HIGH ENERGY ASTROPHYSICS

Compulsary Home Exercises. Problem set 2. Solutions.

Problems

2.1: Show that the accretion rate of the wind accreting object is

$$\dot{M} = \dot{M}_w \left(\frac{M_x}{M_n}\right)^2 \frac{(v/v_w)^4}{[1+(v/v_w)^2]^{3/2}},$$

where $\dot{M}_w$ is the mass loss rate, M_x and M_n are the masses of the X-ray and normal star, respectively, v is the orbital velocity of a compact object around a companion and v_w is the wind velocity.

Solution:

Let us use the equations shown in class. First, we write the equation for the accretion radius R_{acc}:

$$(1) \quad \frac{mv_{rel}^2}{2} = \frac{GM_x m}{R_{acc}}.$$

The relative velocity (because wind and neutron star move perpendicular to each other) is

$$(2) \quad v_{rel}^2 = v^2 + v_w^2,$$

where v_w is the wind velocity and v is the orbital velocity:

$$(3) \quad v^2 = \frac{GM_n}{a}.$$

The mass accretion rate

$$(4) \quad \dot{M} = \pi R_{acc}^2 v_{rel} \rho_w,$$

where a is the binary separation and ρ_w is the wind density at that distance. The mass conservation law for the wind

$$(5) \quad \dot{M}_w = 4\pi a^2 v_w \rho_w.$$

Now divide (4) by (5):

$$\frac{\dot{M}}{\dot{M}_w} = \frac{R_{acc}^2}{4a^2}\frac{v_{rel}}{v_w}.$$

Substituting expressions for R_{acc} from eq (1) and for a from eq (3), we get

$$\frac{\dot{M}}{\dot{M}_w} = \frac{M_x^2}{v_{rel}^4}\frac{v^4}{M_n^2}\frac{v_{rel}}{v_w} = \frac{M_x^2}{M_n^2}\frac{v^4}{v_w^4}\frac{v_w^3}{v_{rel}^3} = \frac{M_x^2}{M_n^2}\frac{(v/v_w)^4}{[v_{rel}^2/v_w^2]^{3/2}}.$$

And finally using eq (2), we get what was desired.

2.2: (a) In class we derived the Eddington limiting luminosity assuming the accretion of pure ionized hydrogen. Show that more generally the Eddington limit can be written as

$$L_{\rm Edd} = \frac{4\pi GMc}{\kappa},$$

where κ is the *mass absorption coefficient*. It has units of cm^2 g^{-1} and is the absorption cross-section per unit mass, $\kappa = \sigma/m$, where σ is the cross-section, m is the particle mass.

(b) What is the Eddington limit for a plasma composed entirely of completely ionized helium? Compute the numerical coefficient x in $L_{\rm Edd} = (x \text{ erg s}^{-1})(M/M_\odot)$.

(c) What is the Eddington limit for a plasma composed entirely of electron-positron pairs? Compute the numerical coefficient x in $L_{\rm Edd} = (x \text{ erg s}^{-1})(M/M_\odot)$. Note that the positron will also now scatter photons. The small Eddington limit here is one of the reasons people believe some jets may have large number of electron-positron pairs.

(d) What is the (pure ionized hydrogen) Eddington limit for an $\sim 10M_\odot$ black hole (like Cygnus X-1), an $\approx 4 \times 10^6 M_\odot$ black hole (like on our Galactic center), and an $\approx 10^9 M_\odot$ black hole (like in a luminous quasar)?

Solutions:

In a general case, the balance between gravity force and radiation pressure force is

$$\frac{L\sigma}{4\pi r^2 c} = \frac{GMm}{r^2}.$$

Defining $\kappa = \sigma/m$, we get

$$L_{\rm Edd} = \frac{4\pi GMc}{\kappa}.$$

For an atom of charge Z and atomic mass A, the total electron scattering cross-section is $\sigma = Z\sigma_T$ and the total mass $m = Am_p$. Since the proton mass $m \approx (5/3)10^{-24}\,$g and the Thomson cross-section is $\sigma_T \approx (2/3)10^{-24}\,$cm^2, we get $\kappa \approx 0.4(Z/A)\,cm^2\,$g^{-1}.

(a) For the hydrogen gas, $A = 1$, $Z = 1$, we have $\kappa \approx 0.4\,$cm$^2\,$g^{-1} and

$$L^{\rm H}_{\rm Edd} = 1.26 \times 10^{38}(M/M_\odot)\,{\rm erg\,s^{-1}}.$$

(b) Helium has four nucleons and two electrons, so $\kappa = 0.2\,$cm$^2\,$g^{-1} and $L^{\rm He}_{\rm Edd} = 2L^{\rm H}_{\rm Edd} = 2.52 \times 10^{38}(M/M_\odot)\,{\rm erg\,s^{-1}}$.

(c) Here we just take $m = m_e$ and $\sigma = \sigma_T$ and get $\kappa = 0.4(m_p/m_e)\,$cm$^2\,$g^{-1} and

$$L^{\rm e^\pm}_{\rm Edd} = (m_e/m_p)L^{\rm H}_{\rm Edd} = 6.86 \times 10^{34}(M/M_\odot)\,{\rm erg\,s^{-1}}.$$

(d) For $M = 10M_\odot$, the limit is $L^{\rm H}_{\rm Edd} = 1.26 \times 10^{39}\,{\rm erg\,s^{-1}}$. For the Galactic center black hole $M = 4 \times 10^6 M_\odot$, $L^{\rm H}_{\rm Edd} \approx 5 \times 10^{44}\,{\rm erg\,s^{-1}}$. For a quasar with $M = 10^9 M_\odot$, $L^{\rm H}_{\rm Edd} = 1.26 \times 10^{47}\,{\rm erg\,s^{-1}}$.

2.3: Calculate the terminal velocity, v (i.e. velocity at $r = \infty$), for an electron-positron pair under radiation and gravitational force alone, if it starts from rest at distance $R = 10R_S$ from the black hole. ($R_S = 2GM/c^2$ is the Schwarzschild radius.)

Hint: Write down effective force (gravitational minus radiation force), and use the energy conservation equation. Use the relativistic formula for the electron (positron) energy.

Solutions:

Since luminosity of the black hole was not given, we need to assume something. Let us assume that the luminosity exceeds the Eddington limit for the electron-positron plasma by a factor f. Total force acting on a electron (or positron) is then the difference between the radiation pressure force and the gravitational force:

$$\mathcal{F} = (f-1)\frac{GMm_e}{r^2}.$$

The total work that this force will do when moving electron from radius R to infinity is

$$E = \int_R^\infty \mathcal{F} dr = (f-1)\frac{GMm_e}{R}.$$

This energy will be transferred to the electron kinetic energy $(\gamma - 1)m_e c^2$, where γ is the Lorentz factor. Thus we get

$$\gamma - 1 = \frac{1}{\sqrt{1 - v^2/c^2}} - 1 = \frac{E}{m_e c^2} = (f-1)\frac{GM}{Rc^2} = (f-1)\frac{R_S}{2R} = \frac{f-1}{20}.$$

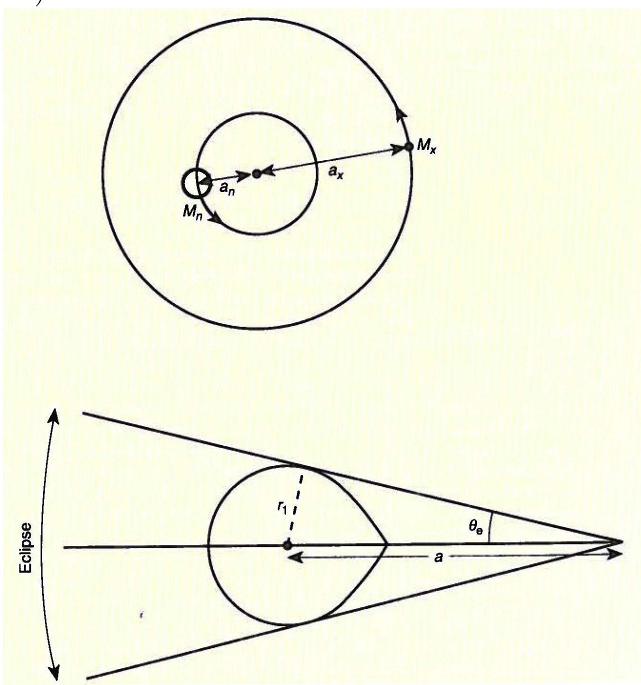
For example, for $f = 21$, we get $\gamma = 2$ and

$$v = c\sqrt{1 - 1/\gamma^2} = 0.866c \approx 2.6 \times 10^{10}\,\mathrm{cm\,s^{-1}}.$$

2.4: Show that the inclination of the binary orbit i and the half-angle θ_e of the eclipse of the central compact object are related by

$$\left(\frac{R_n}{a}\right)^2 = \cos^2 i + \sin^2 i \sin^2 \theta_e,$$

where R_n is the radius of the companion star and a is the binary separation (see figure below).



Solution: At the beginning of the eclipse the line of sight is tangential to the surface of the companion star. Let ψ be the angle between the line of sight and the direction to the center of the companion from the central compact source. Then at the start of the eclipse $\psi = \psi_e$, which is given by the relation

$$R_n = a \sin \psi_e.$$

In the spherical coordinate system centered at the compact object with the z axis perpendicular to the orbital plane, define the unit vector in the direction to the observer

$$\vec{k} = (\sin i, 0, \cos i).$$

The coordinates of the center of the companion are

$$\vec{a} = a(\cos \phi, \sin \phi, 0),$$

where ϕ is the orbital phase (azimuth). The scalar product of the unit vectors is

$$\cos \psi = \frac{\vec{a}}{a} \cdot \vec{k} = \sin i \cos \phi,$$

so that

$$\left(\frac{R_n}{a}\right)^2 = \sin^2 \psi_e = 1 - \cos^2 \psi_e = 1 - \sin^2 i \cos^2 \phi_e = \cos^2 i + \sin^2 i \sin^2 \phi_e,$$

where $\phi_e = \theta_e$ is the half-angle of the eclipse. Thus, we get the desired equation.

2.5: XTE J1807−294 is an accreting millisecond pulsar. Using variations of the arrival time of the pulse to the observer the pulsar mass function was measured

$$f_x = \frac{M_n^3 \sin^3 i}{(M_x + M_n)^2} = 1.49 \times 10^{-7} M_\odot.$$

Solve for the mass of the companion as a function of the inclination $M_n(i)$ and plot that relation. Assume that the neutron star mass is $M_x = 1.4 M_\odot$. What is the minimum value for M_n? Since $\cos i$ is distributed randomly, obtain the upper limit on the mass with 90% confidence.

Solution: Redefine all masses in units of the solar mass, $m = M/M_\odot$. The equation we need to solve is then

$$(1) \quad f'_x = \frac{m_n^3 \sin^3 i}{(m_x + m_n)^2} = 1.49 \times 10^{-7}.$$

Use any programming language you know. First set an equally spaced grid of values for $\cos i$ from 0 to 1. For each $\cos i$ compute i. Solve the equation above by iterations. As a first guess for m_n take $(m_x^2 f'_x)^{1/3}/\sin i$. Iterations of order $k+1$ is then:

$$m_{k+1} = [(m_x^2 + m_k^2) f'_x]^{1/3} / \sin i.$$

Iterate until it converges. Plot m_n versus $\cos i$. The minimum mass for m_n is reached at $\cos i = 0$. There is no upper mass, because for $i = 0$ it diverges, but the probability to have zero inclination is exactly zero. To get the upper limit on the mass with 90% confidence just read the value of m_n at $\cos i = 0.9$.

An interesting thing to notice here is that because f'_x is so small, the companion mass is also very small and much smaller than the neutron star mass. Therefore we can ignore m_n in the denominator of eq. (1) and the zeroth approximation is actually very accurate. Thus the companion mass is

$$m_n \approx (m_x^2 f'_x)^{1/3} / \sin i = 0.0066 \,/ \sin i.$$

The minimum mass is then $0.0066\, M_\odot$ and the 90% upper limit is $0.015 M_\odot$.