HIGH ENERGY ASTROPHYSICS
Compulsary Home Exercises. Problem set 5.
Turn in Exercises by Friday, November 1, 2024

Problems

5.1: Consider spherical accretion onto a black hole.
(a) Show that the sonic point is

$$r_{\rm s} = \frac{GM}{c_{\rm s}^2(\infty)} \frac{5-3\gamma}{4}.$$

(b) Show that the accretion rate is

$$\dot{M} = \pi G^2 M^2 \frac{\rho(\infty)}{c_{\rm s}^3(\infty)} \left[\frac{2}{5-3\gamma}\right]^{\frac{5-3\gamma}{2(\gamma-1)}}.$$

Compute the limit at $\gamma \to 5/3$. Express

$$\dot{M} = C \left(\frac{M}{M_\odot}\right)^2 \left(\frac{\rho(\infty)}{10^{-24}\ \mathrm{g\ cm^{-3}}}\right) \left(\frac{c_{\rm s}(\infty)}{10\ \mathrm{km\ s^{-1}}}\right)^{-3}\ \mathrm{g\ s^{-1}}$$

and find constant C.
(c) Show that $\dot{M} = \pi R_{\rm acc}^2 c_{\rm s}(\infty)\rho(\infty)$ (for $\gamma = 5/3$).
(d) Obtain $\rho(r), T(r)$ for $r \ll r_{\rm s}$. Hint: in this region $v(r)$ is a free-fall velocity.

5.2: Prove that the internal energy density ε of a polytropic gas ($P \propto \rho^\gamma$) satisfied the relation

$$\varepsilon = \frac{P}{\gamma - 1}.$$

Hint: use the 1st law of thermodynamics $\mathrm{d}(\varepsilon V) = -P\ \mathrm{d}V$ and fact $V \propto 1/\rho$.

5.3: Compute γ for ultra-relativistic gas $kT \gg m_{\rm p}c^2$.

5.4: Prove

$$\frac{T(r)}{T_*} \approx \left(\frac{r_*}{r}\right)^{2/3}, \quad r \ll r_*,$$

where $kT_* \approx GM/r_* \approx m_{\rm e}c^2$ is the temperature at $r = r_* = R_{\rm g} m_{\rm p}/m_{\rm e}$.
Thus at $R_{\rm g} = 2GM/c^2$, $kT(R_{\rm g}) \approx m_{\rm e}c^2(m_{\rm p}/m_{\rm e})^{2/3} \sim 70$ MeV.

5.5: M87 is a giant elliptical galaxy in the core of the Virgo cluster. It contains a central supermassive black hole with a mass of $\sim 3 \times 10^9 M_\odot$. The nuclear region also contains a diffuse, hot interstellar gas with density $n = 0.5\ \mathrm{cm^{-3}}$ and sound speed $500\ \mathrm{km\ s^{-1}}$. Some of this gas will accrete onto the black hole. Show that the expected accretion rate is $\sim 0.1 M_\odot/\mathrm{yr}$. What fraction of the Eddington accretion rate $\dot{M}_{\rm Edd} = L_{\rm Edd}/c^2$ is this? What would be the produced luminosity if the radiative efficiency η were 0.1?