HIGH ENERGY ASTROPHYSICS

Compulsary Home Exercises. Problem set 3. Solutions.

Problems

3.1: Show that the Alfven radius is

$$R_A \approx \left(\frac{B_0^2 R_*^6}{2\dot{M}\sqrt{2GM}}\right)^{2/7},$$

where B_0 is the surface magnetic field of the neutron star of radius R_* and mass M, and $\dot{M}$ is the accretion rate.

Solution: The equality between the magnetic pressure and the ram pressure

$$(1) \quad \frac{B^2}{8\pi} = \rho V^2.$$

The dipole magnetic field

$$B = B_0(R_*/R)^3.$$

For the velocity, we take the free-fall velocity

$$V^2 = \frac{2GM}{R},$$

and we can find the density from the mass conservation law

$$\rho = \frac{\dot{M}}{4\pi R^2 V}.$$

Putting everything to eq (1), we get

$$\frac{B_0^2}{8\pi}\frac{R_*^6}{R^6} = \frac{\dot{M}}{4\pi R^2 V}V^2 = \frac{\dot{M}}{4\pi R^2}\left(\frac{2GM}{R}\right)^{1/2}.$$

Collecting terms with R on one side and the rest on the other side, we get

$$R^{7/2} = B_0^2 \frac{R_*^6}{2\dot{M}\sqrt{2GM}},$$

which is identical to what was asked.

3.2: A non-magnetized neutron star of mass $M = 1.4\mathrm{M}_\odot$ and radius $R = 13$ km is accreting matter at $10^{-10}\mathrm{M}_\odot/\mathrm{yr}$ via an accretion disk. How long does it take to spin up the star by the accreting matter from the initially large period to 3 ms? How much mass do you need to accrete to spin the star to such a period? Assume a constant moment of inertia $I = 10^{45}$ g cm^2.

Solution: The angular momentum conservation equation reads

$$I\dot{\Omega} = \dot{M}\sqrt{GMR}.$$

Note that we put R instead of R_m in the formula, because the neutron star is non-magnetized. Assuming that neither mass nor moment of inertia significantly change, we get the evolution of spin rate with time t:

$$\Omega(t) = \frac{2\pi}{P(t)} = \Omega_0 + t\frac{\dot{M}\sqrt{GMR}}{I}.$$

Neglecting Ω_0, we get the time needed to reach period P:

$$t = \frac{2\pi}{P}\frac{I}{\dot{M}\sqrt{GMR}}.$$

The mass accretion rate is $\dot{M} \approx 6.3 \times 10^{15}$ g s^{-1}. Substituting the numbers we get $t = 6.8 \times 10^8$ yr. The total accreted mass $\Delta M = \dot{M}t = 0.068\mathrm{M}_\odot$ is much smaller than the neutron star mass.

3.3: The period of the X-ray pulsar Cen X-3 has changed from 1971 to 1975 from 4.844 to 4.837 seconds. Estimate the magnetic field of the pulsar if its average luminosity is $L \approx 2 \times 10^{37}$ erg s^{-1}.

Solution: The period derivative is

$$\dot{P} = \frac{(4.837 - 4.844)\ \mathrm{s}}{4\ \mathrm{yr}} = \frac{-7\ \mathrm{ms}}{4\ \mathrm{yr}} = -5.5 \times 10^{-11} \mathrm{s\,s^{-1}}.$$

Note, that $\dot{P}$ is **negative**! For estimation of the magnetic field, one cannot use magnetic dipole radiation formula, because it is not a radio pulsar and it is not spinning down! Change in the rotational frequency occurs due to the angular momentum brought in by the accreting gas. Accounting only for the accretion torque, the angular momentum conservation equation reads

$$(1) \quad 2\pi I \frac{\dot{P}}{P^2} = -\dot{M}\sqrt{GMR_m}.$$

The luminosity is related to the accretion rate

$$L = \eta \dot{M} c^2.$$

Taking the accretion efficiency of 0.15, we can estimate the accretion rate $\dot{M} \approx 1.5 \times 10^{17}$ g s^{-1}. From eq (1), we get the magnetospheric radius

$$R_m = \xi R_A = \frac{1}{GM}\left(-2\pi \frac{\dot{P}}{P^2}\frac{I}{\dot{M}}\right)^2 = 5 \times 10^7\ \mathrm{cm},$$

where we assumed $I = 10^{45}$ g cm^2 and $M = 1.5 M_\odot$. Using the formula for the Alfven radius we get

$$B_0^2 = \left(\frac{R_m}{\xi}\right)^{7/2} \frac{2\dot{M}\sqrt{2GM}}{R_*^6} = 6 \times 10^{22} R_{*,6}^{-6}\ \mathrm{G}^2,$$

where we assumed $\xi = 0.5$. Thus we get $B_0 \approx 2.5 \times 10^{11}$ G.

From the observed cyclotron line at $\sim$30 keV, however, one can deduce the magnetic field of $B_0 \approx 3.5 \times 10^{12}$ G. This discrepancy likely results from our wrong assumption that the spin up is fully determined by the accretion torque. In reality, there is a magnetic torque that tries to decelerate the star, and the magnetospheric radius is larger than we have estimated.

3.4: A small spot of area S at the magnetic pole of the neutron star radiates as a black body (i.e. radiation intensity is the same in all direction). Assume the frequency integrated intensity is I_0. Derive first the formula for the frequency-integrated flux F observed from such a spot by an observer at distance D from the star as a function of inclination i (angle between the direction to the observer and the rotational axis), the angle between the rotational and magnetic pole θ, and the phase of the pulsar. Assume flat space-time (i.e. no photon bending) and neglect gravitational redshift. Compute then the amplitude of pulsation

$$A = \frac{F_{\max} - F_{\min}}{F_{\max} + F_{\min}},$$

where $F_{\max}$ and $F_{\min}$ are the maximum and minimum of the flux, respectively. Consider the case when the spot is visible all the time. How the result changes when gravitational bending is accounted for? Use approximate Beloborodov's formula for the light bending.

Solution: Let us choose coordinate system with the z-axis along the rotation axis and the observer line of sight lying in the plane $x - z$, see Fig. 1. The unit vector towards the observer is then

$$\vec{k} = (\sin i, 0, \cos i).$$

The radius vector of the spot varies with phase as

$$\vec{r} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta).$$

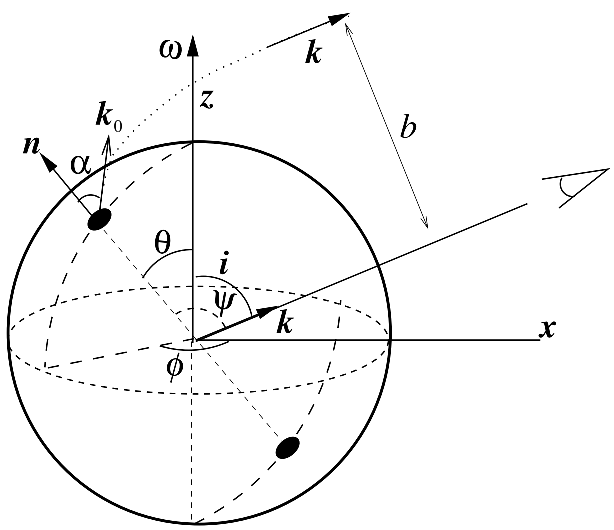

Figure 1: Geometry of the problem.

The angle between these vectors ψ is given by

$$\cos\psi = \vec{k}\cdot\vec{r} = \cos i\cos\theta + \sin i \sin\theta\cos\phi.$$

In flat space the spot is observed at the same angle $\alpha = \psi$. The observed flux is

$$F = I_0 d\Omega = I_0 \frac{S}{D^2}\cos\psi.$$

The maximum/minimum flux is reached when $\cos\phi = \pm 1$. The amplitude of pulsation is then

$$A_{\rm flat} = \frac{\cos\psi_{\rm max} - \cos\psi_{\rm min}}{\cos\psi_{\rm max} + \cos\psi_{\rm min}} = \frac{\sin i \sin\theta}{\cos i \cos\theta} = \tan i \tan\theta.$$

Accounting for light bending using Beloborodov's formula gives the observed flux

$$F \propto \cos\alpha = u + (1-u)\cos\psi,$$

where $u = R_S/R$. Again the extrema of F are reached at $\cos\phi = \pm 1$ and we get

$$A_{\rm GR} = \frac{\cos\alpha_{\rm max} - \cos\alpha_{\rm min}}{\cos\alpha_{\rm max} + \cos\alpha_{\rm min}} = \frac{(1-u)\sin i \sin\theta}{u + (1-u)\cos i \cos\theta} = \frac{\sin i \sin\theta}{\frac{u}{1-u} + \cos i \cos\theta} < A_{\rm flat}.$$

3.5: Computer exercise. Swift J1749.4−2807 is an accreting millisecond pulsar ($\nu =$ 518 Hz) discovered in 2010. It shows X-ray eclipses. From variations of the pulsar frequency the following quantities have been measured (Altamirano et al. 2011): the orbital period $P_{\rm orb} = 31740.73$ s, projected semimajor axis $a_x \sin i = 1.89953$ light seconds, the pulsar mass function was measured $f_x = M_n^3 \sin^3 i/(M_n + M_x)^2 = 0.0545278 M_\odot$. The duration of the eclipse was determined to be 2172 seconds (Markwardt & Strohmayer 2010). Determine the inclination of the orbit i, the mass M_n and the radius R_n of the companion star. Assume neutron star mass of $M_x = 1.5 M_\odot$. Use the Faulkner formula for the size of the Roche lobe:

$$R_n \approx R_L = 0.459 a \left(\frac{q}{1+q} \right)^{1/3},$$

where $q = M_n/M_x$ is the mass ratio and a is the binary separation.

Hint: use the results of exercise 2.4, where we showed that the inclination of the binary orbit i and the half-angle θ_e of the eclipse are related by

$$\left(\frac{R_n}{a} \right)^2 = \cos^2 i + \sin^2 i \sin^2 \theta_e.$$

Solution: The duration of the eclipse is related to the orbital period and the half-angle of the eclipse as

$$T_{\rm ecl} = P_{\rm orb} \frac{2\theta_e}{2\pi}.$$

Thus we get

$$(1) \quad \theta_e = \pi \frac{T_{\rm ecl}}{P_{\rm orb}} \approx 0.215\,\text{rad}.$$

The size of the Roche lobe in units of the binary separation from Faulkner formula is

$$(2) \quad \frac{R_n}{a} = 0.459 \left(\frac{q}{1+q} \right)^{1/3} = \left[1 - \sin^2 i \cos^2 \theta_e \right]^{1/2}$$

From the definition of the pulsar mass function we get

$$\frac{f_x}{M_\odot} = \frac{M_x}{M_\odot} \frac{q^3 \sin^3 i}{(1+q)^2} = 0.0545278,$$

or

$$(3) \quad \frac{q^3 \sin^3 i}{(1+q)^2} = \frac{0.0545278}{1.5} = 0.03635187.$$

The two equations (2) and (3) make a system of equation for two unknowns q and $\sin i$. We can solve them, e.g., by iterations. Or one can substitute $\sin i$ from eq (3) to eq (2) and find the root of the function

$$f(q) = 0.459 \left(\frac{q}{1+q} \right)^{1/3} - \left[1 - 0.1047 \frac{(1+q)^{4/3}}{q^2} \right]^{1/2}.$$

The root is $q \approx 0.4322$ and thus we get the companion mass $M_n = qM_x = 0.6483M_\odot$. Substituting q to eq (3), we get $\sin i = 0.973844$ or $i = 76.9°$. The projected size of the pulsar orbit $a_x \sin i = 5.6946 \times 10^{10}$ cm and the binary separation is

$$a = a_x + a_n = (a_x \sin i)\ (1 + 1/q)/\sin i = 1.9377 \times 10^{11}\ \text{cm}.$$

Using Faulkner formula we get

$$R_n = 0.30787\ a = 5.966 \times 10^{10}\ \text{cm}.$$