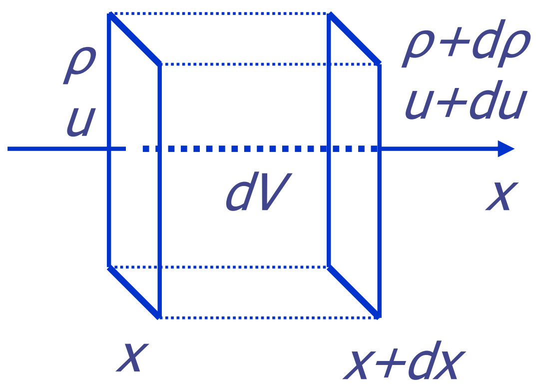
Spherical accretion

Literature:

Frank, King, Raine, Accretion Power in Astrophysics, 2002

Conservation laws

- Gasdynamics can be applied if particles collide many times before crossing the region, i.e. the mean free path λ<< typical size L. Gas can be described by P, ρ, T, u.
- Mass conservation. Continuity equation:



$$\partial(\rho dx) = \partial t[\rho u - (\rho + d\rho)(u + du)] = \partial t[-ud\rho - \rho du]$$

$$\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + \rho\frac{\partial u}{\partial x} = 0 \Rightarrow \frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} = 0$$

$$\text{if } \frac{\partial}{\partial t} = 0 \Rightarrow \rho u = \text{const}$$

- In 3D:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{u}) = 0$$

Momentum conservation. Euler's equation.

- We assume that the gas pressure is the only force acting on the gas.
- In astrophysics other forces are often important: magnetic, gravitational, viscous, and radiation pressure forces

$$\frac{\partial(\rho u dx)}{\partial t} = [\rho u^2 - (\rho + d\rho)(u + du)^2] + P - (P + dP)$$

$$\frac{\partial(\rho u)}{\partial t} = -\frac{\partial P}{\partial x} - u^2\frac{\partial \rho}{\partial x} - 2\rho u\frac{\partial u}{\partial x}$$

$$\rho\frac{\partial u}{\partial t} + u\frac{\partial \rho}{\partial t} = \rho\frac{\partial u}{\partial t} + u\left[-u\frac{\partial \rho}{\partial x} - \rho\frac{\partial u}{\partial x}\right] \quad \Leftarrow \text{ from mass conservation}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = -\frac{1}{\rho}\frac{\partial P}{\partial x}$$

$$\text{For } \frac{\partial}{\partial t} = 0 \text{ and using mass conservation } \rho u = \text{const} \Rightarrow \rho u^2 + P = \text{const}$$

- In 3D:

$$\frac{\partial \vec{u}}{\partial t} + (\vec{u} \cdot \nabla)\vec{u} = -\frac{1}{\rho}\nabla P + \frac{\vec{F}}{m}$$

- **Conservation of energy.** Most complicated equation. Consider two simple limiting cases.
- **Adiabatic flow.** No radiation losses or input, no conduction. Volume element perform work on surrounding. The equation of state:

$$P = K\rho^{\gamma}, \quad \gamma = c_p / c_V \text{ is the ratio of specific heats.}$$

$$\gamma = 5/3 \text{ monoatomic gas}, \quad = 7/5 \text{ for diatomic gas.}$$

- **Isothermal flow.** Temperature is determined by heating=cooling balance. T is adjusted much faster than dynamical time-scale.

$$T = \text{const}$$

Sound waves

- Instead of the energy equation: $P = K\rho^{\gamma}$ $\gamma = 5/3$ - adiabatic, $\gamma = 1$ - isothermal
- Consider gas at rest $P = P_0 = const,\ \rho = \rho_0 = const,\ u = 0$

and small perturbations

$$P = P_0 + P_1,\ \rho = \rho_0 + \rho_1,\ u = u_1$$

$$P_1 = K\gamma\rho_0^{\gamma-1}\rho_1 = \gamma\frac{P_0}{\rho_0}\rho_1,\ \text{define}\ \ c_s^2 \equiv \gamma\frac{P_0}{\rho_0}$$

- Continuity equation:

$$\frac{\partial(\rho_0+\rho_1)}{\partial t} + u_1\frac{\partial\rho_0}{\partial x} + (\rho_0+\rho_1)\frac{\partial u_1}{\partial x} = 0 \Rightarrow \frac{\partial\rho_1}{\partial t} + \rho_0\frac{\partial u_1}{\partial x} = 0 \Rightarrow \frac{1}{\rho_0}\frac{\partial^2\rho_1}{\partial t^2} + \frac{\partial^2 u_1}{\partial t\partial x} = 0$$

- Momentum equation:

$$\frac{\partial u_1}{\partial t} + u_1\frac{\partial u_1}{\partial x} = -\frac{1}{\rho}\frac{\partial(P_0+P_1)}{\partial x} \Rightarrow \frac{\partial u_1}{\partial t} + \frac{1}{\rho_0}\frac{\partial P_1}{\partial x} = \frac{\partial u_1}{\partial t} + \frac{1}{\rho_0}c_s^2\frac{\partial\rho_1}{\partial x} = 0$$

$$\Rightarrow \frac{\partial^2 u_1}{\partial t\partial x} + \frac{c_s^2}{\rho_0}\frac{\partial^2\rho_1}{\partial x^2} = 0$$

- The wave equation

$$\frac{\partial^2\rho_1}{\partial t^2} - c_s^2\frac{\partial^2\rho_1}{\partial x^2} = 0,\quad \frac{\partial^2 u_1}{\partial t^2} - c_s^2\frac{\partial^2 u_1}{\partial x^2} = 0$$

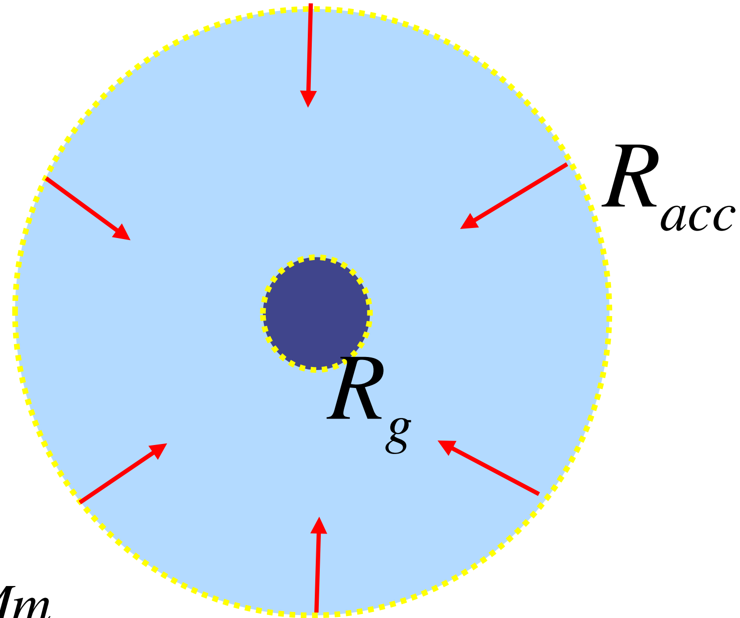
- Solution: $\rho_1 = f(x \pm c_s t)$ if $u_0 \neq 0$ sound propagates with $u_0 \pm c_s$.
- Note

$$c_s \propto \rho^{(\gamma-1)/2},\ \ \gamma = 5/3,\ \ c_s \propto \rho^{1/3}\ \text{adiabatic;}$$

$$\gamma = 1,\ \ c_s = \text{const} - \ \text{isothermal}$$

Spherical accretion

- Given a large cloud with $T(\infty)$ and $\rho(\infty)$. Black hole (BH) in the cloud is an ideal "vacuum cleaner".
- $R_g = 2GM/c^2$ is the radius of the absorbing surface, the event horizon.
- R_{acc} is the radius of which the BH gravitational pull dominates over thermal motions in the cloud:
- For $M = M_\odot$ and $c_s = 10$ km/s, $R_{acc} = 1.3\ 10^{14}$ cm $= 10$ au.



$$\frac{GMm_p}{R_{acc}} \approx kT \Rightarrow$$

$$R_{acc} \approx \frac{GMm_p}{kT} = \frac{GM}{kT/m_p} \approx \frac{GM}{P/\rho} \approx \frac{GM}{c_s^2}$$

$$\left[P = \frac{\rho}{\mu m_p} kT \Rightarrow c_s^2 = \frac{\partial P}{\partial \rho} \right]$$

- What is the accretion rate $\dot{M}$?
- What is the inflow velocity profile $V(r)$?
- What is the gas temperature profile $T(r)$?
- What is the emission from the flow?

- Consider steady-state solution for a spherically symmetric flow.
- Simple equations of hydrodynamics

Continuity equation

$$\frac{\partial\rho}{\partial t}+\nabla\cdot(\rho\vec{V})=0 \quad\Rightarrow\quad \frac{1}{r^2}\frac{d}{dr}\left(r^2\rho V\right)=0$$

$$(1)\quad \frac{d\dot{M}(r)}{dr}=0,\quad \dot{M}=4\pi\, r^2\rho(r)V(r)$$

Euler equation

$$\rho\left(\frac{\partial\vec{V}}{\partial t}+(\vec{V}\cdot\nabla)\vec{V}\right)=-\nabla P+\vec{f}\Rightarrow$$

$$\rho\frac{dV}{dt}=-\frac{dP}{dr}+f$$

External force (gravitational) per cm^3

- ***V***- speed of a given element. But if we are interested in velocity profile at a given space coordinate, we have to make a transformation:

$$dV=\frac{\partial V}{\partial t}dt+\frac{\partial V}{\partial r}dr$$

- Change of velocity consists of two parts.

- change of the velocity in a given point during time *dt*
- Difference in velocities (at the same moment *t*) in two points separated by *dr*, which is the distance the elements moves in *dt*:

$$\frac{dV}{dt}=\frac{\partial V}{\partial t}+\frac{dr}{dt}\frac{\partial V}{\partial r}$$

In steady-state

$$\rho\left[\frac{\partial V}{\partial t}+V\frac{\partial V}{\partial r}\right]=-\frac{dP}{dr}+f\Rightarrow\rho V\frac{dV}{dr}=-\frac{dP}{dr}-\frac{GM\rho}{r^2}\quad(2)$$

Equation of state

(3) $P = \dfrac{\rho k T}{\mu m_H}$ - ideal gas, μ is the mean mass per particle

$\mu = 1$ for neutral H, $\mu = 1/2$ for ionized H.

The Mach number

$\mathbf{M} = \dfrac{V}{c_s}$, where c_s is the sound speed

$\mathbf{M} < 1$- subsonic flow. In the limit $\mathbf{M} << 1$ hydrostatics

$\mathbf{M} > 1$- supersonic flow. In the limit $\mathbf{M} >> 1$ free-fall

Polytropic flow

$P = K\rho^{\gamma}$

Examples: adiabatic flow, $\gamma = 5/3$; isothermal flow ($T = const$), $\gamma = 1$.

$$c_s^2 \equiv \frac{\partial P}{\partial \rho} = \gamma \frac{P}{\rho}$$, c_s - sound speed

$$\left.\begin{aligned}
&\frac{dP}{dr} = \frac{dP}{d\rho}\frac{d\rho}{dr} = c_s^2 \frac{d\rho}{dr} \\
&(1) \Rightarrow \frac{d\rho}{dr} = -\frac{\rho}{V}\frac{dV}{dr} - \frac{2\rho}{r} \\
&(2) \Rightarrow \rho V \frac{dV}{dr} = -\frac{dP}{dr} - \frac{GM\rho}{r^2}
\end{aligned}\right\} \Rightarrow V\frac{dV}{dr}\left[1-\mathbf{M}^{-2}\right] = \frac{2c_s^2}{r}\left[1-\frac{GM}{2rc_s^2}\right] \quad (4)$$

at $r \to \infty$, $\mathbf{M} \to 0$ $\quad [\ \mathrm{V} \to 0,\ c_s \to c_s(\infty)\ ]$

at $r \to R_g$, $\mathbf{M} > 1$ $\quad \left[V\frac{dV}{dr}<0,\ 1-\frac{c^2}{c_s^2}<0 \Rightarrow 1-\mathbf{M}^{-2}>0\right]$

Hence at some point $\mathbf{M}=1$ - transonic flow

Let $\mathbf{M}=1$ at $r=r_s$ (sonic point)

Then from (4) at $r=r_s$ $\quad \frac{dV}{dr}(r_s)=\infty$ or $\boxed{c_s^2(r_s) = \frac{GM}{2r_s}}$ ← We choose a physical solution

[The general "regularity" condition for any transonic flow. After expressing explicitely $V\frac{dV}{dr}=\frac{\mathrm{N}(r)}{\mathrm{D}(r)}$. There exists such r_s that $\mathrm{N}(r)=\mathrm{D}(r)=0$]

For a polytropic gas, one can integrate the Euler equation (2) to get the Bernoulli equation

$$\frac{V^2}{2}+\frac{c_s^2}{\gamma-1}-\frac{GM}{r}=\text{const}$$

The constant can be found by considering the limit $r \to \infty$. Then $V \to 0, GM/r \to 0$ and const $= c_s^2(\infty)/(\gamma-1)$.

We thus get a set of algebraic equations describing the accretion problem:

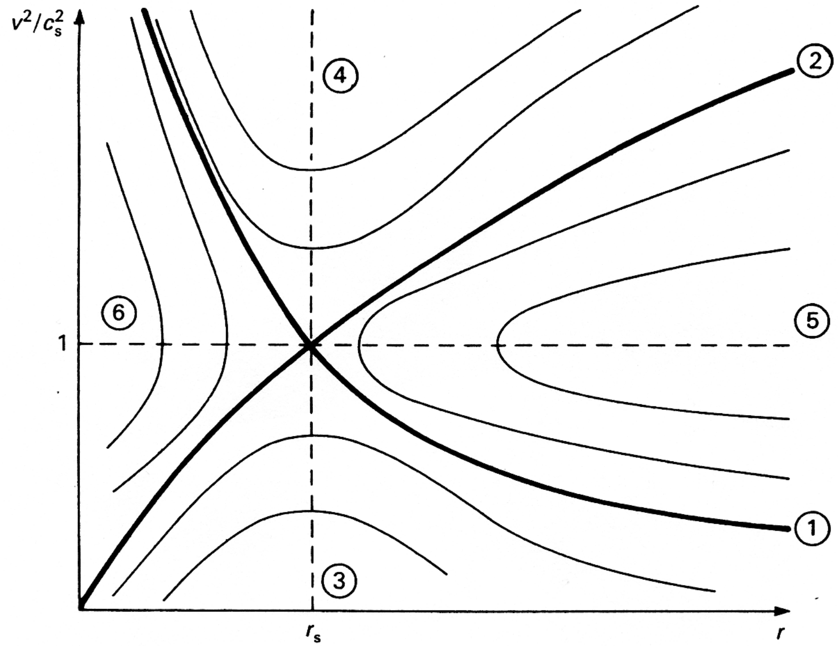


Figure 1. Spherical adiabatic gas flows $v^2(r)/c_s^2(r)$ in the gravitational field of a star. For $v < 0$ these are accretion flows, while for $v > 0$ they are winds or 'breezes'. The two trans-sonic solutions 1, 2 divide the remaining solutions into the families 3–6 described in the text.

(1) $\dot{M} = 4\pi r^2 \rho V = \text{const}$ - to be found (continuity)

(2) $\frac{V^2}{2}+\frac{c_s^2}{\gamma-1}-\frac{GM}{r}=\frac{c_s^2(\infty)}{\gamma-1}$ (Bernoulli equation)

(3) $\left[\frac{\rho}{\rho(\infty)}\right]^{\gamma-1} = \left[\frac{c_s}{c_s(\infty)}\right]^2$ (the polytropic condition)

(4) $c_s(r_s) = V(r_s)$

(5) $c_s^2(r_s) = \frac{GM}{2r_s}$

(the regularity conditions)

The case γ=5/3 (adiabatic accretion)

At home you will show that

$$(a)\quad \dot{M} = \pi G^2 M^2 \frac{\rho(\infty)}{c_s^3(\infty)} \left[\frac{2}{5-3\gamma}\right]^{(5-3\gamma)/2(\gamma-1)}$$

$$(b)\quad r_s = \frac{GM}{c_s^2(\infty)} \frac{5-3\gamma}{4}$$

When $\gamma \to 5/3,\ r_s$ decreases from $GM/c_s^2(\infty)$ to $r_s = 0$.

As a result, the transonic region get "stretched out" into the whole range of radii

$r << GM/c_s^2(\infty)$.

The flow is near the regularity condition.

$$\left.\begin{array}{l} V(r) \approx c_s(r) \\ c_s^2(r) \approx \dfrac{GM}{2r} \end{array}\right] \text{at } r << \frac{GM}{c_s^2(\infty)}$$

$$\boxed{\mathbf{M} \approx 1,\ V(r) \approx \sqrt{\frac{GM}{2r}}}$$

The accretion velocity is about virial, $V_{vir} \approx \sqrt{GM/r}$, and the gas temperature is about virial $kT_{vir} \approx GMm_p/r$ (since $c_s^2 \approx kT_{vir}/m_p$). The gas gets heated with decreasing r ($T \propto 1/r$) owing to "adiabatic heating", $dU = -PdV_{vol}$ (1st law of thermodynamics).

The temperature reaches $kT \approx m_e c^2$ at $r_* \approx \frac{GM}{c^2} \frac{m_p}{m_e} \approx R_g \frac{m_p}{m_e}$

Inside r_* the electrons are relativistic (their internal energy $\gg m_e c^2$).

Then the adiabatic index changes.

One can show that internal energy density of a polytropic gas satisfies the relation $\varepsilon = P/(\gamma - 1)$

Ionized non-relativistic plasma, $kT \ll m_e c^2$

$$\left.\begin{aligned} \varepsilon &= \frac{3}{2} nkT + \frac{3}{2} nkT = 3nkT \\ P &= nkT + nkT = 2nkT \end{aligned}\right\} \Rightarrow \gamma = 5/3.$$

Ionized relativistic plasma, $m_e c^2 \ll kT \ll m_p c^2$

$$\left.\begin{aligned} \varepsilon &= \frac{3}{2} nkT + 3nkT = \frac{9}{2} nkT \\ P &= nkT + \frac{1}{3} 3nkT = 2nkT \end{aligned}\right\} \Rightarrow \gamma = 13/9.$$

At r_* the index γ changes from 5/3 to 13/9. Since $13/9 < 5/3$, r_s shifts from 0 to a value $\le r_*$ and the flow becomes supersonic at $r \ll r_*$.

Emission from the accretion flow

Most of the energy is released in the very vicinity of the BH, at radii $r \approx R_g$

The innermost region makes the main contribution to the observed luminosity.

Therefore, to estimate the luminosity and spectrum, one needs to know the flow parameters at $r \approx R_g << r_*$. Here the gas is nearly free-falling (**M**>>1) with velocity $\approx c$.

The density can be found from the continuity equation (mass conservation):

$$\dot{M} = 4\pi r^2 \rho V \Rightarrow \rho = \frac{\dot{M}}{4\pi r^2 V} \approx \frac{\dot{M}}{4\pi R_g^2 c}$$

The temperature is determined from the Bondi solution with γ=13/9 which should match the solution with γ=5/3 at $r \approx r_*$.

At home you proove: $\frac{T(r)}{T_*} \approx \left(\frac{r_*}{r}\right)^{2/3}$, at $r << r_*$.

Thus at R_g, $T(R_g) \approx m_e c^2 \left(\frac{m_p}{m_e}\right)^{2/3} \approx 70$ MeV.

The observed spectrum is produced by a plasma cloud of size R_g with temperature $T(R_g)$ and density $\rho(R_g)$.

For the densities and temperatures of interest, the dominant emission is free-free emission (if no magnetic field).

$\Lambda_{ff} = \Lambda_{ei} + \Lambda_{ee}$ erg/cm^3 / s (emission rate per unit volume).

$$\Lambda_{ei} = 12\alpha r_e^2 n^2 ckT\left[\frac{3}{2} + \ln\frac{2kT}{m_e c^2} - 0.577\right]$$

$$\Lambda_{ee} = 24\alpha r_e^2 n^2 ckT\left[\frac{5}{4} + \ln\frac{2kT}{m_e c^2} - 0.577\right], \text{ for } kT >> m_e c^2.$$

Here $\alpha = \frac{e^2}{\hbar c} = 1/137$ - the fine structure constant, r_e is the classical electron radius.

The luminosity from the accretion flow:

$$L_{ff} = \int_{R_g}^{\infty} \Lambda_{ff} 4\pi r^2 dr \approx \Lambda_{ff}\frac{4\pi}{3}R_g^3 \approx 10^{21}\left(\frac{n_\infty}{1\text{ cm}^{-3}}\right)^2\left(\frac{T_\infty}{10^4\text{ K}}\right)^{-3}\left(\frac{M}{M_{sun}}\right)^3 \text{erg/s}$$

The resulting (free-free) spectrum is $L_\nu \propto \nu^0$ with an exponential cutoff at $h\nu \approx kT \approx 70$ MeV.

Note: the luminosity is low $\frac{L_{ff}}{L_{Edd}} \approx 10^{-17}$.

In the presence of magnetic field, an important emission mechanism is synchrotron emission. The synchrotron emissivity of hot relativistic ($kT >> m_e c^2$) electrons:

$$\Lambda_{syn} = \frac{16}{3}\frac{e^2}{c}\left(\frac{eB}{m_e c}\right)^2\left(\frac{kT}{m_e c^2}\right)^2 n = \frac{16}{3} r_e^2 c B^2 n \left(\frac{kT}{m_e c^2}\right)^2 \ \text{erg/cm}^3\text{/s}$$

The plausible magnetic field (Shvartsman 1971), is the "equipartition" field:

$$\frac{B^2}{8\pi} \approx \frac{GM\rho}{r}.$$

Show that the resulting synchrotron luminosity from the accretion flow:

$$L_{syn} \approx 10^{27}\left(\frac{n_\infty}{1\ \text{cm}^{-3}}\right)\left(\frac{T_\infty}{10^4\ \text{K}}\right)^{-3}\left(\frac{M}{M_{sun}}\right)^3 \text{erg/s}$$

The luminosity from spherical accretion is thus low even in the presence of the equipartition magnetic field.

The radiative efficiency of accretion

$$\eta = \frac{L_{syn}}{\dot{M}c^2} \leq 10^{-4}.$$

- Where this theory is applicable? In situations where a black hole is surrounded by low angular momentum gas
  - Center of the Milky Way : the central black hole is surrounded by gas originating from stellar winds. Accretes with little angular momentum onto black hole.
  - Center of elliptical galaxies : interstellar medium is hot (~5 million K; X-ray emitting) and accretes with low angular momentum onto central supermassive black hole.

- In both these cases the black holes are very quiescent, i.e., they produce little electromagnetic radiation.