HIGH ENERGY ASTROPHYSICS

Compulsary Home Exercises. Problem set 1. Solutions.

1.1: Consider three model mass density profiles in a neutron star of radius R: (1) $\rho(r) = \rho_c$; (2) $\rho(r) = \rho_c\,[1 - (r/R)^2]$, where r is radial coordinate within the star and ρ_c is the central density. Take the canonical neutron star model with $M = 1.4\,M_\odot$ and $R = 10$ km. Find ρ_c (expressed in units of standard nuclear matter density $\rho_0 = 2.8 \times 10^{14}$ g cm^{-3}).

Solution:

$$M = \int_0^R 4\pi r^2 \rho(r) dr = \frac{4\pi}{3} R^3 \bar{\rho} = V\bar{\rho}$$

where $\bar{\rho} = M/V = 6.65 \times 10^{14}$g cm^{-3}=$2.38\rho_0$ is the average density.

Case 1:

$$M = \frac{4\pi}{3} R^3 \rho_c = V\rho_c,$$

as thus

$$\rho_c = \bar{\rho} = 2.38\rho_0.$$

Case 2:

$$M = \int_0^R 4\pi r^2 \rho_c\,[1 - (r/R)^2] dr = 4\pi R^3 \rho_c \int_0^1 x^2(1 - x^2) dx = 4\pi R^3 \rho_c \frac{2}{15} = \frac{2}{5} V\rho_c,$$

where $x = r/R$ and $\int_0^1 x^2(1 - x^2)dx = 1/3 - 1/5 = 2/15$. Thus we get

$$\rho_c = \frac{5}{2}\bar{\rho} = 5.95\rho_0.$$

1.2: Calculate moment of inertia I of a neutron star of M and radius R for the two model density profiles (from problem 1.1, neglecting the effects of General Relativity). Evaluate I for a canonical neutron star with $M = 1.4\,M_\odot$ and $R = 10$ km.

Solution: $MR^2 = 1.4 \times 2 \times 10^{33} \times 10^{12} = 2.79 \times 10^{45}\,\mathrm{g\,cm^2}$. General expression for I:

$$I = \int_0^R r^2 dr \int_0^\pi \sin\theta d\theta \int_0^{2\pi} d\phi\, \rho(r)(r^2 \sin^2\theta) = 2\pi \int_{-1}^1 (1-\cos^2\theta) d\cos\theta \int_0^R \rho(r) r^4 dr = \frac{8\pi}{3} R^5 \int_0^1 \rho(x) x^4 dx.$$

Case 1: $\rho_c = M/V = 3M/4\pi R^3$,

$$I = \frac{8\pi}{3} R^5 \frac{\rho_c}{5} = \frac{8\pi}{15} R^5 \frac{3M}{4\pi R^3} = \frac{2}{5} MR^2 = 1.11 \times 10^{45}\,\mathrm{g\,cm^2}.$$

Case 2: $\rho_c = 5M/2V = 15M/8\pi R^3$, $\int_0^1 x^4(1-x^2)dx = 1/5 - 1/7 = 2/35$.

$$I = \frac{8\pi}{3} R^5 \rho_c \left(\frac{1}{5} - \frac{1}{7}\right) = \frac{8\pi}{3} R^5 \frac{2}{5 \cdot 7} \frac{15M}{8\pi R^3} = \frac{2}{7} MR^2 = 0.78 \times 10^{45}\,\mathrm{g\,cm^2}.$$

1.3: Estimate at what density inside a neutron star neutrons become degenerate. Assume temperature of the neutron star of 10^9 K. At what density neutrons become relativistic?

Solution: The neutron concentration when they become degenerate is

$$n_n = \frac{8\pi}{3} \frac{(3m_n kT)^{3/2}}{h^3} = 1.66 \times 10^{34}\ \mathrm{cm}^{-3},$$

this corresponds to the density

$$\rho = n_n m_n = 2.78 \times 10^{10}\ \mathrm{g\,cm}^{-3},$$

They become relativistic when the Fermi momentum

$$p_F = \left(\frac{3h^3 n_n}{8\pi}\right)^{1/3}$$

becomes comparable to $m_n c$. This happens at

$$n_n = \frac{8\pi}{3}\left(\frac{m_n c}{h}\right)^3 = 3.6 \times 10^{39}\ \mathrm{cm}^{-3},$$

corresponding to the density

$$\rho = n_n m_n = 6.1 \times 10^{15}\ \mathrm{g\,cm}^{-3},$$

which is not achieved in neutron star cores.

1.4: Suppose a spherical body of a radius R_1 is spinning originally at a rate ν_1 Hz collapses to a body of radius R_2 conserving its mass M and angular momentum L. Express the ratio of the new and old spin rates ν_2/ν_1 and the new and old rotational energies E_2/E_1 in terms of the ratio R_2/R_1. You may assume the moment of inertia for a homogeneous sphere. By what factor would the star spin faster if it were to collapse from a radius typical of a white dwarf to the dimensions typical of a neutron star? By what factor would the rotational energy increase in such a collapse? Where ultimately does this energy come from?

Solution: The angular momentum is

$$L = I\Omega,$$

where $\Omega = 2\pi\nu$, $I = (2/5)MR^2$. Thus

$$\nu_1 R_1^2 = \nu_2 R_2^2 \Rightarrow \frac{\nu_2}{\nu_1} = \frac{R_1^2}{R_2^2}.$$

Rotational energy $E = I\Omega^2/2$, thus

$$\frac{E_2}{E_1} = \frac{I_2}{I_1}\frac{\nu_2^2}{\nu_1^2} = \frac{R_2^2}{R_1^2}\frac{R_1^4}{R_2^4} = \frac{R_1^2}{R_2^2}.$$

For WD to become NS, radius changes by a factor $\sim 10^3$, so the frequency and the energy increase by a factor $\sim 10^6$. Energy comes from gravitational energy.

1.5: A neutron star cannot spin with less than a certain period or it will start to shed mass from its equator due to centrifugal force. Consider a neutron star of mass M and radius R. Show that this critical period is

$$P_{\min} = K \left(\frac{1.4M_\odot}{M}\right)^{1/2} \left(\frac{R}{10\,\mathrm{km}}\right)^{3/2} \text{ ms},$$

where K is a constant. Compute K using Newtonian gravity (neglect also any deformation of neutron star due to rotation). Calculations using general relativity give $K = 0.77$. Compute the limit on the radius of the neutron star which has a period $P = 1.4$ ms.

Solution: The neutron star angular frequency should be smaller that the Keplerian one

$$\Omega_K = \sqrt{\frac{GM}{R^3}}.$$

The limiting period is then

$$P_{\min} = \frac{2\pi}{\Omega_K} = \frac{2\pi}{\sqrt{G}} M^{-1/2} R^{3/2} = 0.46\ (1.4M_\odot/M)^{1/2} (R/10\,\mathrm{km})^{3/2}\,\mathrm{ms}.$$

The upper limit on the radius for $P = 1.4$ ms is (taking $K = 0.77$)

$$R = 14.9\ (M/1.4M_\odot)^{1/3}\ \mathrm{km}.$$

1.6: Crab pulsar has period of $P = 0.033$ s and period derivative $\dot{P} = 4.21 \times 10^{-13}$ s/s. Estimate the age of the pulsar. Compare it with the true age (SN in year 1054). Estimate the pulsar magnetic field using magnetic dipole radiation formula.

Solution: The characteristic age is

$$\tau = \frac{P}{2\dot{P}} = 3.9 \times 10^{10}\ \mathrm{s} = 1250\ \mathrm{yr}.$$

The actual age is about 970 yr, i.e. slightly smaller. The magnetic field is

$$B = 6 \times 10^{19} \sqrt{P\dot{P}}\ \mathrm{G} = 7 \times 10^{12}\ \mathrm{G}.$$