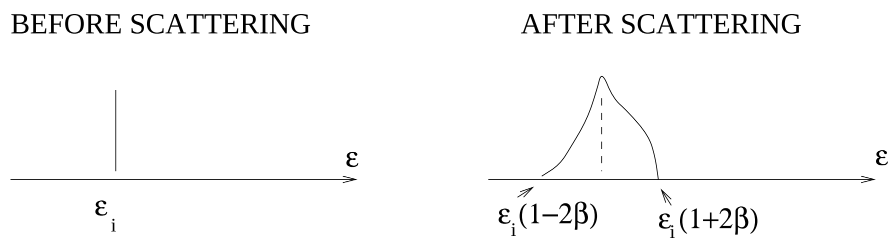
Consider two extreme cases:

(a) Before the collision electron and photon moving towards each other ($\theta_i = \pi$) and after the collision the photon is moving in the same direction as the electron ($\theta_f = 0$). The head-on collision gives maximal energy increase for back-scattered photons:

$$\epsilon_i \to \epsilon_f = \epsilon_i \left(\frac{1 - \beta\cos\theta_i}{1 - \beta\cos\theta_f} \right) = \epsilon_i \left(\frac{1+\beta}{1-\beta} \right) \approx \epsilon_i(1 + 2\beta). \tag{8.30}$$

(b) Before the collision electron and photons are moving in the same direction ($\theta_i = 0$), while after the collision in exactly opposite directions ($\theta_f = \pi$). The tail-on collision gives maximal energy decrease for back-scattered photons:

$$\epsilon_i \to \epsilon_f = \epsilon_i \left(\frac{1-\beta}{1+\beta} \right) \approx \epsilon_i(1 - 2\beta). \tag{8.31}$$



The small asymmetry (of order $O(\beta^2)$) gives mean increase

$$\langle \frac{\Delta\epsilon}{\epsilon_i} \rangle = \langle \frac{\epsilon_f - \epsilon_i}{\epsilon_i} \rangle = \frac{4}{3}\beta^2. \tag{8.32}$$

The mean over a Maxwell-Boltzmann distribution becomes

$$\langle \frac{\Delta\epsilon}{\epsilon_i} \rangle = 4\frac{kT}{m_e c^2} = 4\frac{T}{5 \times 10^9 \mathrm{K}}. \tag{8.33}$$

8.8 Comptonization

By Comptonization we mean multiple scattering of low energy photons by hot electron gas. A single scattering gives a small change in photon energy. However, many scatterings may give a noticable change. The total relative change after N scatterings becomes

$$\langle \frac{\Delta\epsilon}{\epsilon_i} \rangle \Big|_{\mathrm{tot}} = \langle \frac{\Delta\epsilon}{\epsilon_i} \rangle \Big|_{\mathrm{single}} \times N = 4\frac{kT}{m_e c^2} \times N. \tag{8.34}$$

The scattering optical depth, $\tau_T = R/\text{mean free path} = n_e \sigma_T R$, in a cloud of size R and electron density n_e needs to be larger than unity for a large fraction of photons to scatter many times before escaping. The photons then diffuse out of the cloud (random walk).

In a random walk, the mean displacement after N scatterings is $\sqrt{N}\times$mean free path. Therefore, for the mean displacement of a photon to be R requires

$$N = \left(\frac{R}{\text{mean free path}} \right)^2 = (R n_e \sigma_T)^2 = \tau_T^2 \tag{8.35}$$

number of scatterings. This is valid for $\tau_T \gg 1$. For $\tau_T \leq 1$, $N \sim \tau_T$. Thus for any τ_T, one can write approximately $N \approx \tau_T(1 + \tau_T)$.

The relative energy change of escaping photons then becomes

$$\langle\frac{\Delta\epsilon}{\epsilon_i}\rangle\Big|_{\text{tot}} = 4\frac{kT}{m_e c^2}\tau_T^2 \equiv y, \tag{8.36}$$

which is called Compton y-parameter (for no obvious reason). We call it the Kompaneets parameter. $y \gtrsim 1$ is required for photons to get non-negligible energy increase, i.e. either kT or τ_T, or both must be sufficiently large. When the photons diffuse along the energy axis (i.e. sometimes they lose, but more often they gain energy), then the time evolution is described by a diffusion equation. Most easily this equation is written in terms of the phase-space density (occupation number) $n(x)$ of photons of energy $x \equiv h\nu/kT = \epsilon/kT$. The photon density $\propto x^2 n(x)$ and the intensity and energy density is $\propto x^3 n(x)$. This diffusion equation

$$\frac{1}{n_e c\sigma_T}\frac{\partial n(x)}{\partial t}\Big|_{\text{Comp}} = \frac{kT}{m_e c^2}\frac{1}{x^2}\frac{\partial}{\partial x}\left[x^4\left(\frac{\partial n}{\partial x} + n\right)\right] \tag{8.37}$$

is called the Kompaneets equation. It was derived by A.C. Kompaneets in Soviet Union aroud 1950 but was classified due to bomb research until 1956. Since the advent of X-ray astronomy in $\approx$ 1970, it has been heavily used by astronomers.

8.9 Comptonization spectra

Let us inject photons with energy $x_i \equiv \frac{\epsilon_i}{kT} \ll 1$ in a gas cloud with radius R and electron density n_e, and optical depth $\tau_T = n_e\sigma_T R > 1$. The number of photons per unit phase space volume escaping the cloud per unit time is given by

$$\frac{\partial n(x)}{\partial t}\Big|_{\text{escape}} \approx -\frac{n(x)}{t_{\text{esc}}}, \tag{8.38}$$

where the escape time is given by

$$t_{\text{esc}} \approx \frac{N \times \text{mean free path}}{c} = \frac{NR/\sqrt{N}}{c} = \frac{R}{c}\tau_T. \tag{8.39}$$

If the cloud had been optically thin ($\tau_T \ll 1, N \approx \tau_T$ scatterings), then $t_{esc} \approx R/c$. Now, since $\tau_T > 1$ the photon suffers $N \approx \tau_T^2$ scatterings and the escape time is prolonged by a factor τ_T.

The total time evolution is desribed by

$$\frac{\partial n(x)}{\partial t} = \left.\frac{\partial n(x)}{\partial t}\right|_{\text{Comp}} + \left.\frac{\partial n(x)}{\partial t}\right|_{\text{esc}} \tag{8.40}$$

for $x_i \ll x \ll 1$. Now we get

$$\frac{1}{n_e c \sigma_T}\frac{\partial n}{\partial t} = \frac{kT}{m_e c^2}\frac{1}{x^2}\frac{\partial}{\partial x}\left[x^4\left(\frac{\partial n}{\partial x} + n\right)\right] - \frac{n}{\tau_T^2}. \tag{8.41}$$

Consider $x \ll 1$, then $\frac{\partial n}{\partial x} \approx n/x \gg n$ and the steady state, i.e. $\frac{\partial n}{\partial t} = 0$. Assume a power-law solution such as photon intensity $I(x) \propto x^{-\alpha}$, where α is spectral index. Then the phase space density $n(x) \propto I(x)/x^3 \propto x^{-(\alpha+3)}$.

$$\frac{1}{x^2}\frac{\partial}{\partial x}\left[x^4\frac{\partial}{\partial x}\left(x^{-(\alpha+3)}\right)\right] - \frac{x^{-(\alpha+3)}}{\frac{kT}{m_e c^2}\tau_T^2} = 0. \tag{8.42}$$

We get

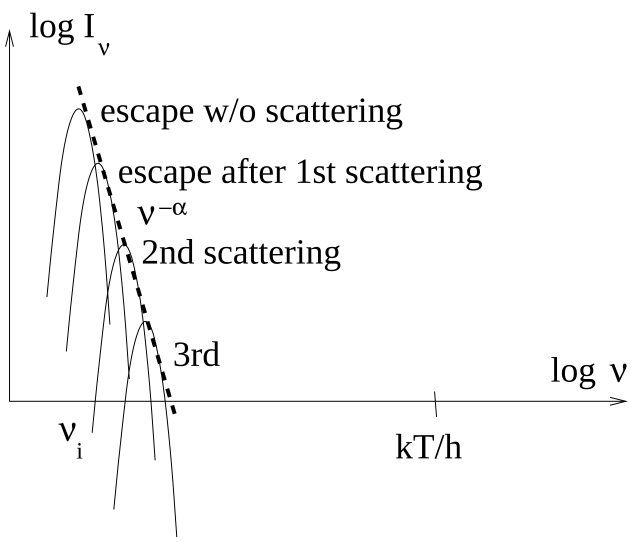
$$(\alpha+3)\alpha - \frac{4}{y} = 0 \tag{8.43}$$

and

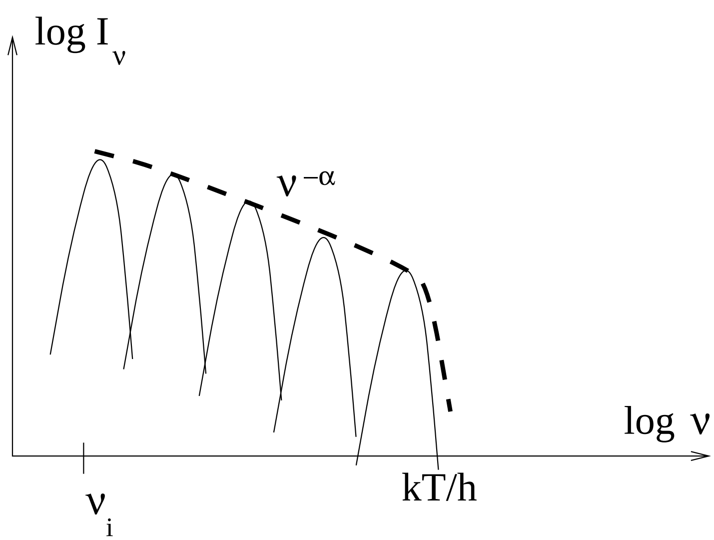
$$\alpha = -\frac{3}{2} \pm \sqrt{\frac{9}{4} + \frac{4}{y}}. \tag{8.44}$$

There are three typical cases.

(a) Very unsaturated, i.e. $y \ll 1$, therefore $\alpha \approx \frac{2}{\sqrt{y}} \gg 1$. Example: Sunyaev-Zeldovich effect, i.e. scattering of the microwave background radiation on the hot electron gas in clusters of galaxies.

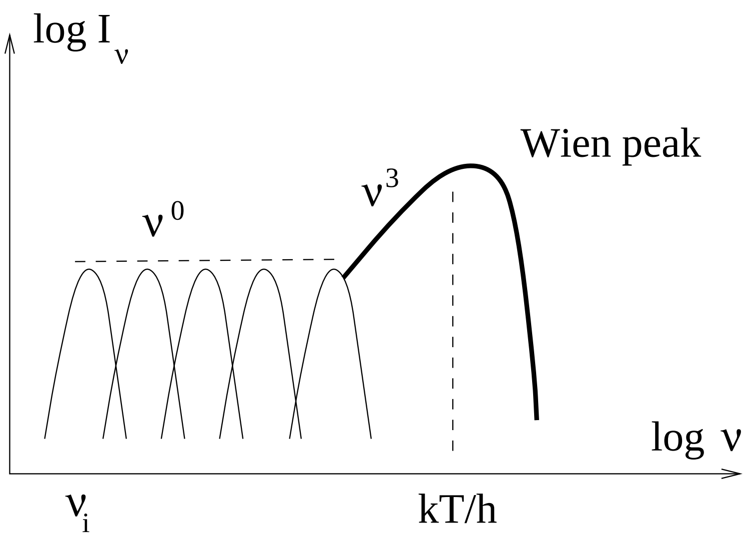


(b) Unsaturated, $y \approx 1$, then $\alpha \sim 1$. Examples: hard X-ray spectra of Galactic black hole candidates, accreting neutron stars, some active galactic nuclei (Seyferts).

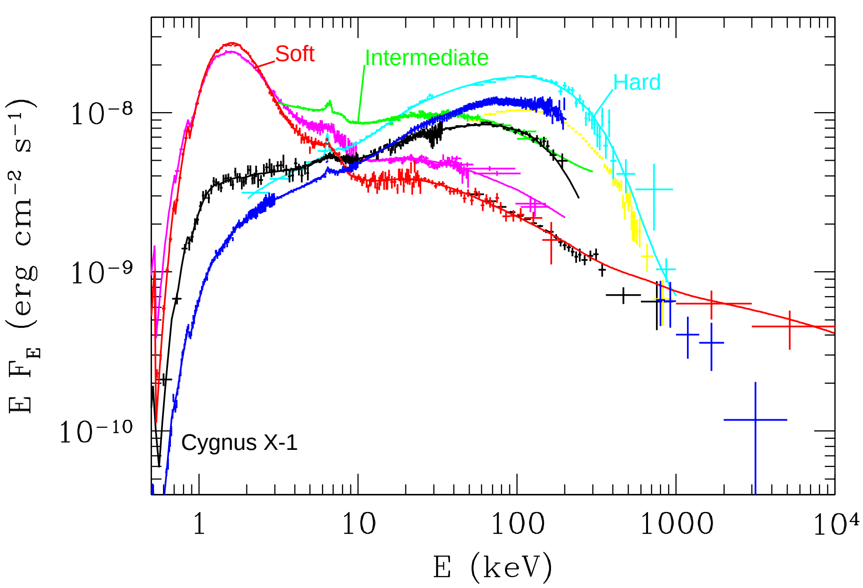


(c) Saturated, $y \gg 1$, then two solutions $\alpha_1 = -\frac{3}{2}+\frac{3}{2} = 0$ and $\alpha_2 = -\frac{3}{2}-\frac{3}{2} = -3$. The first solution descibed the spectrum between $\epsilon_i \ll \epsilon \ll kT$, while the second for ϵ closer to kT. Example: high accretion rate neutron stars or black holes.

When $y \gg 1$ photons stay in the medium very long so that they can scatter into a Bose-Einstein distribution ($\approx$ Wien distribution) before escaping.



We see that thermal plasma cloud can also give rise to a power-law photon spectrum. Observed power-law spectra in compact X-ray sources could be due to either thermal or non-thermal Compton scattering. The hard state spectrum of the Galactic black hole Cyg X-1 (see picture below) is probably formed by thermal Comptonization, while the soft state is better described by Compton scattering of non-thermal electrons distributed according to a powerlaw.



EXERCISES

Problems set 1

1.1 (RL1.3) X-ray photons are produced in a cloud of radius R at the uniform rate Γ (photons per unit volume per unit time). The cloud is a distance d away. Neglect absorption of these photons (optically thin medium). A detector at Earth has an angular acceptance beam of half-angle $\Delta\theta$ and it has an effective area of A.

a. Assume that the source is completely resolved. What is the observed intensity (photons per unit time per unit area per steradian) toward the center of the cloud?

b. Assume that the cloud is complete unresolved. What is the average intensity (in the above units) when the source is in the beam of the detector?

1.2 (RL 1.5) A supernova remnant has an angular diameter $\theta = 4.3$ arcminutes and a flux at 100 MHz of $F_{100} = 1.6 \times 10^{-19}$ erg cm^{-2} s^{-1} Hz^{-1}. Assume that the emission is thermal.

a. What is the brightness temperature T_b? What energy regime of the blackbody curve does this correspond to?

b. The emitting region is actually more compact than indicated by the observed angular diameter. What effect does this have on the derived value of T_b?

c. At what frequency will this object's radiation be maximum, if the emission is blackbody?

d. What can you say about the temperature of the material from the above results?

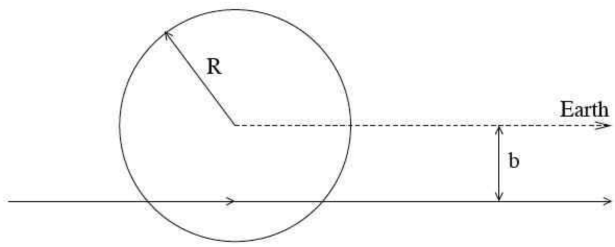


Figure 1: Geometry for exercise 1.3.

1.3 (RL1.8) A certain gas emits thermally at the rate $P(\nu)$ (power per unit volume and frequency range). A spherical cloud of this gas has radius R, temperature T and is a distance d from Earth ($d \gg R$).

a. Assume that the cloud is optically thin. What is the brightness of the cloud as measured on Earth? Give your answer as a function of the distance b away from the cloud center, assuming the cloud may be viewed along parallel rays with different impact parameters b (i.e. the closest distance from the cloud center to the ray, see Fig. 1).

b. What is the effective temperature of the cloud?

c. What is the flux F_ν measured on Earth coming from the entire cloud?

d. How do the measured brightness temperatures compare with the cloud's temperature?

e. Answer parts (a)-(d) for an optically thick cloud.

1.4 In the excercise we examine the radiation from the planet Jupiter.

a. What is the power intercepted by Jupiter from the Sun?

b. We define the albedo A as the ratio of the incident over the reflected flux, $F_{\rm reflected} = A \times F_{\rm in}$. Jupiter has $A = 0.52$. What is the amount of energy from the Sun absorbed per second by Jupiter?

c. We approximate the thermal emission by Jupiter as a pure blackbody. Jupiter rotates fast, once per 10 hrs. This is fast enough to even out any temperature differences between the side illuminated by the Sun and the side turned away from the Sun. Jupiter can therefore be approximated as isothermal. What is the equilibrium temperature of Jupiter in the Sun's radiation field?

d. The observed spectrum from Jupiter can be reasonably approximated by a Planck curve. It peaks at about 7.13×10^{12} Hz. What temperature do you deduce for Jupiter?

e. A possible explanation for this temperature difference is that it is a remnant from the formation stage of Jupiter. The gravitational energy liberated as the protoplanterary material coalesced into Jupiter, is stored as thermal energy (heat) of the gas deep inside Jupiter's interior. For this gas a specific heat capacity of 2.1×10^7 erg g^{-1} K^{-1} can be assumed, appropriate for atomic hydrogen. The energy liberated by the temperature decrease of this material contributes energy to the budget that has to be evaluated to calculate Jupiter's surface equilibrium temperature.

If we assume that the power delivered by Jupiter's interior is constant with time, calculate the minimum temperature of the Jupiter's interior at the formation time 4.5 Gyr ago.

1.5 (RL 1.4) The Eddington limit.

a. Show that the condition that an optically thin cloud of material can be ejected by radiation pressure from a nearby luminous object is that the mass to luminosity ratio (M/L) for the object be less that $\kappa/(4\pi Gc)$, where G = gravitational constant, c = speed of light, κ = mass absorption coefficient of the cloud material (assumed independent of frequency).

b. Calculate the terminal velocity V attained by such a cloud under radiation and gravitational forces alone, if it starts from a rest distance R from the object. Show that

$$V^2 = \frac{2GM}{R}\left(\frac{\kappa L}{4\pi GMc} - 1\right). \tag{1}$$

c. A minimum value for κ may be estimated for pure hydrogen as that due to Thomson scattering of free electrons, when hydrogen is completely ionized. The Thomson cross section is $\sigma_T = 6.65 \times 10^{-25}$ cm^2. The mass scattering coefficient is therefore $> \sigma_T/m_H$, where m_H = mass of hydrogen atom. Show that the maximum luminosity that a central mass M can have and still not spontaneously eject hydrogen by radiation pressure is

$$L_{\rm Edd} = 4\pi GMcm_H/\sigma_T = 1.25 \times 10^{38} \frac{M}{M_\odot} \text{erg s}^{-1}, \tag{2}$$

where $M_\odot = 2 \times 10^{33}$ g is the mass of the Sun. This is called the Eddington limit.

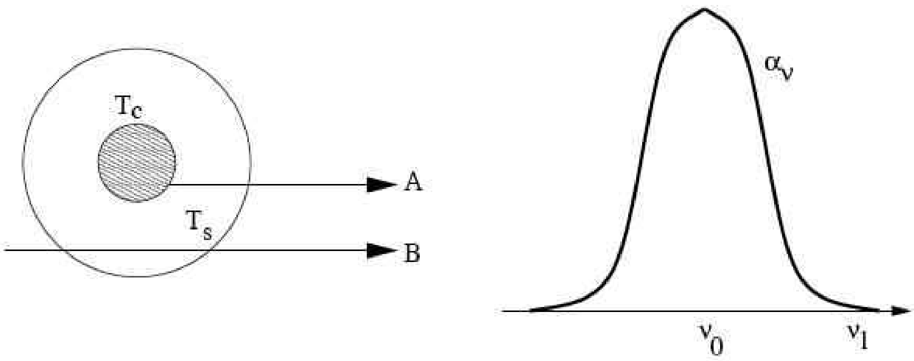


Figure 2: Left: geometry for exercise 1.6. Right: absorption coefficient as a function of frequency.

1.6 (RL1.9) A spherical, opaque object emits as a blackbody at temperature T_c. Surrounding this central object is a spherical shell of material, thermally emitting at a temperature T_s ($T_s < T_c$). This shell absorbs in a narrow spectral line; that is, its absorption coefficient becomes large at the frequency ν_0 and is neglibly small at other frequencies, such as at ν_1: $\alpha_{\nu_0} \gg \alpha_{\nu_1}$. The object is observed at frequencies ν_0 and ν_1. and along two rays A (passing through the center) and B (passing only through the absorbing shell). See Fig. 2. Assume that the Planck function does not vary appreciably from ν_0 to ν_1.

a. At which frequency will the observed brightness be larger when observed along ray A? And along ray B?

b. Answer the preceding questions if $T_s > T_c$.

1.7 Consider a spherical cloud with particles emitting thermal radiation. These particles all have the same temperature $T = 4 \times 10^3$ K. The cloud has a diameter of 0.1 pc. At frequency $\nu_0 = 1.3 \times 10^{15}$ Hz the particles have an absorption coefficient $\alpha_\nu = 5.51 \times 10^{-20}$ cm^{-1} and a scattering coefficient $\sigma_\nu = 9.51 \times 10^{-18}$ cm^{-1}. What is the luminosity at frequency ν_0 emitted by this cloud in all directions together?

1.8 (RL1.10) Consider a semi-infinite half space in which both scattering (σ) and absorption and emission (α_ν) occur. Idealize the medium as homogeneous and isothermal, so that the coefficients σ and α_ν do not vary with depth. Further assume the scattering is isotropic (which is a good approximation for the forward-backward symmetric Thomson differential cross section).

a. Using the radiative diffusion equation with two-stream boundary conditions, find expressions for the mean intensity $J_\nu(\tau)$ in the medium and the emergent flux $F_\nu(0)$.

b. Show that $J_\nu(\tau)$ approaches the blackbody intensity at an effective optical depth of order $\tau_* = \sqrt{3\tau_a(\tau_a + \tau_s)} \sim 1$.

Problems set 2

2.1—The magnetic field **B** is defined using the Lorentz force equation

$$\mathbf{F} = \frac{q}{c}(\mathbf{v} \times \mathbf{B})\,.$$

Performing three experiments gives

$$\mathbf{v} = \mathbf{i}\,, \qquad \frac{c}{q}\mathbf{F} = 2\mathbf{k} - 4\mathbf{j}$$

$$\mathbf{v} = \mathbf{j}\,, \qquad \frac{c}{q}\mathbf{F} = 4\mathbf{i} - \mathbf{k}$$

$$\mathbf{v} = \mathbf{k}\,, \qquad \frac{c}{q}\mathbf{F} = \mathbf{j} - 2\mathbf{i}$$

Determine the magnetic field **B** using these results. **i**, **j**, **k** are unit vectors in the x, y, and z-directions.

2.2— Show the identity

$$\mathbf{E} \cdot (\nabla \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{E}) - \nabla \cdot (\mathbf{E} \times \mathbf{B})$$

that was used when deriving Poynting's theorem. Do it by direct expansion in cartesian coordinates.

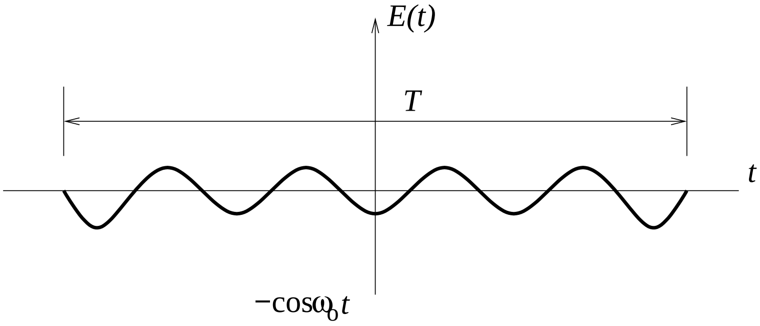
2.3— Show the identity

$$\nabla \times (\nabla \times \mathbf{E}) = \nabla(\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E}.$$

2.4— Derive relations (3.37).

2.5— Two oscillating quantities $A(t)$and $B(t)$ are represented as the real parts of the complex quantities $\mathcal{A}e^{i\omega t}$ and $\mathcal{B}e^{i\omega t}$. Show that the average of AB is given by

$$\langle AB \rangle = \frac{1}{2} Re(\mathcal{A}^* \mathcal{B}) = \frac{1}{2} Re(\mathcal{A}\mathcal{B}^*). \tag{3}$$



2.6 — A sine-shaped pulse is given by the figure above and is described by the following expression

$$E(t) = -\cos\omega_0 t, \qquad |t| < \frac{N\pi}{2\omega_0},$$

$$E(t) = 0, \qquad |t| > \frac{N\pi}{2\omega_0},$$

where $N = 9$. Determine the frequency spectrum $|\hat{E}(\omega)|^2$. Make an illustrative figure showing the frequency spectrum. Choose reasonable units for the frequency and a reasonable normalization for the spectrum.

(Hints: (i) $E(t) = E(-t) \Longrightarrow$ use the cosine transform. (ii) See Example 15.3.1 in Arfken.)

2.7—(From Jackson, Exercise 7.1) For each set of Stokes parameters given below deduce the amplitude of the electric field in the base $\hat{\mathbf{x}}$ och $\hat{\mathbf{y}}$ (see Fig. 3.3). Make a drawing showing the ellipse, the lengths of the axes, and the orientation.

(a) $I = 3$, $Q = -1$, $U = 2$, $V = -2$

(b) $I = 25$, $Q = 0$, $U = 24$, $V = 7$

2.8—Show that equations (3.64) for the Stokes parameters follow from equations (3.63).

Problems set 3

3.1—Prove relation (4.24).

3.2—In this problem, we derive the electromagnetic fields associated with the Lienard-Wiechert potentials which are given by the following expression

$$\begin{pmatrix} \phi(\vec{x},t) \\ \vec{A}(\vec{x},t) \end{pmatrix} = \frac{q}{R - \vec{R}\cdot\vec{u}/c} \begin{pmatrix} 1 \\ \vec{u}/c \end{pmatrix}, \tag{4}$$

where

$$\begin{aligned} t_{\rm ret} &= t - R(t_{\rm ret})/c, \\ \vec{R}(t_{\rm ret}) &= \vec{x} - \vec{r}(t_{\rm ret}), \end{aligned} \tag{5}$$

and $\vec{u} = \dot{\vec{r}}(t_{\rm ret})$ is also evaluated at the retarded time $t_{\rm ret}$. Calculations of various derivatives over time and space are complicated by the fact that the rhs of equation (4) depends not on t and $\vec{x}$, but on the retarded time through equations (5).

The electric and magnetic fields are given by

$$\vec{B} = \nabla \times \vec{A}, \tag{6}$$

$$\vec{E} = -\nabla\phi - \frac{1}{c}\frac{\partial \vec{A}}{\partial t}. \tag{7}$$

We would need to compute $\nabla t_{\rm ret}$ and $\frac{\partial t_{\rm ret}}{\partial t}$, where ∇ operates at constant t and $\frac{\partial}{\partial t}$ operates at constant $\vec{x}$.

1. Noticing that

$$R^2 = \vec{R}\cdot\vec{R} = x^2 + \vec{r}\cdot\vec{r} - 2\vec{x}\cdot\vec{r}, \tag{8}$$

and taking derivative over $t_{\rm ret}$ show that

$$\frac{\partial R}{\partial t_{\rm ret}} = -\vec{n}\cdot\vec{u}, \tag{9}$$

where $\vec{n} = \vec{R}/R$. Differentiating equation (5) with respect to t, show that

$$\frac{\partial t_{\rm ret}}{\partial t} = \frac{1}{1 - \vec{n}\cdot\vec{u}/c}. \tag{10}$$

2. Take the gradient of the expression for R^2 to obtain:

$$2R\nabla R = 2\vec{x} + 2\vec{r}\cdot\vec{u}\nabla t_{\rm ret} - 2\vec{r} - 2\vec{x}\cdot\vec{u}\nabla t_{\rm ret}. \tag{11}$$

Hint: write down the i^{th} component of the gradient of e.g. a scalar product $\vec{x}\cdot\vec{r}$, i.e. $\partial_i x_j r_j$. Differentiate this product and notice that $\partial_i x_j = \delta_{ij}$ and $\partial_i r_j = \frac{\partial r_j}{\partial t_{\rm ret}}\partial_i t_{\rm ret}$.
On the other hand, taking gradient of equation (5), we get

$$\nabla R = -c\nabla t_{\rm ret}. \tag{12}$$

Collecting terms show that

$$\nabla t_{\rm ret} = -\frac{\vec{n}/c}{1 - \vec{n}\cdot\vec{u}/c}. \tag{13}$$

3. Consider now taking gradient of ϕ, i.e. the first raw of equation (4):

$$\nabla\phi = -\frac{q}{(R - \vec{R}\cdot\vec{u}/c)^2}\left[\nabla R - \nabla(\vec{R}\cdot\vec{u}/c)\right]. \tag{14}$$

   Remember that $\nabla R = -c\nabla t_{\rm ret}$ and show that

$$\nabla(\vec{R}\cdot\vec{u}) = \vec{u} + \frac{\partial(\vec{R}\cdot\vec{u})}{\partial t_{\rm ret}}\nabla t_{\rm ret} = \vec{u} + [-u^2 + \vec{R}\cdot\dot{\vec{u}}]\nabla t_{\rm ret}. \tag{15}$$

   Thus get:

$$\nabla R - \nabla(\vec{R}\cdot\vec{u}/c) = (-c + u^2/c + \vec{R}\cdot\dot{\vec{u}}/c)\nabla t_{\rm ret} - \vec{u}/c. \tag{16}$$

   Obtain now the final expression for gradient of ϕ:

$$\nabla\phi = -\frac{q}{(R - \vec{R}\cdot\vec{u}/c)^3}\left[\vec{R}\left(1 - \frac{u^2}{c^2} - \vec{R}\cdot\frac{\dot{\vec{u}}}{c^2}\right) - \frac{\vec{u}}{c}\left(R - \vec{R}\cdot\frac{\vec{u}}{c}\right)\right]. \tag{17}$$

4. In a similar manner obtain

$$\frac{1}{c}\frac{\partial\vec{A}}{\partial t} = -\frac{q}{(R - \vec{R}\cdot\vec{u}/c)^3}\left[\left(-\vec{R}\cdot\frac{\vec{u}}{c} + R\frac{u^2}{c^2} - R\vec{R}\cdot\frac{\dot{\vec{u}}}{c^2}\right)\frac{\vec{u}}{c} - R\left(R - \vec{R}\cdot\frac{\vec{u}}{c}\right)\frac{\dot{\vec{u}}}{c^2}\right]. \tag{18}$$

5. Substitute equations (17) and (18) into equation (7) to obtain the expression for the electric field:

$$\vec{E} = \frac{q}{(R - \vec{R}\cdot\vec{u}/c)^3}\left\{\left(1 - \frac{u^2}{c^2}\right)\left(\vec{R} - R\frac{\vec{u}}{c}\right) + \vec{R}\times\left[\left(\vec{R} - R\frac{\vec{u}}{c}\right)\times\frac{\dot{\vec{u}}}{c^2}\right]\right\}. \tag{19}$$

6. Convince yourself that the magnetic field $\vec{B} = \nabla\times\vec{A}$ is given by the following expression:

$$\vec{B} = \frac{\vec{R}}{R}\times\vec{E}. \tag{20}$$

3.3—A pulsar can be described as a rotating neutron star. It has a strong magnetic field B_0 since it traps lines of force during the collapse. If the magnetic axis of the neutron star does not line up with the rotational axis, there will be magnetic dipole radiation from the time-changing magnetic dipole $\vec{m}(t)$. Assume the mass and radius of neutron star is M and R, the angle between the magnetic and rotational axes is α, and the rotational velocity is ω.

1. Find an expression for the radiated power P in terms of ω, R, B_0 and α.

2. Assuming that the rotational energy of the pulsar is the ultimate source of the radiated power, find an expression for the slow-down time-scale $\tau = \omega/\dot{\omega}$ of the pulsar.

3. For $M = 1.4M_\odot$, $R = 10$ km, $B_0 = 10^{12}$ G, $\alpha = 90^{\rm o}$, find P and τ for $\omega = 10^4, 10^3, 10^2$ s^{-1}. The highest rate $\omega = 10^4$ s^{-1} is believed to be typical of newly born pulsars.

4. Assume $B_0 = 10^9$ G, compute P and τ for $\omega = 10^4$ s^{-1}. Such small magnetic field is believed to correspond to the so called recycled millisecond pulsars, which are old stars spun-up by the accreting matter from the companion.

5. Show that
$$\dot{\omega} = -C\omega^3, \tag{21}$$
and derive coefficient C. Solve this equation to show that the actual pulsar age is less than 1/2 of the spin-down time τ.

6. Show that the so called breaking index is
$$n \equiv \frac{\omega\ddot{\omega}}{\dot{\omega}^2} = 3. \tag{22}$$

Hints: radiated power is
$$P = \frac{2}{3}\frac{|\ddot{\vec{m}}|^2}{c^3}$$
and the magnetic field is related to the magnetic dipole moment as
$$B_0 = \frac{2m}{R^3}.$$
Assume homogeneous sphere to calculate the moment of inertia of the star.

3.4–A particle of mass m and charge e moves at constant, nonrelativistic speed u in a circle of radius a.

1. What is the power emitted per unit solid angle in a direction at angle θ to the axis of the circle?

2. Describe qualitatively and quantitatively the polarization of radiation as a function of the angle θ.

3. What is the spectrum of the emitted radiation?

4. Suppose a particle is moving nonrelativistically in a constant magnetic field B. Show that the frequency of circular motion is $\omega_B = eB/mc$ and that the total emitted power is
$$P = \frac{2}{3} r_e^2 c (u_\perp/c)^2 B^2, \tag{23}$$
(here $u_\perp$ is the velocity component perpendicular to the field) and is emitted solely at the frequency ω_B. This nonrelativistic form of synchrotron radiation is called cyclotron radiation.

3.5–Consider a medium containing a large number of radiating particles (e.g. electrons). Each particle emits a pulse of radiation with an electric field $E_0(t)$ as a function of time. An observer will detect a series of such pulses, all with the same shape but with random arrival times $t_1, t_2, ..., t_N$. The measured electric field will be
$$E(t) = \sum_{i=1}^{N} E_0(t - t_i). \tag{24}$$

1. Show that the Fourier transform of $E(t)$ is

$$\hat{E}(\omega) = \hat{E}_0(\omega) \sum_{i=1}^{N} e^{i\omega t_i}, \tag{25}$$

   where $\hat{E}_0(\omega)$ is the Fourier transform of $E_0(t)$.

2. Argue that

$$\left| \sum_{i=1}^{N} e^{i\omega t_i} \right|^2 = N. \tag{26}$$

   Hint: Use the fact the arrival times are random and consider a random walk in complex plane.

3. Thus show that the measured spectrum is simply N times the spectrum of an individual pulse. (Note that this result still holds if the pulses overlap.)

4. By contrast, show that if all particles are in a region much smaller than a wavelength they produce and they emit their pulses simultaneously, then the measured spectrum will be N^2 times the spectrum of an individual pulse.

Problems set 4

4.1—Derive formulae (5.16)–(5.17).

4.2—Consider a car moving straight with velocity V directed along plane surface. Let a photon be emitted at angle α' (in the car comoving frame) relative to the vertical direction (as viewed in the car frame) and projection of the photon momentum to the surface makes angle ϕ' with the direction of motion. Compute the direction of the photon propagation in the static frame of the surface. What is the relation between α (angle the photon makes with the surface normal) and α'?

4.3—In astrophysics it is frequently argued that a source radiation which undergoes a fluctuation of duration Δt must have a physical diameter of order $D \leq c\Delta t$. This argument is based on the fact that even if all portions of the source undergo a disturbance at the same instant and for an infinitesimal period of time, the resulting signal at the observer will be smeared out over the time interval $\Delta t_{\rm min} \sim D/c$ because of the finite light travel time across the source. Suppose, however, that the source is an optically thick spherical shell of radius $R(t)$ that is expanding with relativistic velocity $\beta \sim 1, \gamma \gg 1$ and energized by a stationary point at its center. By consideration of relativistic beaming effect show that if the observer sees a fluctuation from the shell of duration Δt at time t, the source may actually be of radius

$$R < 2\gamma^2 c\Delta t,$$

rather that the much smaller limit given by the nonrelativistic considerations. In the rest frame of the shell surface, each surface element may be treated as isotropic emitter.

This later argument has been used to show that the active regions in quasars may be much larger than $c\Delta t \sim 1$ light month across, and thus avoid much energy being crammed into so small a volume.

4.4—Let two different uniformly moving observers have velocities $\vec{u}_1$ and $\vec{u}_2$ in units where $c = 1$. Show that their relative velocity, as measured by one of the observers, satisfies

$$V^2 = \frac{(1 - \vec{u}_1 \cdot \vec{u}_2)^2 - (1 - u_1^2)(1 - u_2^2)}{(1 - \vec{u}_1 \cdot \vec{u}_2)^2}.$$

A straight application of velocity transformation is painfully tedious, but an application of 4-vector invariants (e.g. scalar product of two 4-velocities) is trivial!

Consider now velocities u_1 and u_2 very close to speed of light. Derive the relation between the relative Lorentz factor $\gamma = 1/\sqrt{1 - V^2}$ and the corresponding $\gamma_1 = 1/\sqrt{1 - u_1^2}$ and $\gamma_2 = 1/\sqrt{1 - u_2^2}$. Consider the relative motion in the same direction and in the opposite directions. What can you say about the relative Lorentz factor in these two cases?

4.5—(a) Show that an observer moving with respect to a blackbody field of temperature T will see blackbody radiation with a temperature that depends on angle according to

$$T' = \frac{(1 - V^2/c^2)^{1/2}}{1 - (V/c)\cos\theta'} T,$$

where θ' is the viewing angle, i.e. angle between the line of sight and direction of motion.

(b) The isotropy of the 2.7 K microwave background radiation at $\lambda = 3$ cm has been established to about one part in 10^3. What is the maximum velocity that the Earth can have with respect to the frame in which the radiation is isotropic? Isotropy is measured by the ratio $(I_{\rm max} - I_{\rm min})/(I_{\rm max} + I_{\rm min})$.

Hint: the ratio I_ν/ν^3 (where I_ν is the specific intensity) is Lorentz invariant.

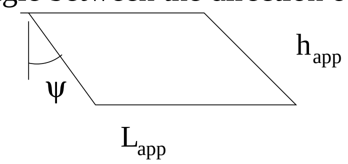
4.6—A small, cubical, red-flashing traffic light illuminated by sunlight is hanging over the super-super-highway. Riding in a convertible with the top down, you approach this traffic light along a straight road from a great distance at a constant road speed $\beta = 0.866$ (i.e. $\gamma = 2$). You pass directly under this traffic light without slowing down and continue to a great distance beyond it, keeping you eyes on the traffic light as you go.

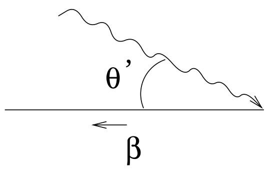
Calculate and describe the apparent appearence (geometry and color) of the traffic light during this ride. Consider the distributed hints.

Helps and Hints

Read first the distributed paper by Ghisellini ”Special relativity at action in the Universe”.

The problem is most easily considered in the observer’s rest frame. Neglect that the traffic light is an extended object, i.e. assume that the distance to the observer is always $\gg$ the size of the traffic light. Assume that the traffic light is a cube of size L. In the observer’s frame the angle between the direction of motion as the line of sight (photon arrival direction) is θ'.





Determine as a function of θ' (and answer the questions):

a) the apparent length $L_{\rm app}/L$ of the bottom side of the traffic light in the direction parallel to the superhighway.

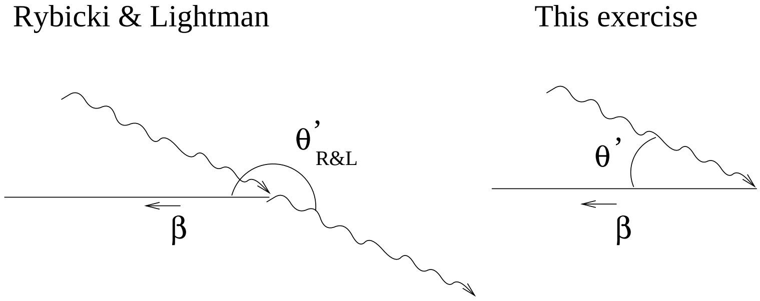
b) the apparent slope, ψ, and apparent height, $h_{\rm app}/L$, of the traffic light.

c) the projected sizes, $L_{\rm proj}/L$, and $h_{\rm proj}$, on the plane of the sky. You must do some geometrical considerations in order to derive the projected sizes. For what θ' is only the bottom side visible?

d) the blue or red shift of the light from the traffic light, which only reflects visible light (4000-7000 Å). Between which angles, θ', is the traffic light visible to the car driver? Between which angles, θ', is the blinking red light (= 6000 Å) visible? How does the blinking period vary with θ'? Assume that the red light blinks once per second for an observer standing next to the traffic light.

e) Use the aberration formula to determine the angles θ in the street frame, that correspond to the angles θ' in the car frame. Note that the angle θ' as we defined it in this exercise differs by π

from the definition in Rybicki & Lightman (R&L) and our lecture notes, i.e. for the same θ we have $\theta'_{\rm R\&L} = \theta' + \pi$:



We have:

$$\cos\theta_{\rm R\&L} = \frac{\cos\theta'_{\rm R\&L} + \beta}{1 + \beta\cos\theta'_{\rm R\&L}}.$$

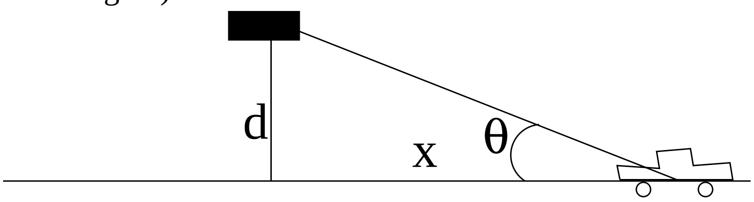
Converting to θ' gives

$$-\cos\theta = \frac{-\cos\theta' + \beta}{1 + \beta\cos\theta'},$$

or

$$\cos\theta = \frac{\cos\theta' - \beta}{1 - \beta\cos\theta'}.$$

f) Finally, one can determine the position of the car, x/d, relative to the traffic light (x and d are defined in figure).



Make many figures to show your results. In particular, show $L_{\rm proj}/L$, and $h_{\rm proj}/L$, as a function of θ', and also as a function of x/d. Mark on the curves where interesting things happen, e.g. where the traffic light is visible, and where the blinking red light is visible. Final hint: Most of the effects above are simple classical Doppler and aberration effects. Do not forget the special relativistic effects of Lorentz contraction and time dilation (these involve an extra factor of γ).

Problems set 5

5.1—Compute the integral in equation (5.64) and prove the relation.

5.2—In this exercise we will try to obtain a more accurate expression for the bremsstrahlung emissivity. Consider a Cartesian coordinate system (with units vectors along the axes $\vec{e}_x, \vec{e}_y, \vec{e}_z$) with the electron moving along the z-axis with velocity $\vec{u} = u(0,0,1)$. Assume the observer is in the direction $\vec{k} = (\sin\theta\ \cos\phi, \sin\theta\ \sin\phi, \cos\theta)$, i.e. the angle between $\vec{k}$ and $\vec{u}$ is θ. The heavy ion is at position $\vec{b} = b(1,0,0)$, where b is the impact parameter. We assume that deviation of the electron trajectory from the straight line are small.

1) Compute the Coulomb force $\vec{F}$ (which is a vector) acting on the electron as a function of time t (with $t = 0$ corresponding to the electron passing the ion at the closest distance). This force has two components along x- and z-axes. Compute corresponding acceleration $\ddot{\vec{r}}$ of the electron.

2) Compute the electric field at the position of the observer at distance r from the charges as a function of time

$$\vec{E}(t) = \frac{1}{c^2 r}\left[\left(\ddot{\vec{d}} \times \vec{k}\right) \times \vec{k}\right] \tag{27}$$

where $\ddot{\vec{d}} = e\ddot{\vec{r}}$. Show that the result can be represented as a sum of two terms

$$\vec{E}(t) = \vec{E}_1(t) + \vec{E}_2(t), \tag{28}$$

with

$$\vec{E}_1(t) = \frac{Ze^3}{m_e c^2 r}\frac{b}{[b^2 + (ut)^2]^{3/2}}\left(\sin\theta\ \cos\phi\ \vec{k} - \vec{e}_x\right), \tag{29}$$

$$\vec{E}_2(t) = \frac{Ze^3}{m_e c^2 r}\frac{ut}{[b^2 + (ut)^2]^{3/2}}\sin\theta(-\cos\theta\ \cos\phi\ \vec{e}_x - \cos\theta\ \sin\phi\ \vec{e}_y + \sin\theta\ \vec{e}_z). \tag{30}$$

3) Compute the Fourier transform of both terms:

$$\hat{\vec{E}}_{1,2}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega t}\vec{E}_{1,2}(t)dt. \tag{31}$$

Hint:

$$\int_0^\infty \frac{\cos ax}{(b^2 + x^2)^{3/2}}dx = \frac{a}{b}K_1(ab),\ \int_0^\infty \frac{x\ \sin ax}{(b^2 + x^2)^{3/2}}dx = aK_0(ab), \tag{32}$$

where $K_n(x)$ are the modified Bessel functions

$$K_n(x) = \int_0^\infty e^{-x\ \mathrm{ch}t}\mathrm{ch}(nt)\ dt. \tag{33}$$

4) Compute the square of the Fourier transform:

$$|\hat{\vec{E}}(\omega)|^2 = |\hat{\vec{E}}_1(\omega) + \hat{\vec{E}}_2(\omega)|^2 \tag{34}$$

to obtain

$$|\hat{\vec{E}}(\omega)|^2 = \frac{Z^2 e^6}{m_e^2 c^4 r^2}\frac{1}{\pi^2}\frac{1}{(bu)^2}\left(\frac{\omega b}{u}\right)^2\left[(1 - \sin^2\theta\cos^2\phi)K_1^2(\omega b/u) + \sin^2\theta\ K_0^2(\omega b/u)\right]. \tag{35}$$

Make sure that the cross-term $\propto K_0 K_1$ disappears.

5) Integrate the previous expression over the surface of the sphere of radius r (with Ω being the solid angle),

$$\frac{dW}{d\omega} = P_\omega = c \int |\hat{\vec{E}}(\omega)|^2 r^2 d\Omega, \tag{36}$$

to get the the total energy (per units frequency) radiated in all directions passing through that sphere

$$P_\omega = \frac{8}{3\pi} \frac{Z^2 e^6}{m_e^2 c^3} \frac{1}{(bu)^2} \left(\frac{\omega b}{u}\right)^2 \left[K_1^2(\omega b/u) + K_0^2(\omega b/u)\right]. \tag{37}$$

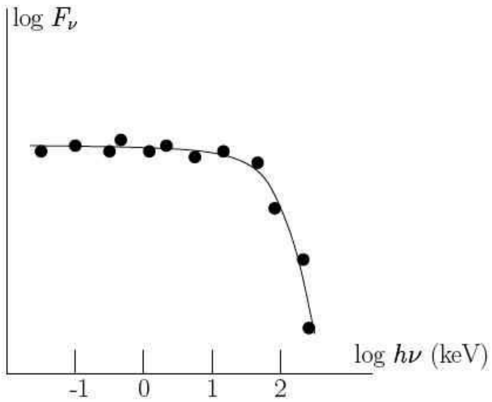
6) Compare this expression to that derived in class (equation 6.10). Remember that $P_\nu = 2\pi P_\omega$. Using the asymptotic expansions of $K_n(x)$ for small and large x:

$$K_0(x) \sim K_1(x) \sim \sqrt{\frac{\pi}{2x}} \exp(-x), \quad x \gg 1, \tag{38}$$

$$K_0(x) \sim -\ln x, \quad K_1(x) \sim 1/x, \quad x \ll 1, \tag{39}$$

compute the limiting expression for P_ω for $\omega \ll u/b$ and $\omega \gg u/b$. How much wrong we were in our derivation of the power in class?

5.3–Suppose X-rays are received from a source of known distance L with a flux F (erg s^{-1} cm^{-2}). The X-ray spectrum has the form as sketched in the figure below.



It is proposed that these X-rays are due to bremsstrahlung from an optically thin, hot plasma cloud, which is in hydrostatic equilibrium around a central mass M. This means that the pressure must balance the gravity, or $3kT \sim GmM/R$, where m is the typical mass of the gas particles. Assume that the cloud thickness ΔR is roughly its radius, $\Delta R \sim R$. Find R and the density of the cloud ρ in terms of the known observations and the conjectured mass M.

a. If $F = 10$ erg s^{-1} cm^{-2}, $L = 10$ kpc, what are the constraints on M such that the source would indeed be effectively optically thin (for self-consistency)?

b. Does electron scattering play any role?

5.4—An ultrarelativistic, $\gamma \gg 1$, electron emits synchrotron radiation. Solve Eq. (7.12) and show that its energy decreases with time according to

$$\gamma = \gamma_0 (1 + A\gamma_0 t)^{-1}, \quad A = \frac{2e^4 B_\perp^2}{3m^3 c^5}.$$

Here γ_0 is the initial value of γ, $B_\perp = B\sin\alpha$, and α is the pitch angle. Show that the time for the electron to lose half its energy is

$$t_{1/2} = (A\gamma_0)^{-1} = \frac{5.1 \times 10^8}{\gamma_0 B_\perp^2}.$$

How does one reconcile the decrease of γ here with the result of constant γ implied by Eq. (7.1)?

5.5—Prove that the charged particle trajectory in the homogeneous magnetic field is a helical curve given by Eq. (7.6).

5.6— The Radio Lobes of Cygnus A. Read the section *Application to Radio Galaxies* in Shu, p. 179-181. The observational data consist of (i) a radio map at 6 cm of Cyg A (Fig.18.5 in Shu), and (ii) the radio spectrum of the lobes of Cyg A (shown below).

Hubble's law is given by $v = cz = H_0 d$ km/s, where v is the expansion velocity, $H_0 = 50$ km/s/Mpc, d is the distance, and $z = \lambda_{obs}/\lambda_{lab} - 1$ is the redshift. The redshift was determined to be $z = 0.0566$ from the optical spectrum.

a) Determine the distance to Cyg A.

b) Make an estimate of the size of the radio lobes. Assume spherical lobes. Calculate the volume.

c) Assume that the number density of electrons has the energy distribution $N(\gamma) = n_0\gamma^{-p}$ cm^{-3}. Calculate an expression for the volume emissivity, j_ν, from the $N(\gamma)$ electrons, under the assumption that one single electron radiates $4\pi j_\nu = \langle P_{\rm em}\rangle \delta(\nu - \gamma^2\nu_{\rm L})$ erg/s/Hz. Determine p, $\nu_{\rm min}$, and $\nu_{\rm max}$ using the observed radio spectrum.

d) Calculate n_0, $\gamma_{\rm min}$, B using the following three relations:

1) Assume that the radio lobes contains the minimum possible energy, i.e. assume equipartition. Use Eq. (18.14). Note the sign error in Eq. (18.13).

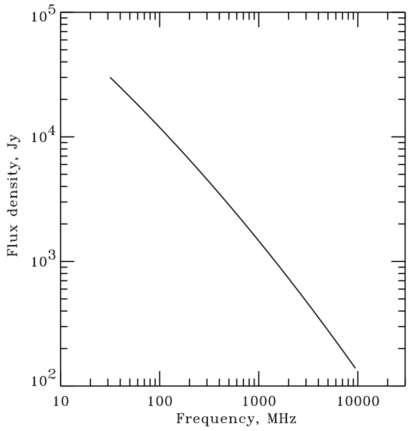
2) Relate $\gamma_{\rm min}$ to $\nu_{\rm min}$ and B.

3) Determine using j_ν and the volume V, an expression for the total monochromatic radio luminosity, L_ν (see Eq. 18.12), as a function of n_0, B, and $\nu_{\rm min}$ (or $\gamma_{\rm min}$).

e) Calculate the total energy in the radio lobes.

f) The center of the galaxy (i.e. the central black hole in Cyg A) radiates $\sim 10^{44}$ erg/s from radio to gamma ray wavelengths. Assume that the black hole feeds the same power into the radio lobes through the jets. How long time has it taken to fill the lobes with the energy that we deduce to be contained in magnetic fields and electrons? Assume that the lobes have not lost much energy through radiation.

g) Calculate the cooling time (i.e. the radiative lifetime) for electrons at $\gamma_{\rm min}$? At $\gamma_{\rm max}$? Compare with the results from f). Discuss the implications.



Radio spectrum of Cygnus A radio galaxy. 1 Jy = 10^{-23} erg s^{-1} cm^{-2} Hz^{-1}.