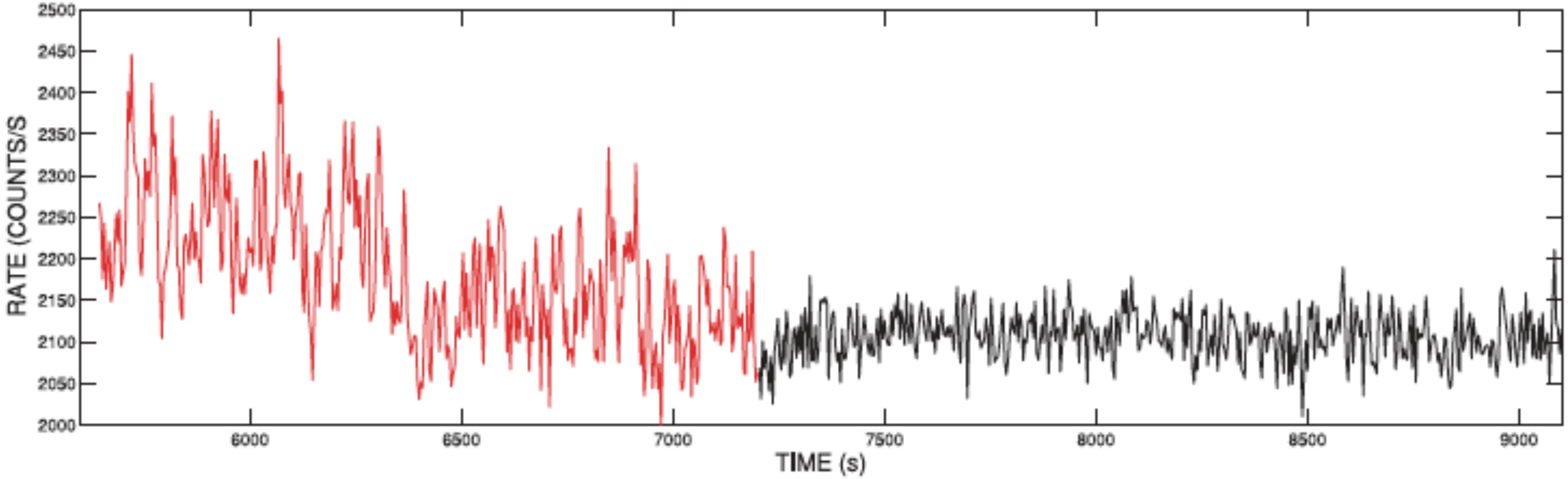
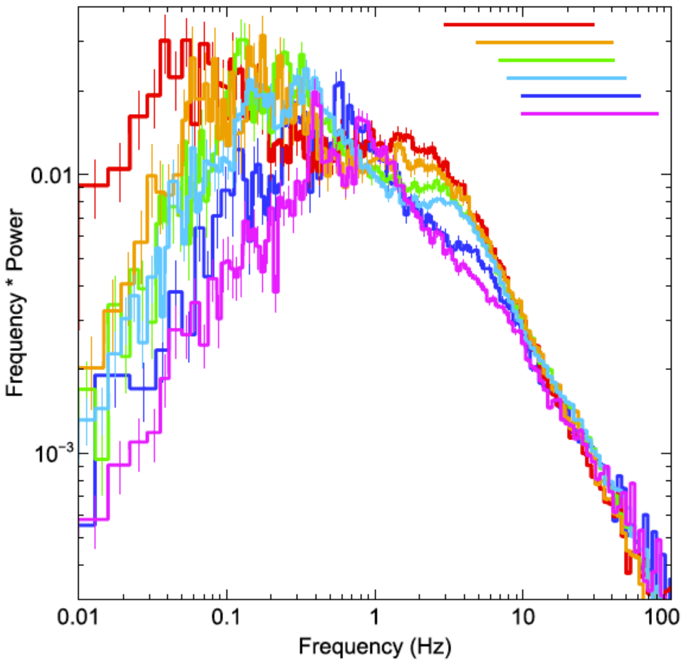
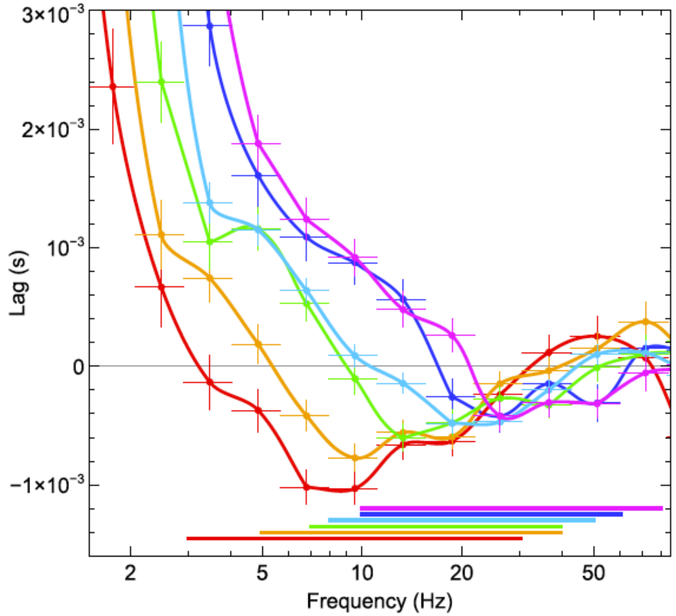


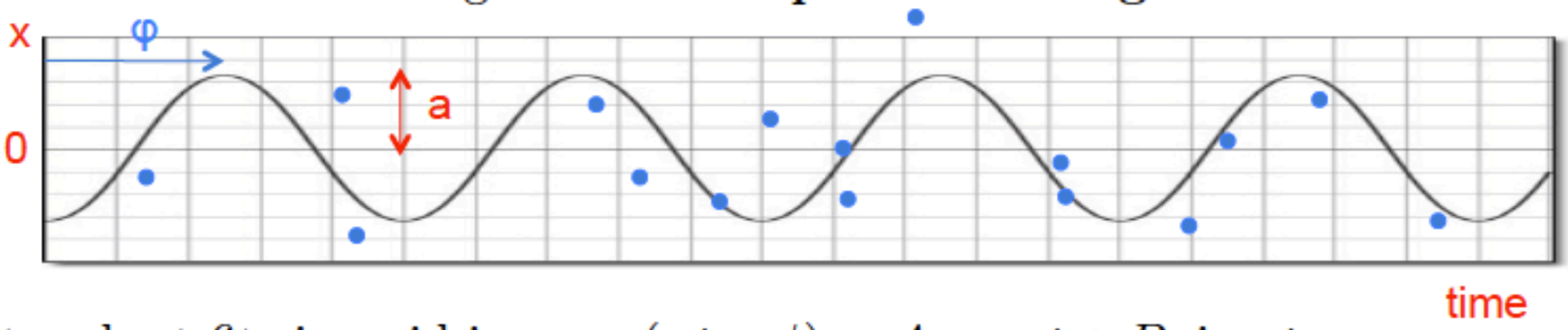
Fast variability of X-ray binaries





Fourier transform

Fourier transform of signal = **decomposition of signal into sine waves.**



At ω, best-fit sinusoid is: $a\cos(\omega t - \phi) = A\cos\omega t + B\sin\omega t$
$(a = \sqrt{A^2 + B^2}$ and $\tan\phi = -B/A)$

Do this at many frequencies ω_j, then

$$x(t) = \frac{1}{N}\sum_j a_j \cos(\omega_j t - \phi_j) = \frac{1}{N}\sum_j (A_j \cos\omega_j t + B_j \sin\omega_j t)$$

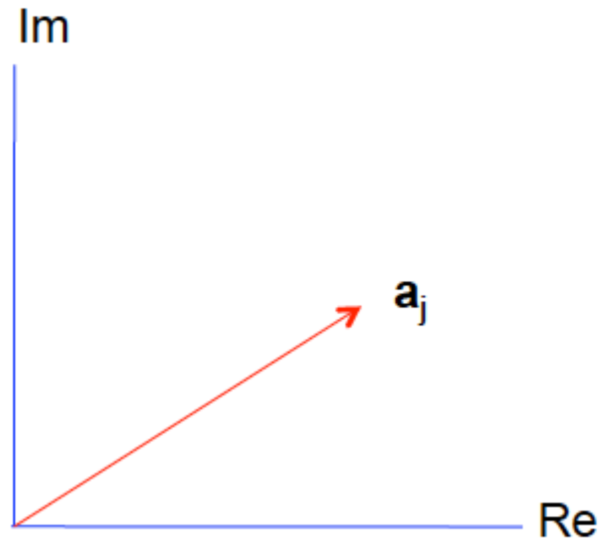
Fourier: $A_j = \sum_k x_k \cos\omega_j t_k$; $B_j = \sum_k x_k \sin\omega_j t_k$

So: **correlate** data with sine and cosine wave.
Good correlation: large A, B — bad correlation: small

Fourier transform

A **way of handling** the two numbers $(A, B$ or $a, \phi)$ you get at each ω.

$$a_j = \sum_k x_k e^{i\omega_j t_k}$$
$$x_k = \frac{1}{N} \sum_j a_j e^{-i\omega_j t_k}$$

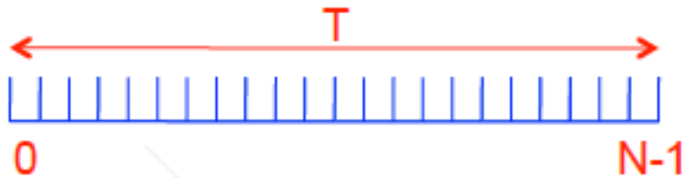


The **Fourier amplitudes** a_j are complex numbers:
$a_j = |a_j| e^{i\phi_j} = |a_j|(\cos\phi_j + i \sin\phi_j)$

If the signal x_k is real then imaginary terms at $+j$ and $-j$ cancel out in $\sum_j$,

to produce strictly real terms $2|a_j| \cos(\omega_j t_k - \phi_j)$

Discrete Fourier transform



Time series: $x_k, \quad k = 0, \ldots, N-1$

Transform: $a_j, \quad j = -\dfrac{N}{2}+1, \ldots, \dfrac{N}{2}$

$$a_j = \sum_{k=0}^{N-1} x_k e^{2\pi ijk/N} \qquad j = -\frac{N}{2}+1, \ldots, \frac{N}{2}$$

$$x_k = \frac{1}{N} \sum_{j=-N/2+1}^{N/2} a_j e^{-2\pi ijk/N} \qquad k = 0, \ldots, N-1$$

Time step $\delta t = \dfrac{T}{N}$; $\boxed{\text{Frequency step } \delta\nu = \dfrac{1}{T}}$

x_k refers to time $t_k = \dfrac{kT}{N}$; a_j refers to frequency $\omega_j = 2\pi\nu_j = \dfrac{2\pi j}{T}$

So, for $e^{i\omega_j t_k}$ we have written $e^{2\pi ijk/N}$

Discrete Fourier transform

- Fourier theorem: transform gives **complete** description of signal
- Highest frequency you need for this is the **Nyquist frequency**

$$\nu_{Ny} = \nu_{N/2} = \frac{N}{2T} = \text{half the sampling frequency } \frac{1}{\delta t} = \frac{N}{T}, \text{ as}$$

 "up-down" is the fastest observable frequency.

$a_{N/2} = \sum_k x_k e^{i\pi k} = \sum_k x_k (-1)^k$ for real x_k is always real

- Lowest frequency (>0) = frequency of first frequency step $= \dfrac{1}{T} =$
  = frequency of sinusoid that fits exactly once on T
- At zero frequency you get $a_0 = \sum_k x_k$, also always real for real x_k. (Called the **DC component**)
- At all frequencies in between you get complex Fourier amplitudes a_j, so:
- N, the number of input values $x_k \equiv$ number of output values; count them: a_0; $(|a_j|, \phi_j)$ pairs for $j = 1, \ldots, N/2 - 1$; $a_{N/2}$.
- Orthogonal, if the x_k are uncorrelated then the a_j are uncorrelated.

Continuous Fourier transform

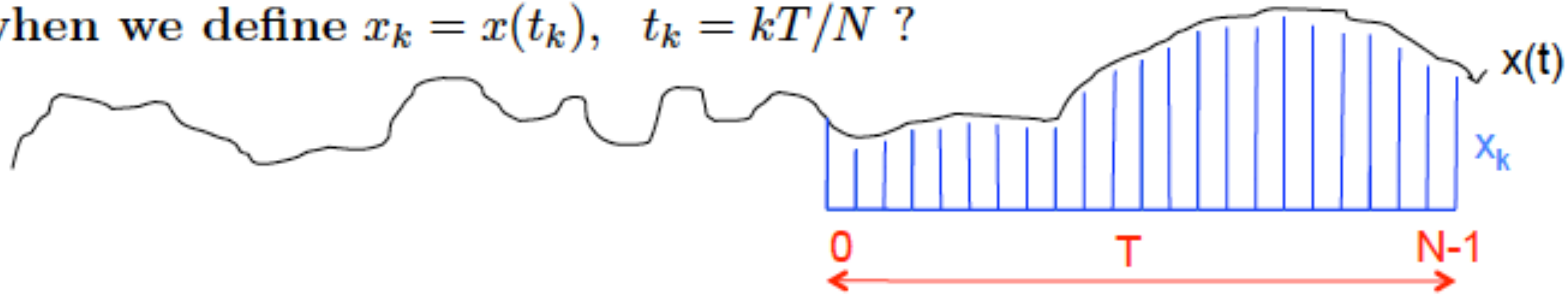
Decomposes a **function** into an **infinite** number of sinusoidal waves.

Signal $x(t) \quad -\infty < t < \infty$

Transform $a(\nu) \quad -\infty < \nu < \infty$

$$a(\nu) = \int_{-\infty}^{\infty} x(t) e^{2\pi\nu i t}\, dt \qquad -\infty < \nu < \infty$$
$$x(t) = \int_{-\infty}^{\infty} a(\nu) e^{-2\pi\nu i t}\, d\nu \qquad -\infty < t < \infty$$

What is the relation of this 'ideal case' with the discrete Fourier transform **when we define** $x_k = x(t_k), \quad t_k = kT/N$?



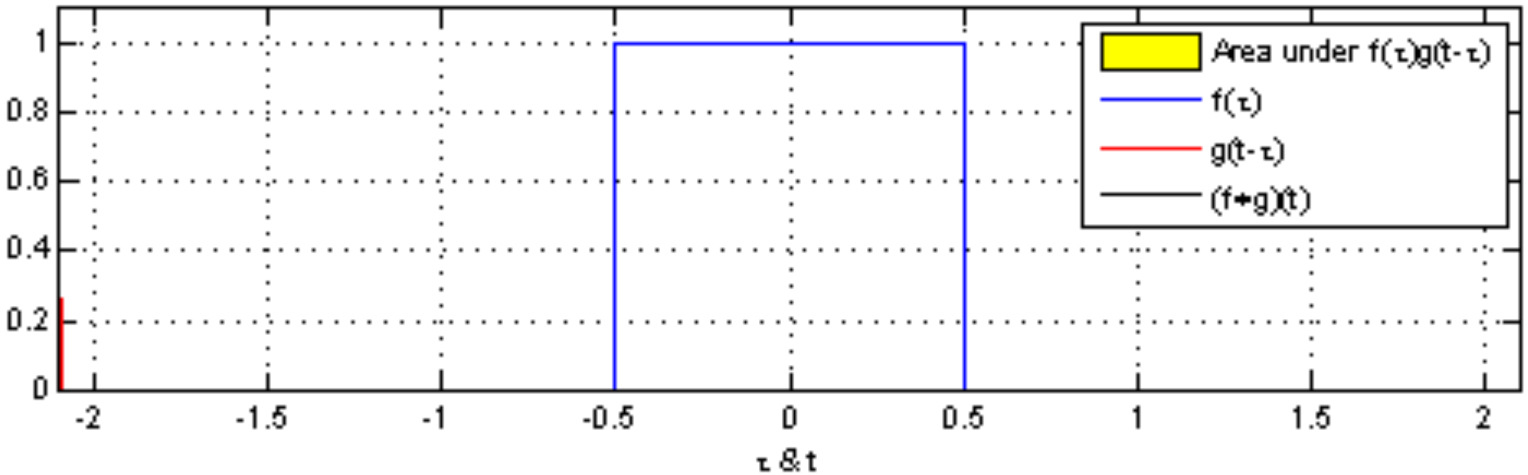
CONVOLUTION THEOREM

If $a(\nu)$ is the Fourier transform of $x(t)$ and
$b(\nu)$ is the Fourier transform of $y(t)$ then:

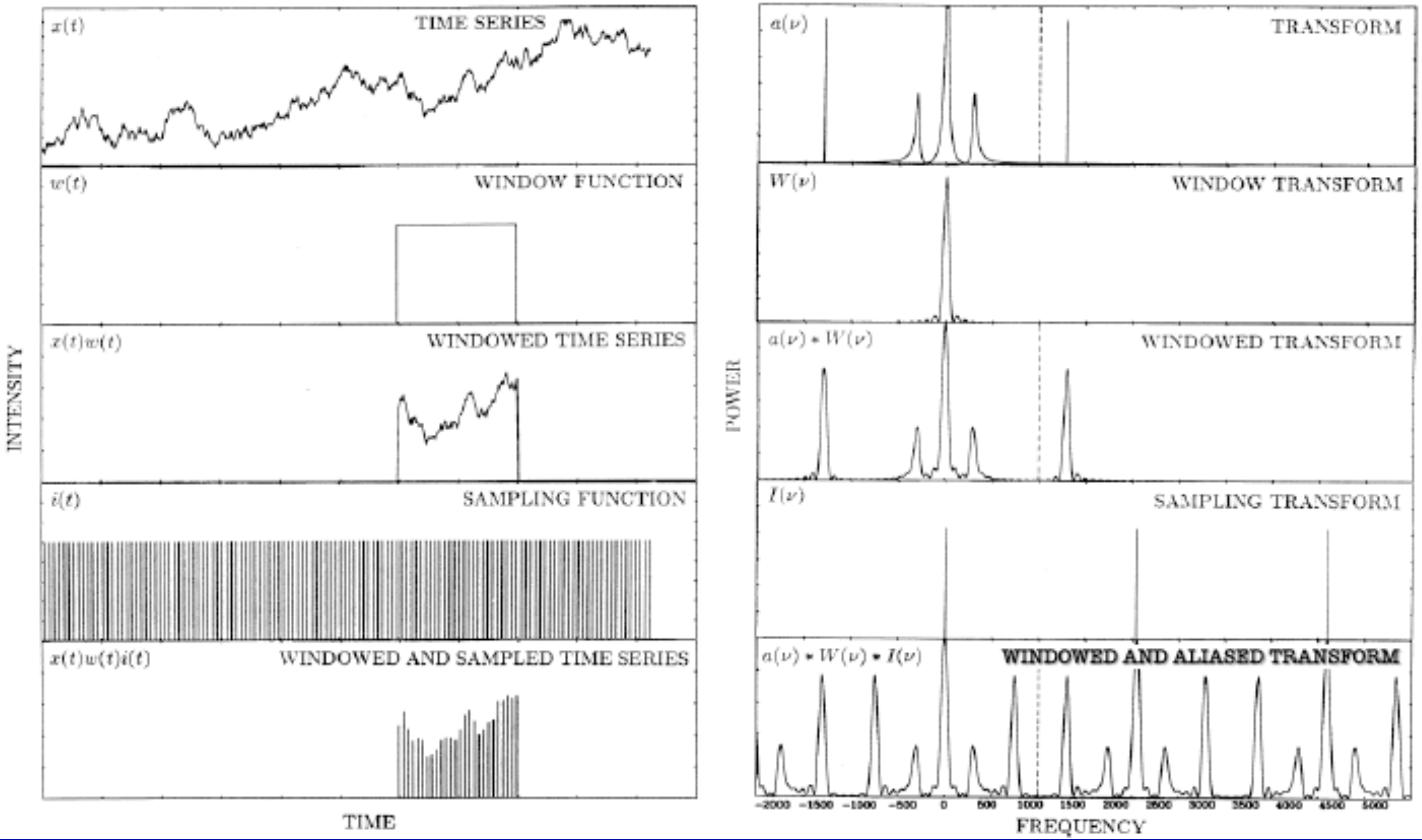
the transform of the product $x(t) \cdot y(t)$ is the convolution of $a(\nu)$ and $b(\nu)$:

$$a(\nu) \circledast b(\nu) \equiv \int_{-\infty}^{\infty} a(\nu')b(\nu - \nu')d\nu'$$

"the transform of the product is the convolution of the transforms" (and vv).
[Convolution denoted by $\circledast$]



Continuous vs Discrete FT



So: the discrete Fourier amplitudes are values **at the Fourier frequencies** of the **windowed** and **aliased** continuous Fourier transform.
Windowing: due to finite duration of the data convolve with window transform.
Aliasing: due to discrete sampling of data reflect around Nyquist frequency.

Power spectrum. Leahy normalization

Leahy et al. (1983)

Parseval's theorem: $\sum_k x_k^2 = \frac{1}{N}\sum_j |a_j|^2$

Variance in the real time series x_k:

$$\mathrm{Var}(x_k) \equiv \sum_k (x_k - \overline{x})^2 = \sum_k x_k^2 - \frac{1}{N}\left(\sum_k x_k\right)^2 = \frac{1}{N}\sum_j |a_j|^2 - \frac{1}{N}a_0^2$$
$$= \frac{1}{N}\sum_{j\neq 0} |a_j|^2$$

Leahy normalized power spectrum

$P_j \equiv \frac{2}{N_{ph}}|a_j|^2 \; ; \quad j = 0, \ldots, \frac{N}{2} \; ; \quad$ where $N_{ph} = \sum_k x_k = a_0$

Then: $\mathrm{Var}(x_k) = \frac{N_{ph}}{N}\left(\sum_{j=1}^{N/2-1} P_j + \frac{1}{2}P_{N/2}\right)$: variance is sum of powers.

As a_j has the same dimension as x_k, the dimension of $P_j \propto |a_j|^2/a_0$ is also the same as x_k: $[P_j] = [a_j] = [x_k]$.

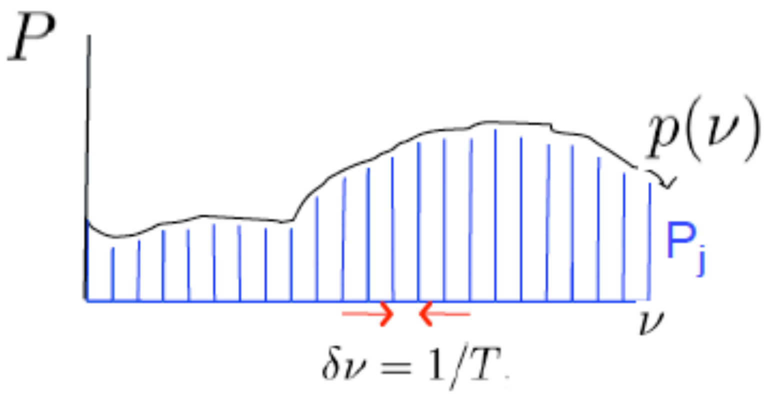
Power spectrum. Leahy normalization

Leahy et al. (1983)

Power density gives power per unit of frequency (i.e., per Hz), so that integral over power density spectrum is sum of powers:

$$\int_{\nu_{j1}}^{\nu_{j2}} p(\nu)d\nu = \sum_{j=j1}^{j2} P_j$$

Now $\delta\nu = 1/T$, so the Leahy normalized power density at ν_j is: $p(\nu_j) \equiv P_j/\delta\nu = TP_j$. Dimension: $[p(\nu)] = [x_k/\nu]$



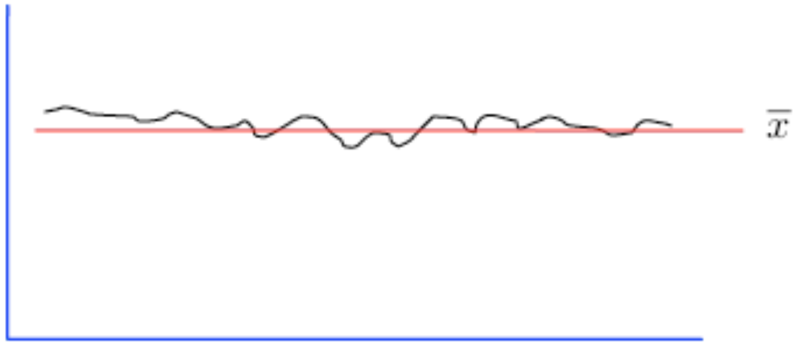
Power spectrum. Rms normalization

Miyamoto et al. (1999)

Fractional rms amplitude of a signal in a time series:

$$r \equiv \frac{\sqrt{\frac{1}{N} Var(x_k)}}{\overline{x}} = \frac{N}{N_{ph}} \sqrt{\frac{N_{ph}}{N^2} \left(\sum_{j=1}^{N/2-1} P_j + \frac{1}{2} P_{N/2} \right)} = \sqrt{\frac{1}{N_{ph}} \left(\sum_{j=1}^{N/2-1} P_j + \frac{1}{2} P_{N/2} \right)}$$

r is dimensionless and often expressed in %. rms=root mean square variability amplitude

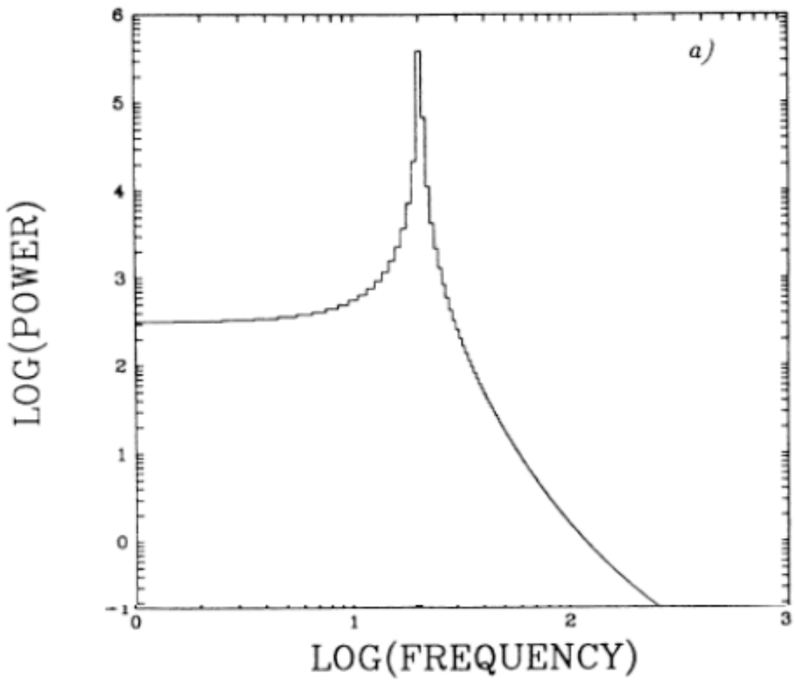


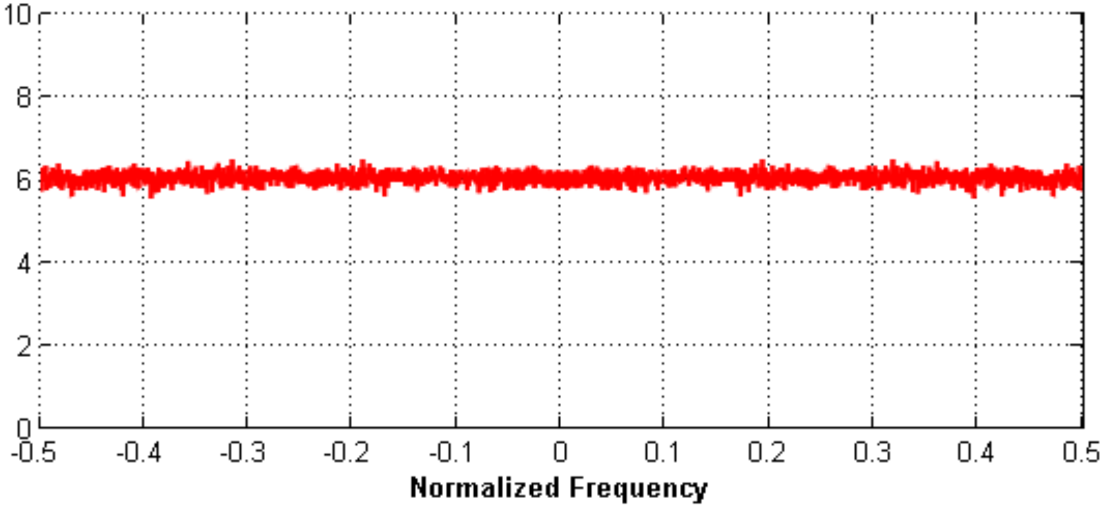
"Rms normalized" power density: $q(\nu_j) \equiv T P_j / N_{ph} = p_j / N_{ph}$

$q(\nu)$ has the nice property that fractional rms is just $\boxed{r = \sqrt{\int q(\nu) d\nu}.}$
Dimension of $q(\nu)$ is $[q] = [1/\nu] = [t]$; physical unit of $q(\nu)$ is $(\mathrm{rms/mean})^2/\mathrm{Hz}$.

Some examples

- PSD of a sine wave
- PSD of white noise





Some further examples

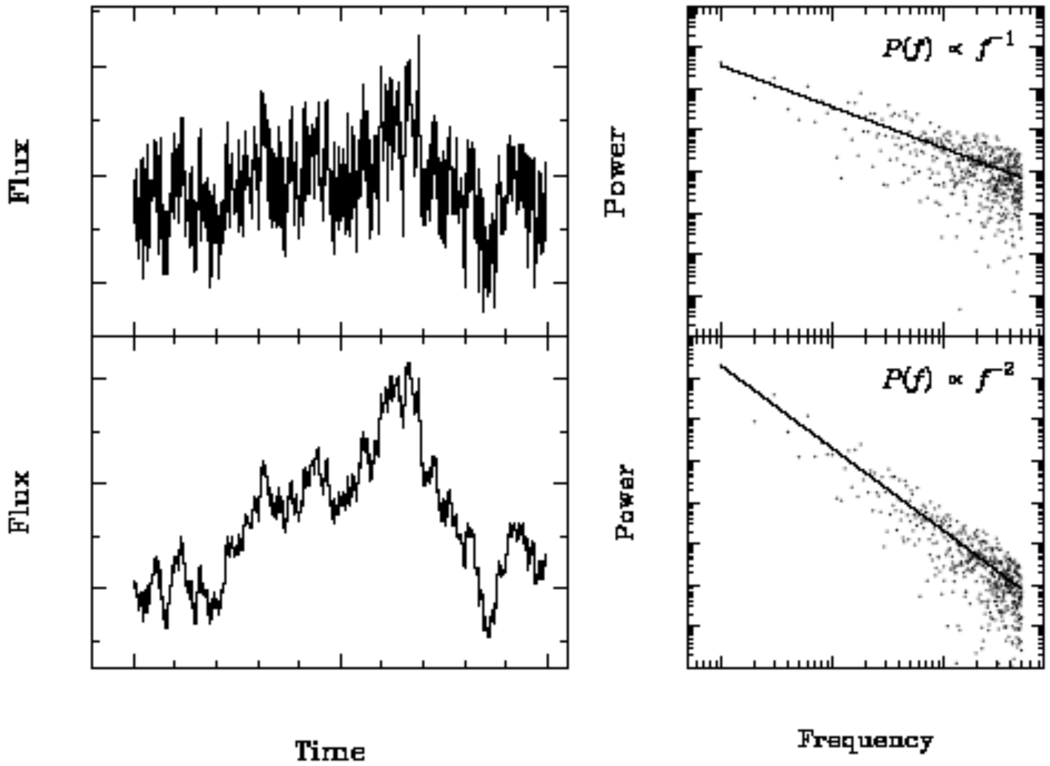
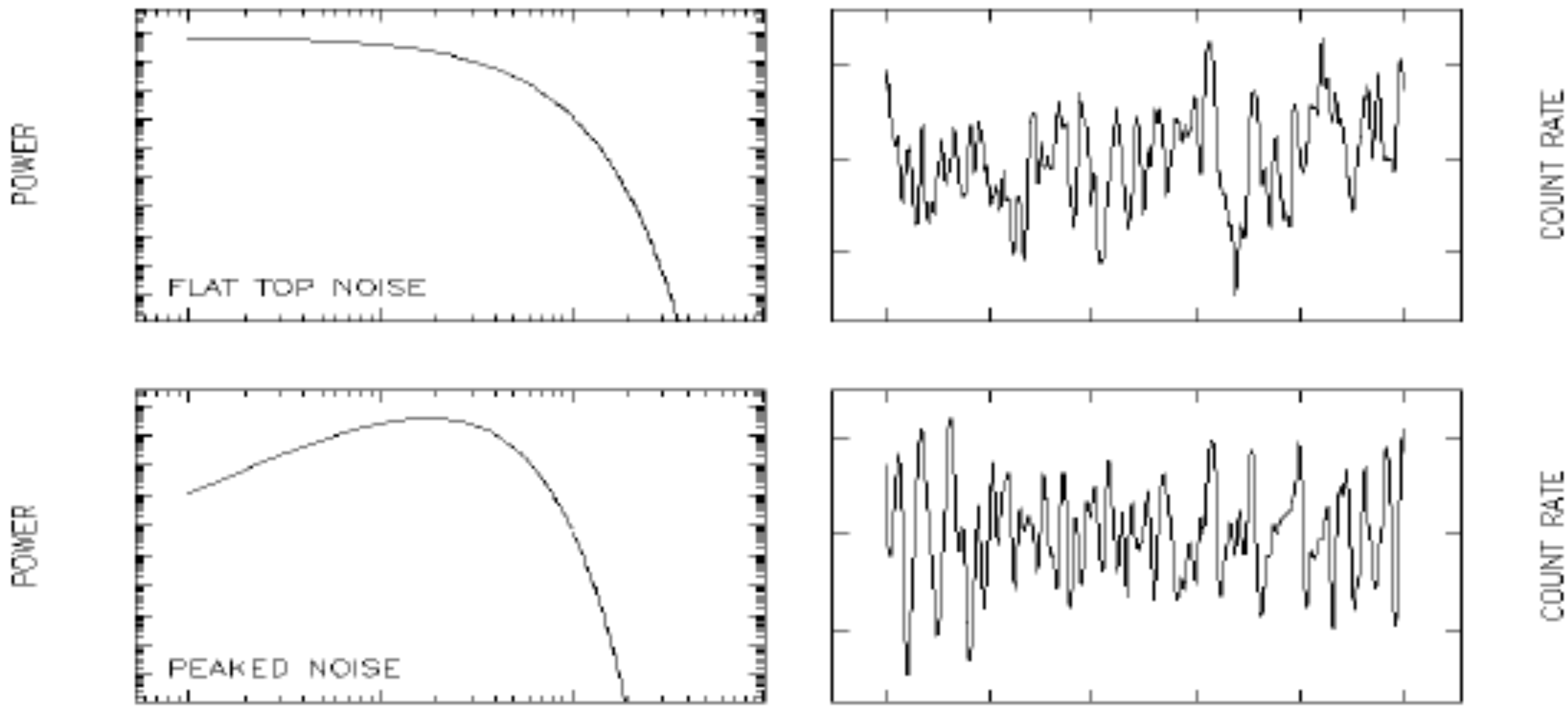


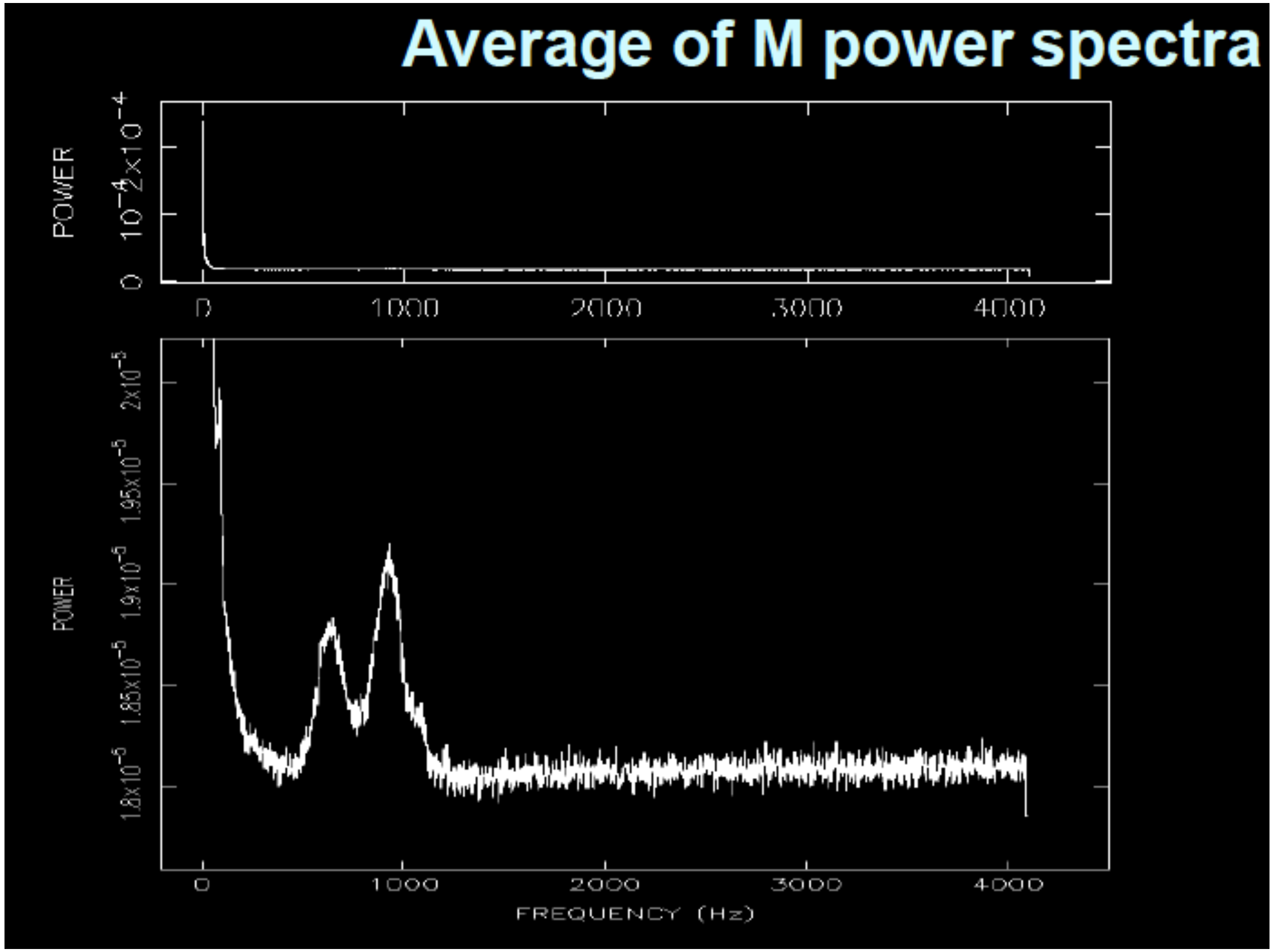
Figure 1. Simulated time series (left) and their periodograms (right). The upper panel shows a 'flicker noise' time series which has a f^{-1} PSD. The lower panel shows a 'random walk' time series with a f^{-2} PSD. Note the large scatter in the periodogram (dots) around the underlying PSD (solid line). It is clear that the time series with the steeper PSD shows more power in long-term variability while the time series with the flatter PSD shows relatively more power in short term variability (flickering). The two series were generated using the same random number sequence.

Vaughan et al. (2003)

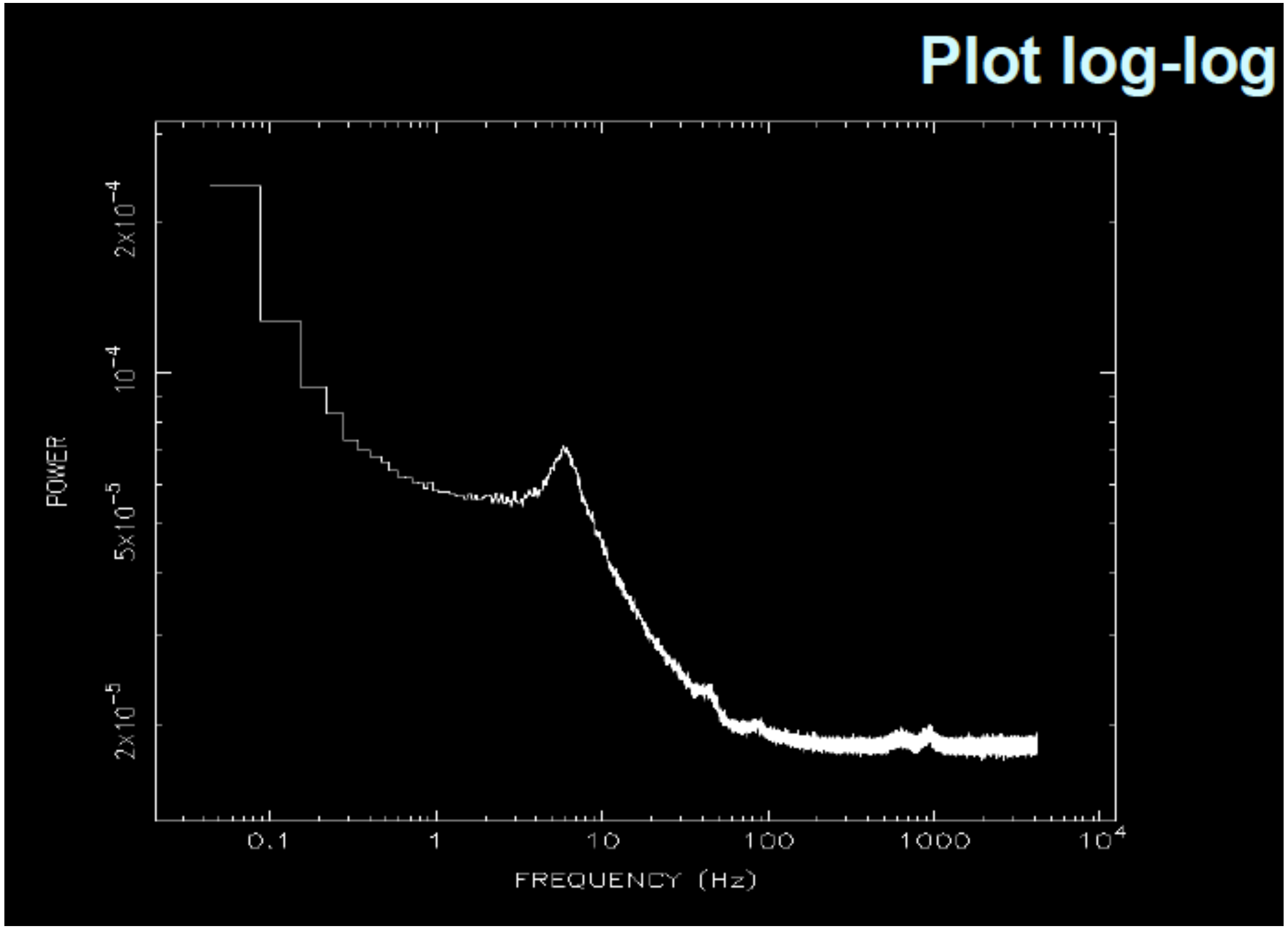
Some further examples



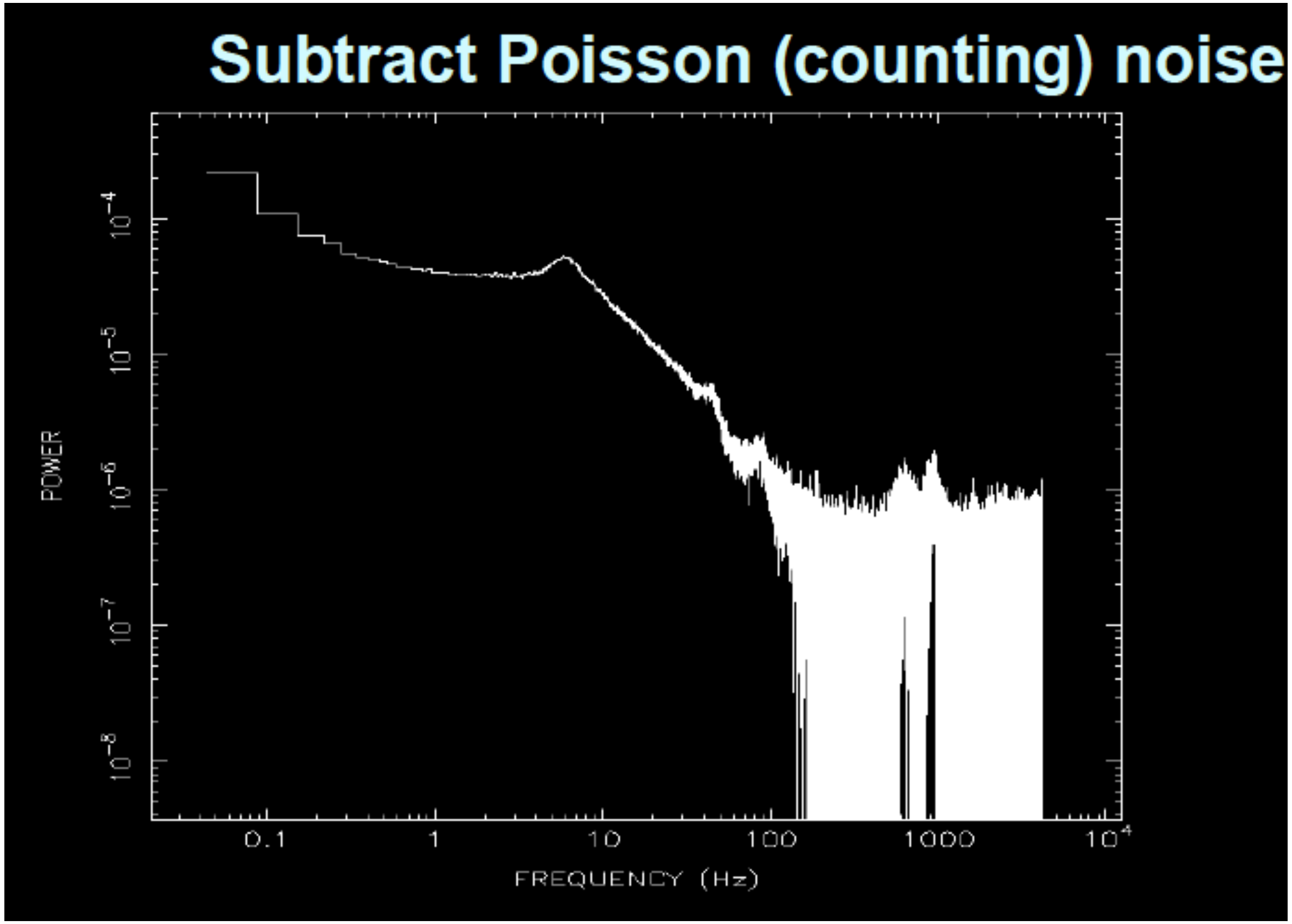
Plotting power spectra



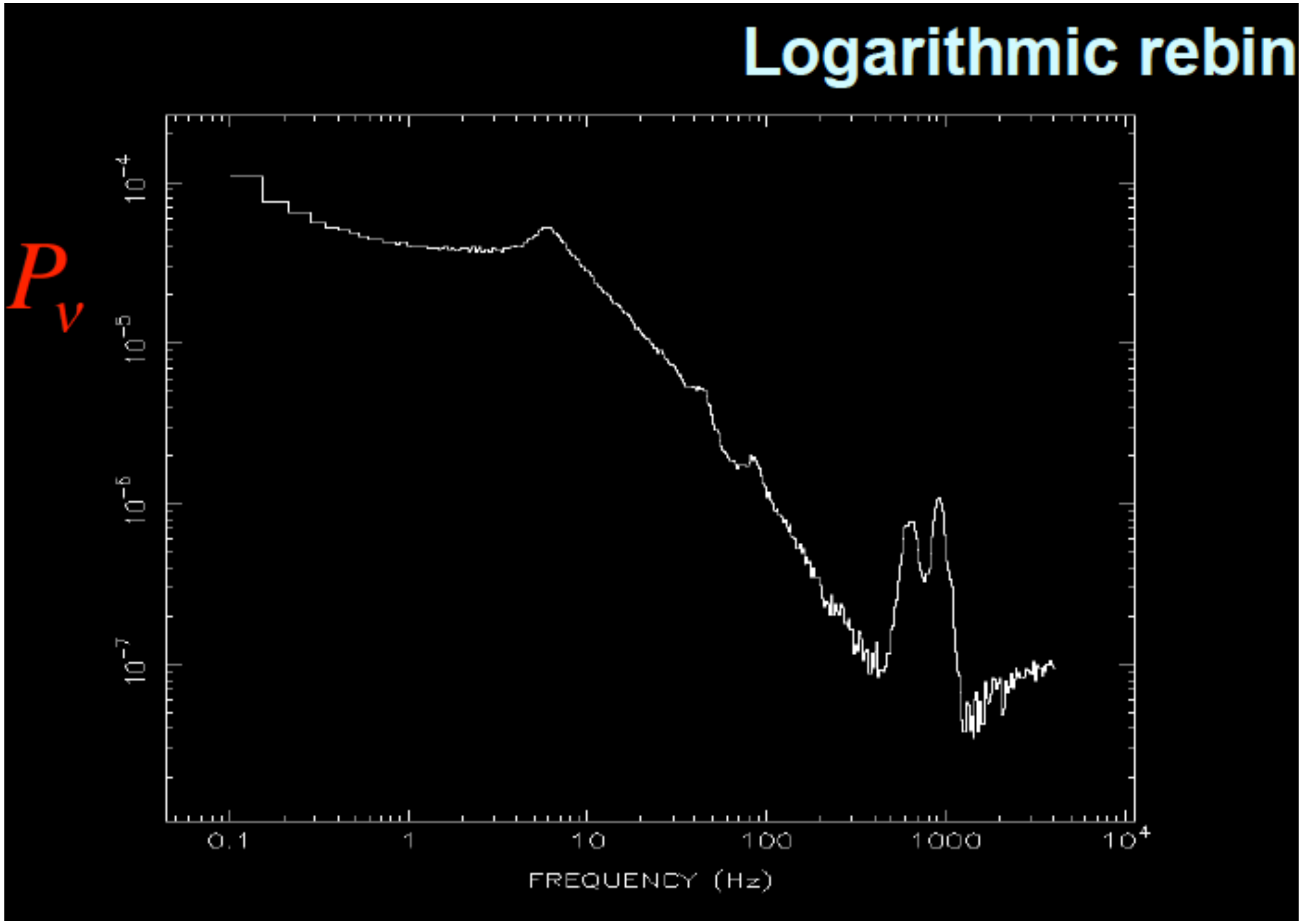
Plotting power spectra



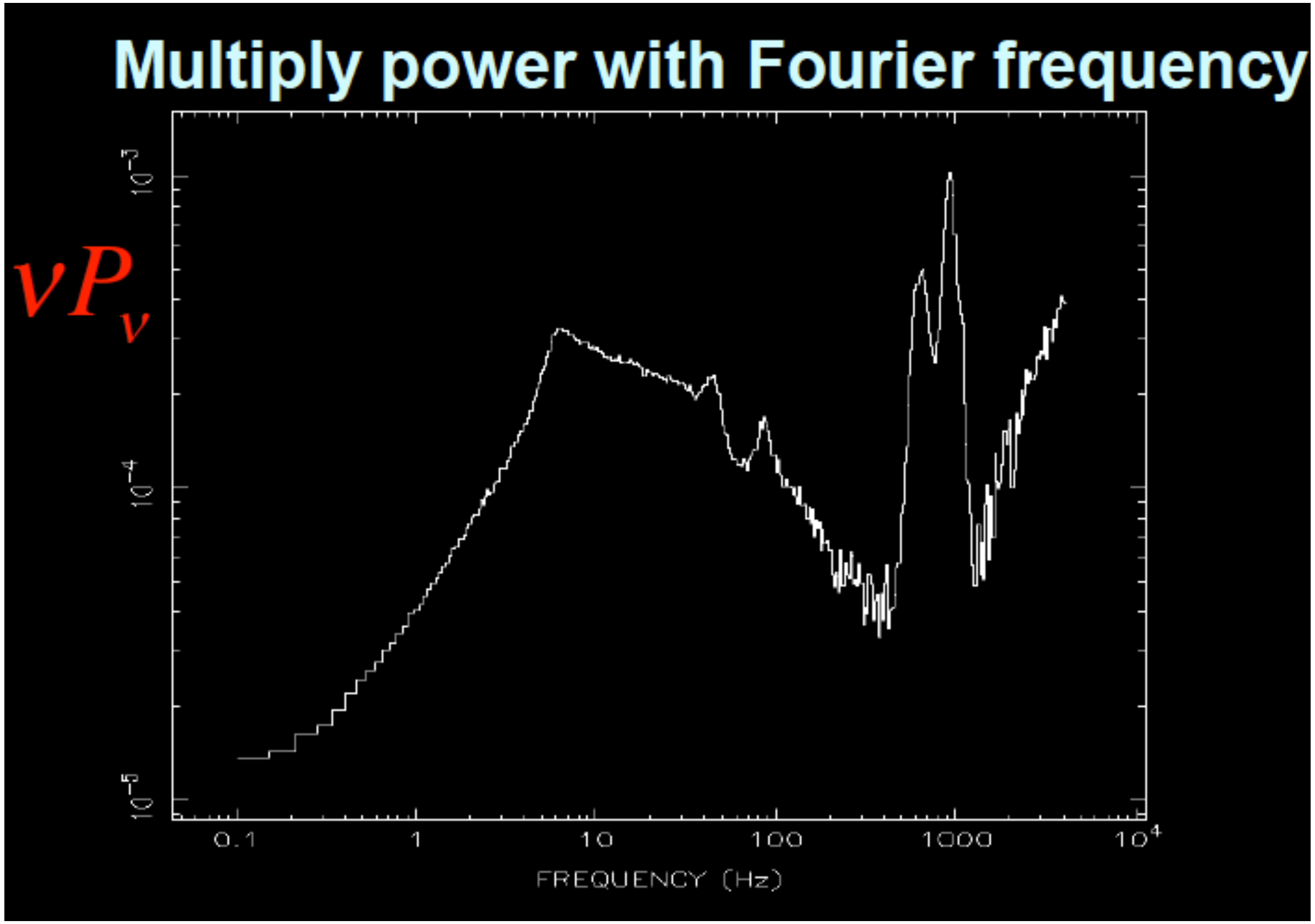
Plotting power spectra



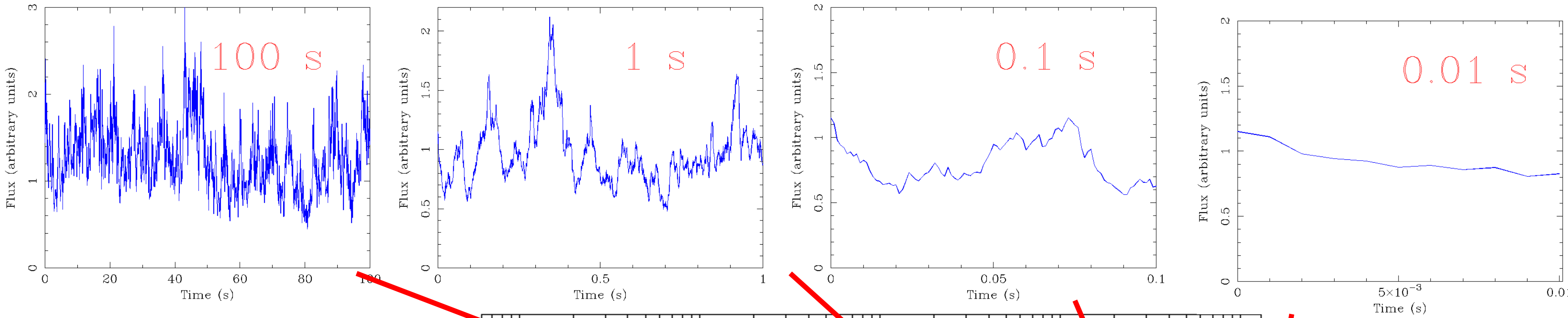
Plotting power spectra



Plotting power spectra

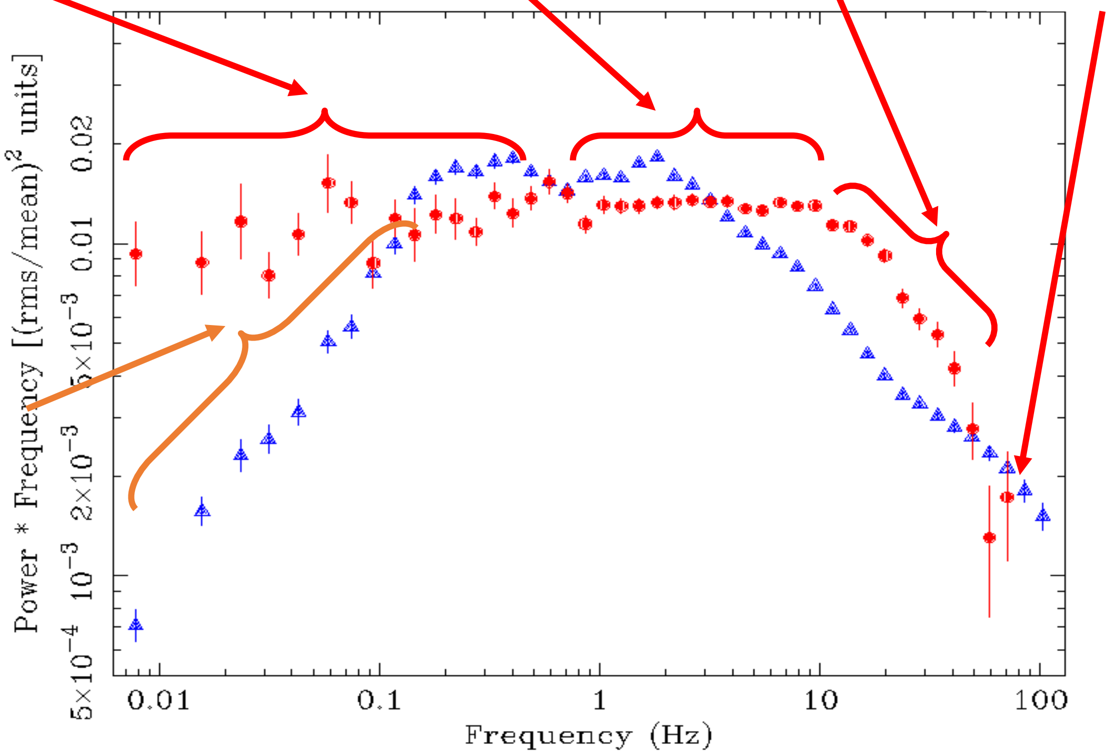


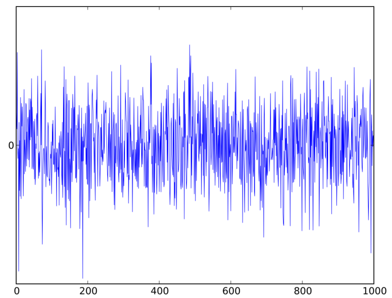
Quantifying variability: PDS of Cyg X-1



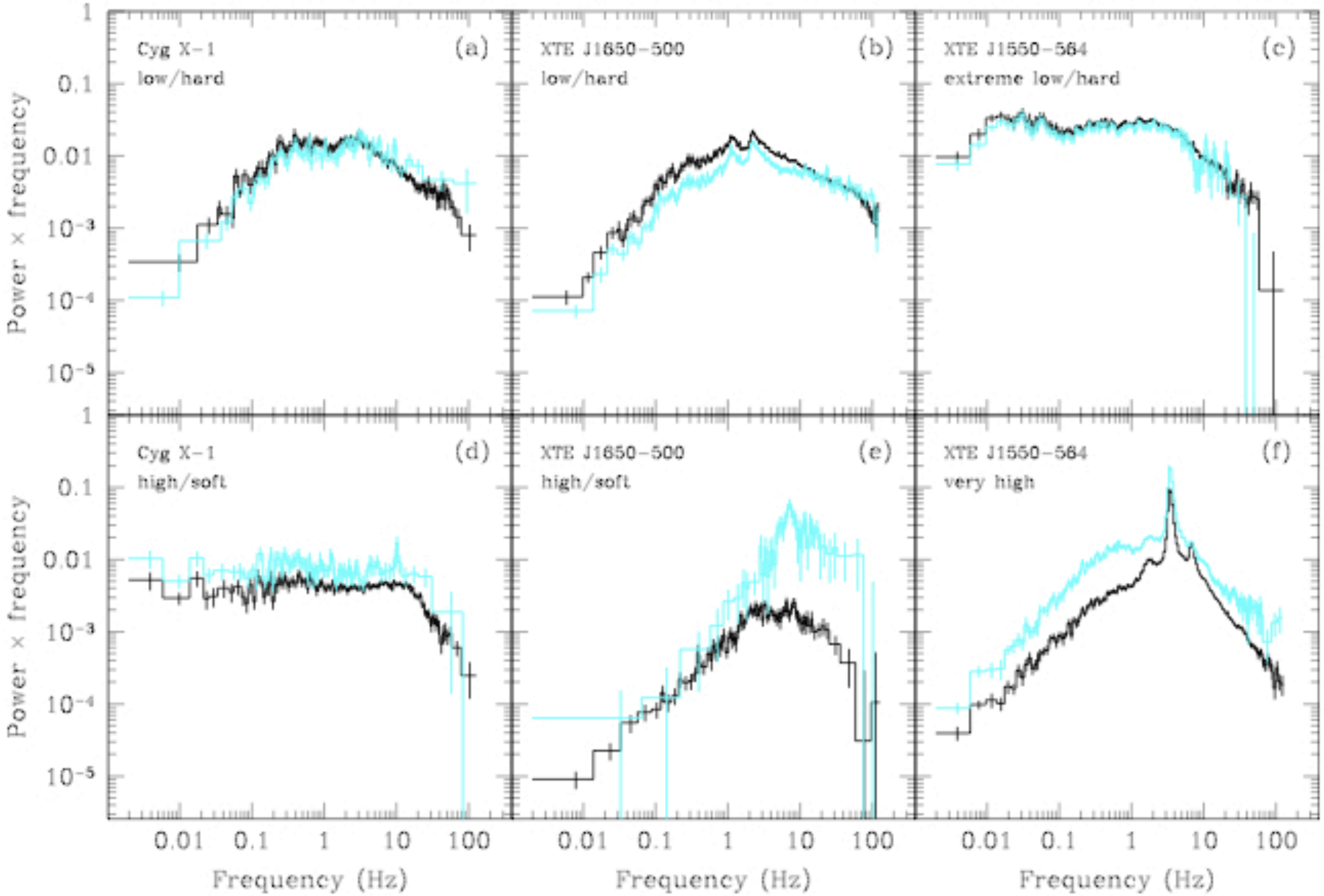
Soft state

Hard state





A gallery of black hole XRB PDSs



Done & Gierlinski 2005

Mathematical shot noise model

- Let the light curve consist of a superposition of uncorrelated shots (flares) of the same shape, $g(t)$. Let λ be the mean number of shots per second. The distribution of shots in time follows Poisson distribution. The probability of appearance of n shots in the interval $(t,t+\tau)$:

$$P_n = \frac{(\lambda\tau)^n}{n!} e^{-\lambda\tau}$$

- For such a process the probability that the time interval between successive shots lies between t and $t+dt$ is

$$P(t)dt = \lambda e^{-\lambda t} dt$$

- The light curve is then $s(t) = \sum_{i=0} g(t-t_i)$

t_i - time of appearance of i^{th} shot

- The PDS of the total light curve is proportional to the $|G(f)|^2$, where $G(f)$ is the Fourier transform of $g(t)$.

Modified shot noise model

- Example

$$g(t) = \begin{cases} e^{-t/\tau}, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

- The Fourier transform

$$G(f) = \frac{1}{2\pi i f - 1/\tau} \Rightarrow |G(f)|^2 = \frac{\tau^2}{1 + (2\pi f \tau)^2}$$

- In Nature usually exist many time-scales, therefore we assume a power-law distribution of time constants τ

$$p(\tau) = \begin{cases} \tau^{-\eta}, & \tau_{\min} < \tau < \tau_{\max} \\ 0, & \text{otherwise} \end{cases}$$

- Then the PDS of such a process

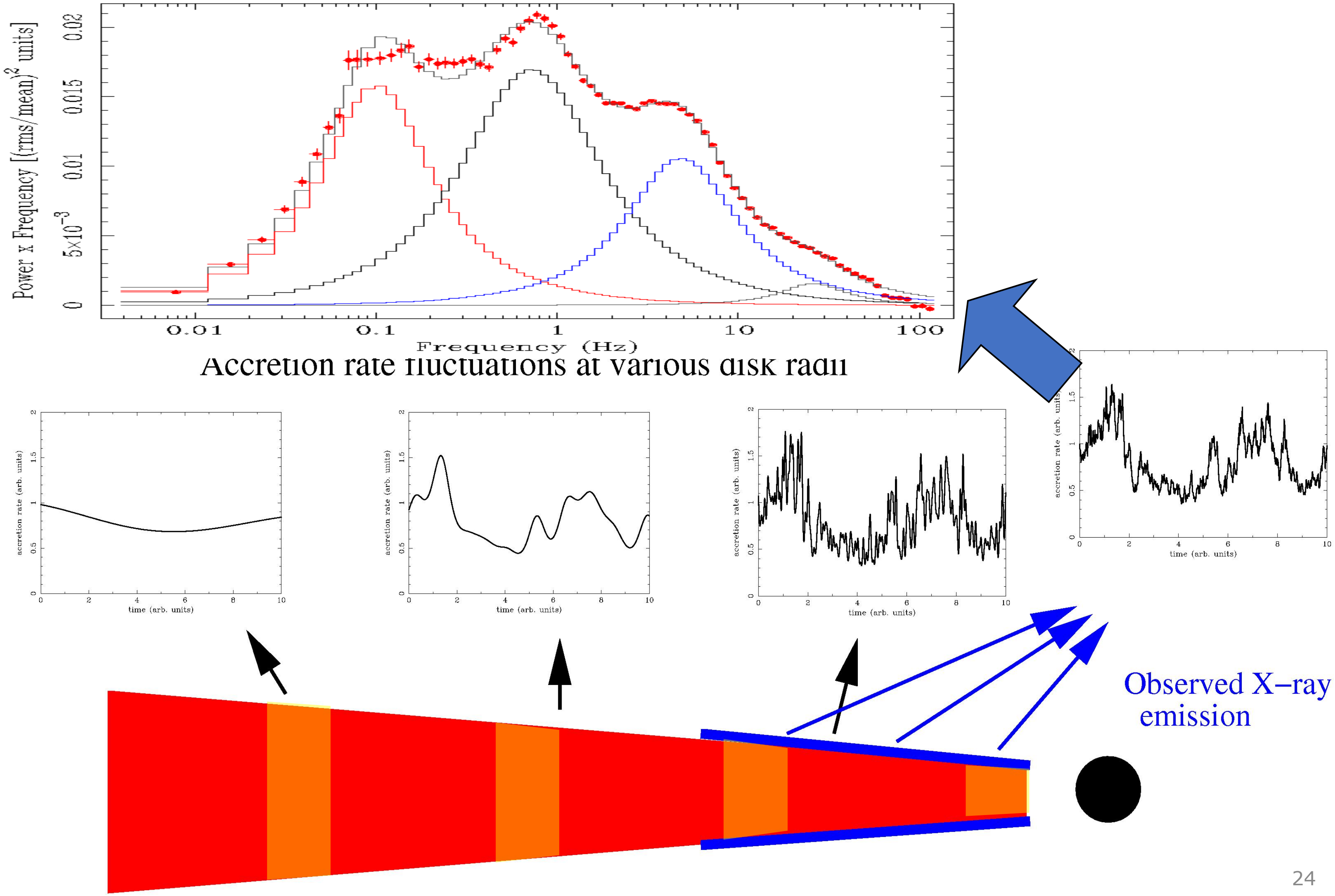
$$PDS(f) = \int_{\tau_{\min}}^{\tau_{\max}} p(\tau) PDS(f,\tau) d\tau$$

- For example, for $\eta=2$,

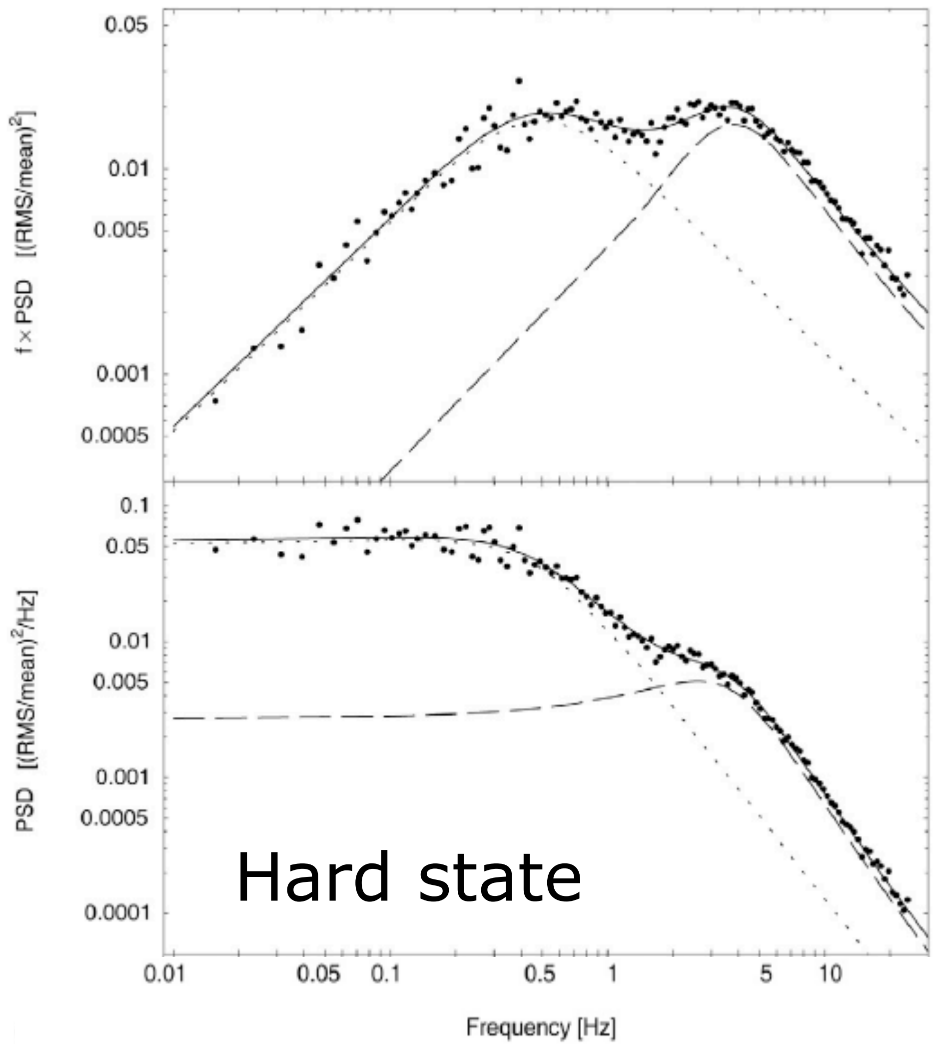
$$PDS_{\eta=2}(f) \propto \int_{\tau_{\min}}^{\tau_{\max}} \tau^{-2} \frac{\tau^2}{1 + (2\pi f \tau)^2} d\tau = \frac{1}{2\pi f}\left[\arctan(2\pi f \tau_{\max}) - \arctan(2\pi f \tau_{\min})\right]$$

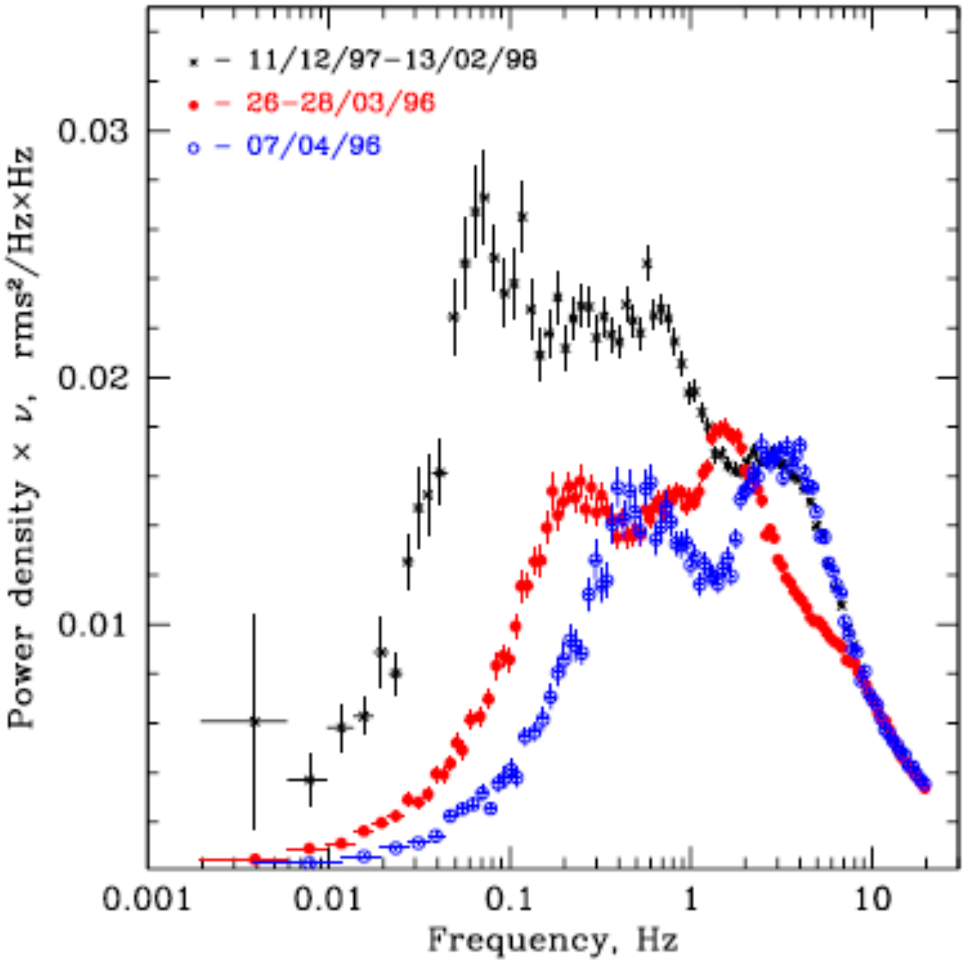
$$\propto f^{-1} \text{ if } 1/2\pi\tau_{\max} << f << 1/2\pi\tau_{\min}$$

Building the PSD from disc rings



PDS of Cyg X-1

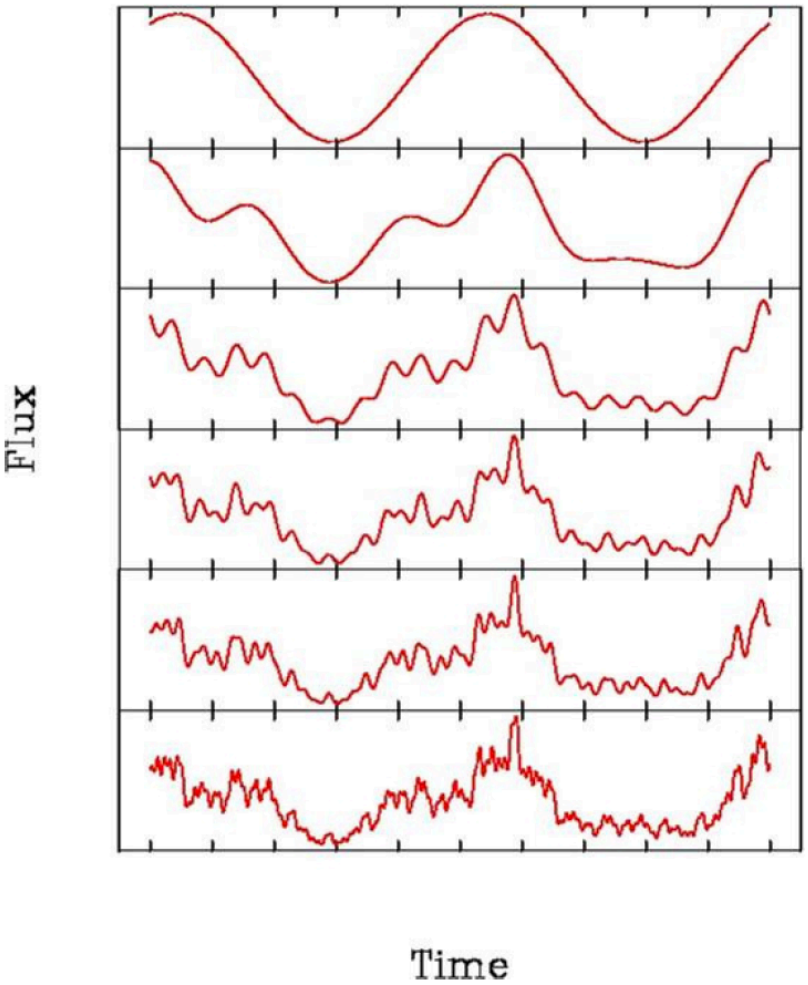




PDS in the hard state can be decomposed into a number of peaks of different time-scales

Propagating fluctuations

Lyubarski 1997

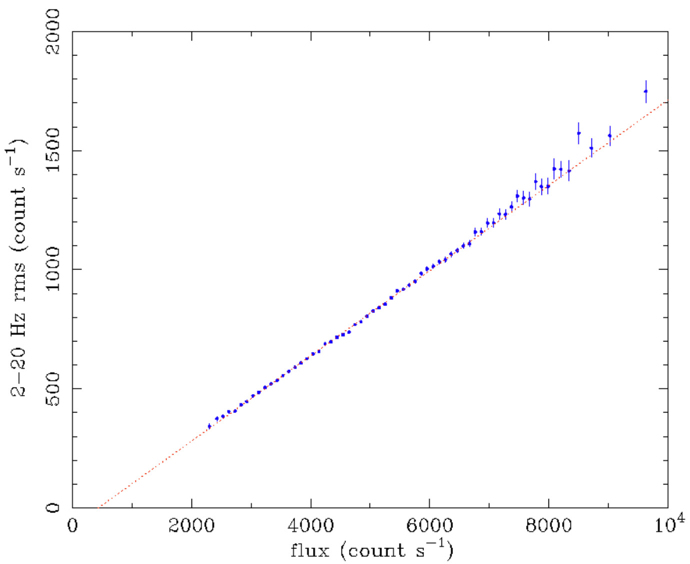


Linear light-curves are sum of sine waves

$$l(t)=1+\Sigma a_i \cos(\omega_i t+\phi_i)$$

But in the accretion disc the fluctuations are multiplied at each radius

$$x(t)=\Pi[1+\Sigma a_i \cos(\omega_i t+\phi_i)]$$



Uttley & McHardy 2001