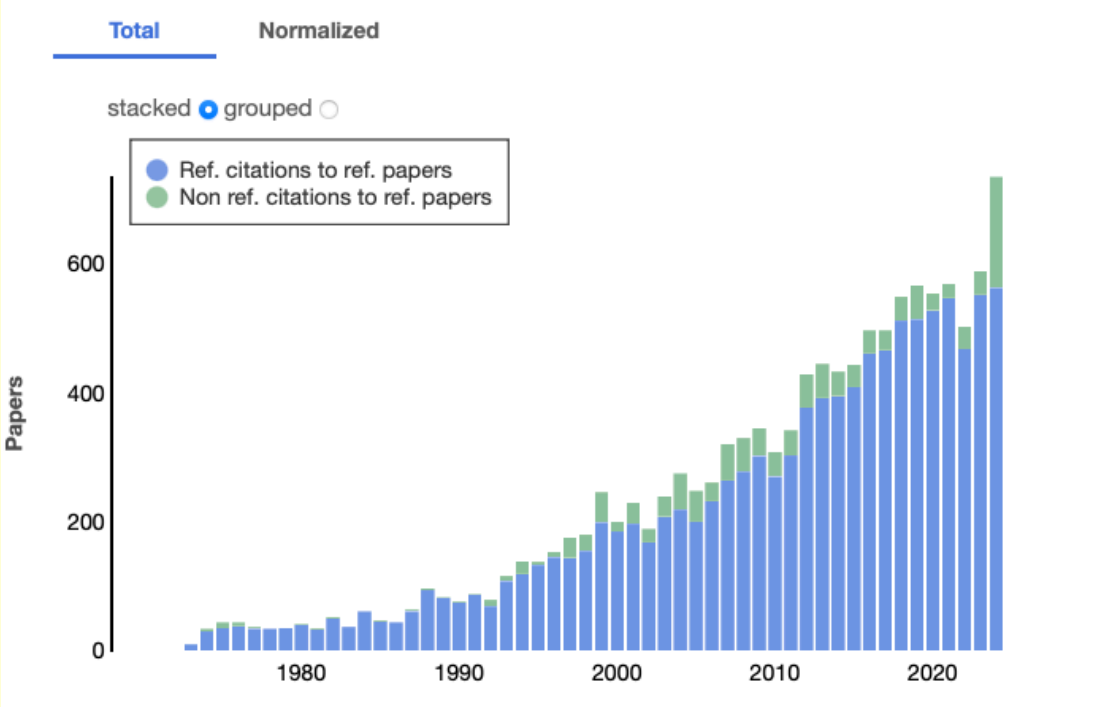
Standard accretion disc theory

references:
Shakura, Sunyaev 1973, A&A, 24, 337
Frank, King, Shu: Accretion power in astrophysics
Kato, Fukue, Mineshige: Black-hole accretion disks

Total citations	12303
Normalized citations	6151.5
Refereed citations	11010
Normalized refereed citations	5505



Accretion discs

- Keplerian rotation in Newtonian and pseudo-Newtonian potentials
- Main equations: mass conservation, angular momentum conservation
- Viscous heating, radiation flux
- alpha-prescription

- Gas in Keplerian rotation

$$\frac{V_\varphi^2}{r} = \frac{GM}{r^2} \Rightarrow V_\varphi(r) = \sqrt{\frac{GM}{r}}$$

the velocity of rotation at a radius r

$$\Omega = \frac{2\pi}{P_K} = \frac{V_\varphi}{r} \Rightarrow \Omega(r) = \sqrt{\frac{GM}{r^3}}$$

the angular velocity

$$l = rV_\varphi \Rightarrow l(r) = \sqrt{GMr}$$

the angular momentum (specific,i.e. per unit mass)

- Shear and viscosity

$\Omega(r)$ increases inwards, $\Omega \propto r^{-3/2}$, i.e. inner rings rotate faster.

$$\text{Shear} = r\frac{d\Omega}{dr} \neq 0$$

Viscous forces lead to an angular momentum exchange between the adjacent rings: the inner fast rings pass their angular momentum to the outer slower rings. As a result, angular momentum is transported outwards.

- Near a black hole or a neutron star, Newtonian gravity is a poor approximation to the real gravitational field. The gas in circular Keplerian rotation in fact has an angular momentum which is different from $\sqrt{GMr}$. The exact $l(r)$ is calculated in General Relativity. It turns out that

$$\frac{dl}{dr} > 0 \text{ at } r > 3R_S \text{ and } \frac{dl}{dr} < 0 \text{ at } r < 3R_S$$

  Therefore, as soon as the gradually spiraling gas reaches $r_* = 3R_S$, it plunges to the central object with a constant angular momentum $l_* = l\,(r_*)$.

- To stay at the circular orbit, the gas would need to increase its angular momentum. Instead, gas just falls freely with constant l.
- The circular orbit of radius r_* at which $\mathrm{d}l/\mathrm{d}r = 0$ is called the marginally stable orbit (*innermost stable circular orbit*).

Pseudo-Newtonian potential (1)

- The ability of a compact object to trap circularly rotating gas from $r_*=3R_S$ can be approximately described by replacing Newtonian gravitational potential

$$\varphi_N = -\frac{GM}{r}$$

by the so called pseudo-Newtonian potential (Paczynski & Wiita 1980):

$$\varphi_{PN} = -\frac{GM}{r - R_S}$$

At $r \gg R_S$, $\phi_{PN} \approx \phi_N$. At $r=R_S$, $\phi_{PN}=-\infty$, which corresponds that nothing can escape from inside $r=R_S$ (effectively, an infinite potential at R_S). Though one cannot describe the exact gravity of compact object just by changing the gravitational potential, ϕ_{PN} is a much better approximation to reality that ϕ_N.

Pseudo-Newtonian potential (2)

- The rotational velocity v_φ on a circular orbit in a gravitational potential is determined by equation

centrifugal acceleration → $$\frac{V_\varphi^2}{r} = \frac{d\varphi}{dr}$$ ← **gravitational acceleration**

- With $\phi = \phi_{PN}$ one gets

$$V_\varphi(r) = \frac{\sqrt{GMr}}{r - R_S}$$

$$\Omega = \frac{V_\varphi}{r} = \frac{\sqrt{GMr}}{r(r - R_S)}$$

$$l(r) = rV_\varphi = \frac{r}{r - R_S}\sqrt{GMr}$$

Check that dl/dr =0 **at** $r=3R_S$

ϕ_{PN} **allows one to mimic the marginally stable orbit = inner edge of the disk**

Mass conservation equation

- The continuity equation $$\frac{\partial \Sigma}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}\left(rV_r\Sigma\right) = 0$$

Σ is the surface density of the disk [g/cm²]

V_r is the radial velocity (accretion velocity) [cm/s]

In steady-state $$\frac{\partial \Sigma}{\partial t} = 0 \Rightarrow \frac{\partial}{\partial r}\left(rV_r\Sigma\right) = 0$$

$$(1)\ \dot{M}(r) = 2\pi r \Sigma V_r = \text{const}$$

$$\frac{\partial \dot{M}(r)}{\partial r} \neq 0$$

would lead to matter accumulation with time at some radius, which is impossible by definition of the steady-state.

Conservation of angular momentum

- The advected angular momentum at a radius r

$$\dot{\mathscr{L}} = \dot{M}\, l(r)$$

$$\dot{M} = \text{const} \quad \text{in steady state}$$

$$l(r) = \sqrt{GMr}$$

assuming $v_r \ll v_\varphi$ i.e. slow accretion, the gas is in almost circular rotation.

Consider a ring $\Delta r = r_2 - r_1$ (static in the lab frame, i.e. Euler coordinate system). The accreting gas enters the ring with specific angular momentum $l_2 = l(r_2)$

and comes out with $l_1 = l(r_1) \Rightarrow$ Angular momentum is pumped away from the ring with the rate

$$\dot{\mathscr{L}}_2 - \dot{\mathscr{L}}_1 = \dot{M}\,(l_2 - l_1)$$

Conservation of angular momentum

It must be done by external forces applied to the ring. There are two external forces due to viscous torques applied to the ring at r_1 and r_2 which have opposite sign

$$\dot{M}\,(l_2 - l_1) = G_2 - G_1 \quad (*)$$

$$\frac{d\vec{\mathscr{L}}}{dt} = \vec{r} \times \vec{f}$$

The torque G at a radius r is $G(r) = rf(r)$

$f(r)$ is the viscous force acting between the two rings with a common boundary at radius r.

$\vec{f}$ is applied along $\vec{e}_\varphi \Rightarrow \vec{f} \perp \vec{r}$

Note that equation (*) is valid for any radius, we can choose r_1 at the inner edge of the disk $r_* = 3R_S$ where $G(r_*) = 0$. Then at arbitrary $r > r_*$ we have

$$(2) \qquad \dot{M}\,[l(r) - l_*] = G(r)$$

Here $l_* \equiv l(r_*)$ is the angular momentum at the inner edge. This l_* is “swallowed ” by the black hole.

Viscous heating

The viscous force is dissipative, i.e. the work it does on the adjacent rings goes into heat. Reminder: the work done by a force $\vec{f}$ which is exerted on a body while the body passes a distance $d\vec{s}$ is $dA = \vec{f} \cdot d\vec{s}$. The power (work per unit time) is $dA/dt = \vec{f} \cdot \vec{V}$. Consider the disk as a set of narrow rings of radii $r_i, i = 1,2,...,N; r_{i+1} - r_i = \Delta r$

Each ring is in Keplerian rotation $\Omega_i = \sqrt{GM/r_i^3}$

In the frame corotating with the i-th ring, the (i+1)-th ring has a shear velocity $V_{\text{shear}} = r\Delta\Omega$ where r is the boundary between the two rings, $\Delta\Omega = \Omega_{i+1} - \Omega_i$

The work done per unit time by viscous force is

$$\frac{dA}{dt} = \vec{f} \cdot \vec{V}_{\text{shear}} = f V_{\text{shear}} < 0$$

The power dissipated per one ring is $\Delta W = -\frac{dA}{dt} = -fr\Delta\Omega$

and the power dissipated per unit radius is

$$(3) \quad \frac{dW}{dr} = -fr\frac{d\Omega}{dr} = -G(r)\frac{d\Omega}{dr}$$

The radiative flux

The main approximation of the standard model is that the locally dissipated heat is radiated away locally.

From eq(3) we get

$$(4) \quad 2F = Q_+ = \frac{1}{2\pi r}\frac{dW}{dr}$$

and from eq(2) $G(r) = \dot{M}\,[l(r) - l_*]$

Hence we have

$$Q_+ = -\frac{\dot{M}}{2\pi r}\,[l(r) - l_*]\frac{d\Omega}{dr}$$

Note the final formula for Q^+ does not depend on the nature of viscous force, the torque $G=rf$ drops out of the problem. Assuming that the disk radiates as a black body, $F = \sigma_{SB} T^4$, we can evaluate the surface temperature as a function of radius

$$T_s(r) = \left(\frac{Q_+}{2\sigma_{SB}}\right)^{1/4}$$

$$\sigma_{SB} = \frac{ac}{4} = 5.67 \times 10^{-5} \mathrm{erg/(cm^2 K^4 s)}$$

Stefan-Boltzmann constant

The radiative efficiency of the disk

In the process of accretion from $r \gg R_S$ down to $r_* = 3R_S$ gas changes its orbital energy from 0 to

(this is the specific energy, i.e. per unit mass).

$$E = \frac{V_\varphi^2(r_*)}{2} + \phi(r_*)$$

Hence, when the rate of mass accretion is $\dot{M}$, the rate of energy release in the disk is

$$L = -\dot{M}\left[\frac{V_\varphi^2(r_*)}{2} + \phi(r_*)\right]$$

(by assumption this power is radiated away)

Inside r_*, no energy is dissipated since the gas is in fact free-fall and viscosity is negligible.

The disk radiative efficiency is by definition

$$\epsilon \equiv \frac{L}{\dot{M}c^2} = -\frac{V_\varphi^2(r_*)}{2c^2} - \frac{\phi(r_*)}{c^2}$$

The height of the disk

Hydrostatic balance in the vertical direction

$$\frac{\partial P}{\partial z} = -\frac{\partial \phi}{\partial z}\rho, \qquad \phi = -\frac{GM}{R} = -\frac{GM}{\sqrt{r^2 + z^2}} \Rightarrow \frac{\partial \phi}{\partial z} = \frac{GMz}{R^3}$$

For a geometrically thin disk, we have $R=r$ (since $z \ll r$)

$$\frac{\partial P}{\partial z} = -\frac{GMz}{r^3}\rho$$

The typical scale-height of the disk can be estimated from equation $\frac{P}{H} = \frac{GMH}{r^3}\rho \rightarrow \frac{H}{r} = \frac{c_s}{V_\varphi}$

where isothermal sound speed $\quad c_s^2 \equiv \frac{P}{\rho}$

alpha-prescription

The viscous force at a radius r can be written in the form $f = 2H \times 2\pi r \times t_{r\varphi}$ where $2H \times 2\pi r$ is the area of the vertical cross-section of the disk

$t_{r\varphi}$ is the <u>viscous stress</u> (= force per unit area), it has dimension of pressure $t_{yx} \propto \frac{dV_x}{dy}$

The α -prescription (Shakura 1972) $t_{r\varphi} = \alpha P$

where P is pressure in the disk and $\alpha < 1$ is a numerical factor

Dynamic and kinematic coefficients of viscosity

$t_{r\varphi}$ can be written as $$t_{r\varphi} = \eta r \frac{d\Omega}{dr}$$

viscous force per unit area of the cross-section is proportional to the shear. The coefficient of proportionality η is called <u>dynamic viscosity</u>.

$\nu = \eta/\rho$ is called <u>kinematic viscosity.</u>

[ν]=cm²/s has the meaning of the diffusion coefficient

In terms of $$t_{r\varphi} = \nu\rho r \frac{d\Omega}{dr} = \nu \frac{\Sigma}{2H} r \frac{d\Omega}{dr}$$

$$G = rf \Rightarrow \quad G = 2\pi r^3 \nu \Sigma \frac{d\Omega}{dr}$$

Turbulent disk

As a result of turbulent pulsations in the disk, each gas element diffuses from one circular orbit to another, with a diffusion coefficient ν

Let V_t be the typical turbulent velocity and d be the typical scale of the turbulent motions. Then the diffusion coefficient is $\nu \approx V_t d$

Compare with the standard problem of a drunk sailor: $\overline{x^2(t)} = Dt$ if $x(0) = 0$.

$D = \frac{(\Delta x)^2}{\tau}$,where Δx is a one random step, τ is the time of one step

$V = \frac{\Delta x}{\tau}$ is the random velocity, $D = V\ \Delta x$

As home you will show that $t_{r\varphi} = \alpha P$ is equivalent to the prescription $\nu = \frac{2}{3}\alpha c_s H$. This scaling is expected

on physical grounds $V_t < c_s$ and $d < H \rightarrow \nu \leq c_s H$

The velocity of accretion

From eq.1, mass conservation, we get $V_r = \frac{\dot{M}}{2\pi r \Sigma}$

The accretion is caused by the torque

$$G = rf = 4\pi H r^2 t_{r\varphi} = 4\pi H r^2 \alpha P$$

On the other hand, from eq.2 (angular momentum conservation), we have $G(r) = \dot{M}\ [l(r) - l_*]$

Thus $4\pi H r^2 \alpha P = \dot{M}\left[l(r) - l_*\right] \Rightarrow \frac{\dot{M}}{\Sigma}\left[l(r) - l_*\right] = 2\pi r^2 \alpha c_s^2 \Rightarrow$

$$\left(c_s^2 \equiv \frac{P}{\rho} = \frac{2HP}{\Sigma}\right)$$

$$V_r = \frac{\alpha c_s^2 r}{l(r) - l_*}$$

Since $l(r) \equiv r V_\varphi$ we get $V_r = \frac{\alpha c_s^2}{V_\varphi[1 - l_*/l(r)]} \sim \alpha V_\varphi \left(\frac{H}{r}\right)^2$

Note that $V_r \propto \alpha, \Sigma \propto \alpha^{-1}$ disks with small α are dense

Radiative cooling

Let T be the temperature inside the disk (in the midplane). If the disk is sufficiently dense (low alpha), so that its opacity to radiation is high, then the accreting gas is a close to a black body=> the radiation density inside is $w = aT^4$

The radiation diffuses out of the disk. The corresponding vertical flux is related to w by

$$F = \frac{cw}{3\tau_0} = \frac{caT^4}{3\tau_0} = \frac{4\sigma_{SB}T^4}{3\tau_0}$$

where $\tau_0 = \Sigma\sigma_T / 2m_p$ is the Thomson optical depth from the disk surface to the midplane. Substituting, we get

$$F = \frac{8m_p\sigma_{SB}T^4}{3\Sigma\sigma_T}$$

Radiative cooling

The stationary diffusion of radiation:

$$F(\tau) = const, \quad 0 \le \tau \le \tau_0 \qquad d\tau = -\frac{\rho}{m_p}\sigma_T dz$$

$$F = \frac{c}{3}\frac{dw}{d\tau} = -\frac{c\lambda}{3}\frac{dw}{dz}$$

where $\frac{c\lambda}{3}$ is the diffusion coefficient,

λ is the photon mean free-path. Integrating, we get

$$w = w_S + \frac{3F}{c}\tau_0$$ where $w_s = w(0)$ at surface $w_s \sim F/c$

At $\tau_0 \gg 1$ one gets $$w \approx \frac{3F}{c}\tau_0$$

The sound speed $c_s^2 \equiv \frac{P}{\rho}$

The pressure of a mixture of ionized gas and radiation is

$$P = 2nkT + \frac{aT^4}{3}$$

ionized gas

radiation

Density $\rho = m_p n = \frac{\Sigma}{2H}$

Thus the sound speed is

$$c_s^2 = \frac{2kT}{m_p} + \frac{2aT^4 H}{3\Sigma}$$

The complete set of equations for the alpha-disk

mass conservation $\dot{M}(r) = 2\pi r \Sigma V_r$

angular momentum conservation plus alpha-prescription $V_r = \frac{\alpha c_s^2}{V_\varphi}[1 - l_*/l(r)]^{-1}$

vertical balance $\frac{H}{r} = \frac{c_s}{V_\varphi}$

heating=cooling balance $2F = \frac{16 m_p \sigma_{SB} T^4}{3\Sigma\sigma_T} = -\frac{\dot{M} l(r)}{2\pi r}\frac{d\Omega}{dr}\left[1 - \frac{l_*}{l(r)}\right]$

Sound speed in a black body gas $c_s^2 = \frac{2kT}{m_p} + \frac{2aT^4 H}{3\Sigma}$

5 unknown Σ, V_r, c_s, H, T and 2 parameters $\dot{M}, \alpha$

How to solve?

- Since the values of the variables vary from 10^{-27} to 10^{33}, it is necessary to introduce dimensionless variables to describe the equations 1-5.

- Radius:
$$x = \frac{r}{r_g} \iff r = x r_g$$
- Velocity:
$$\hat{v} = \frac{v}{c} \iff v = \hat{v} c$$
- Surface density:
$$\tau = \frac{\Sigma \sigma_T}{2 m_p} \iff \Sigma = 2\tau \frac{m_p}{\sigma_T}$$
- Accretion rate:
$$\dot{m} = \frac{\dot{M} c^2}{L_{EDD}} \iff \dot{M} = \dot{m} 2\pi \frac{r_g m_p c}{\sigma_T}$$
- Sound speed:
$$\hat{c}_s^2 = \frac{c_s^2}{c^2} \iff c_s^2 = c^2 \hat{c}_s^2$$
- Scale height:
$$\hat{H} = \frac{H}{r_g} \iff H = \hat{H} r_g$$
- Temperature:
$$\hat{T}^4 = \frac{\sigma_T \sigma_{SB} r_g}{2\pi m_p c^3} T^4 \iff T^4 = \frac{2\pi m_p c^3}{\sigma_T \sigma_{SB} r_g} \hat{T}^4$$

How to solve?

For the Newtonian gravitational potential

- Potential:

$$\varphi_N(x) = -\frac{GM}{xr_g} = -\frac{c^2}{2x}$$

- Rotational velocity:

$$v_{\varphi N}(x) = \sqrt{\frac{GM}{xr_g}} = \frac{c}{\sqrt{2x}}$$

- Angular velocity:

$$\Omega_N(x) = \sqrt{\frac{GM}{x^3 r_g^3}} = \frac{c}{xr_g\sqrt{2x}}$$

- Angular momentum:

$$l_N(x) = \sqrt{GMxr_g}$$

$$\frac{d\Omega_N}{dr} = -\frac{3}{2}\sqrt{\frac{GM}{x^5 r_g^5}} = -\frac{3}{2}\frac{c}{r_g^2\sqrt{2x^5}}$$

How to solve?

For the **Newtonian gravitational potential**

- Mass conservation:

$$\tau = \frac{\dot{m}}{2x\hat{v}}$$

- Angular momentum conservation and α-prescription:

$$\hat{v} = \alpha c_s^2 \sqrt{2x} \left[1 - \sqrt{\frac{3}{x}}\right]^{-1}$$

- Vertical balance:

$$\hat{H} = x(2x)^{1/4} \sqrt{\frac{\hat{v}}{\alpha} \left[1 - \sqrt{\frac{3}{x}}\right]}$$

- Cooling = heating locally:

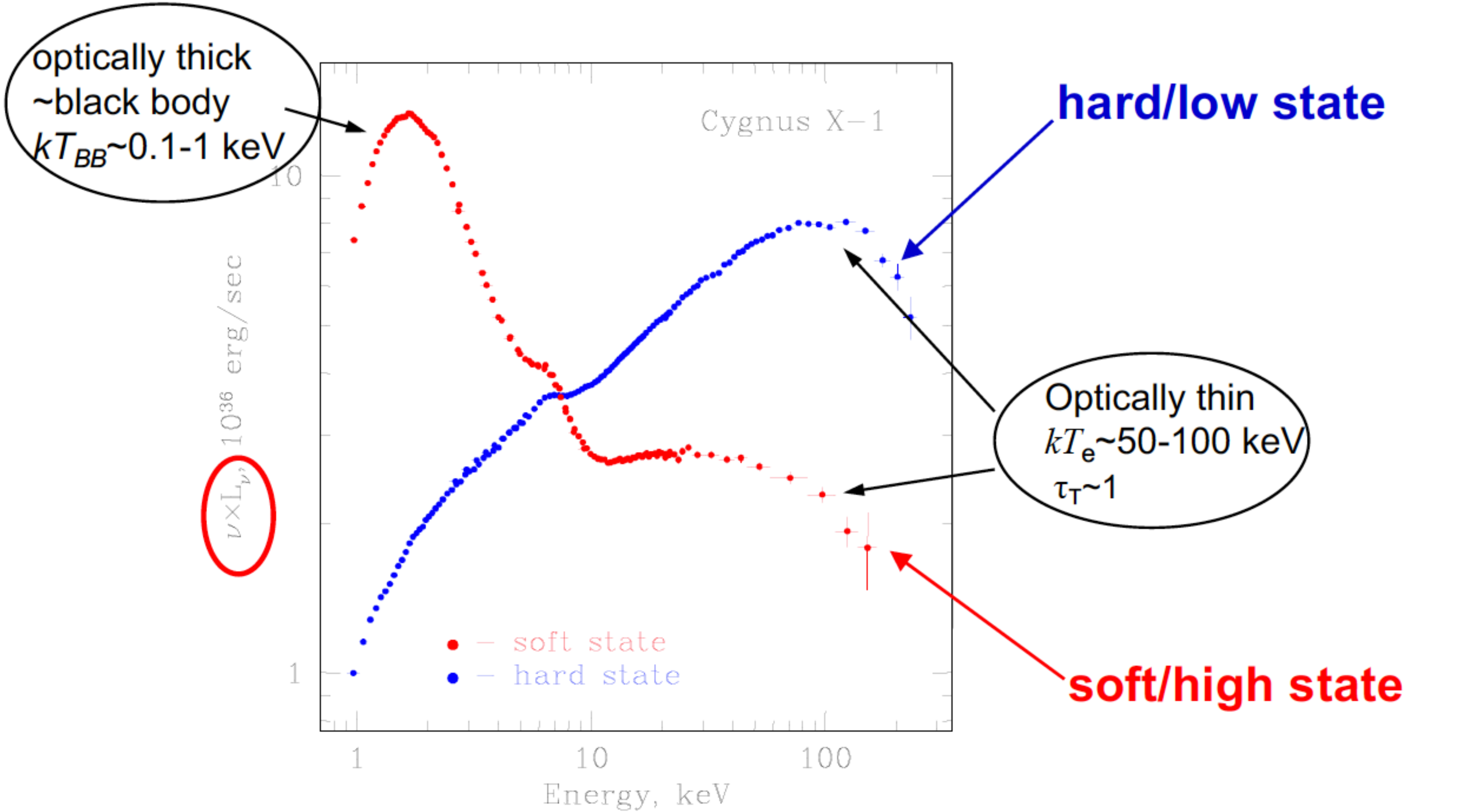
$$\hat{T}^4 = \frac{9\tau\dot{m}}{64\pi x^3} \left[1 - \sqrt{\frac{3}{x}}\right]$$

- Sound speed in black-body gas:

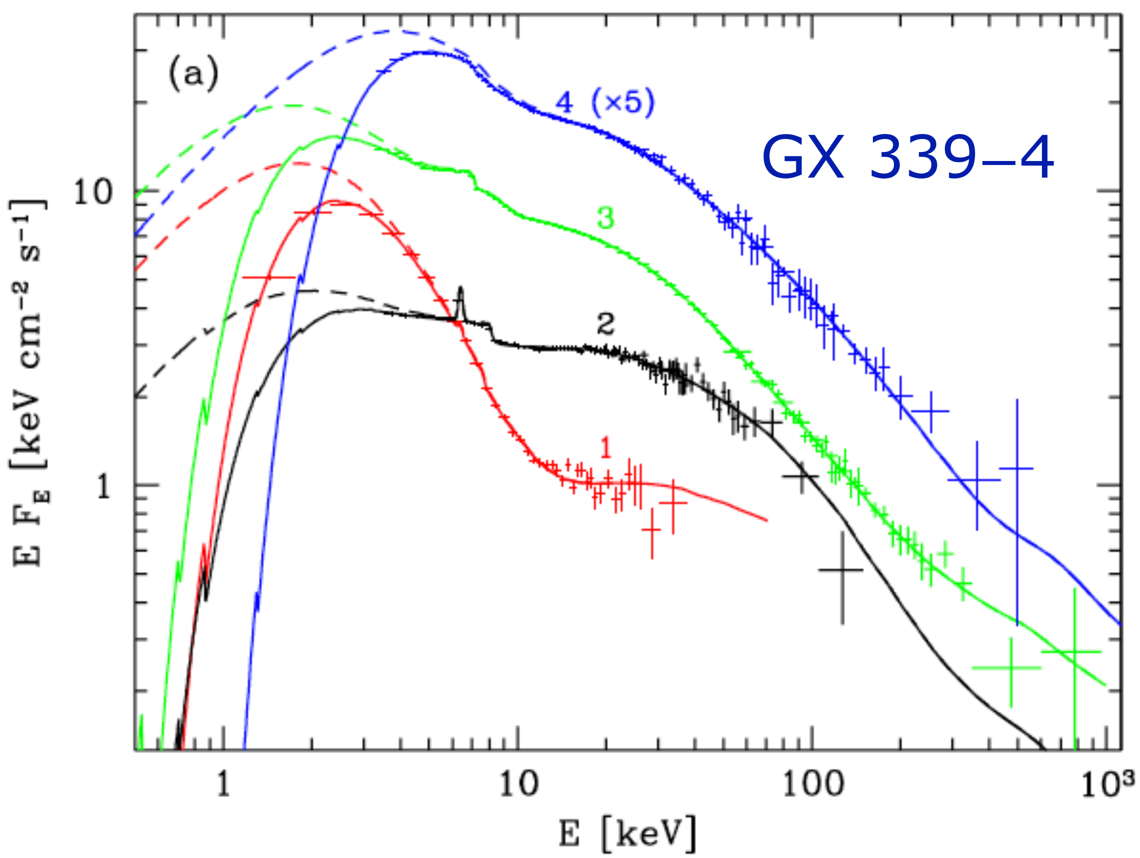
$$\hat{c}_s^2 = A\hat{T} + \frac{8\pi\hat{H}}{3\tau}\hat{T}^4$$

Where the constant $A = \frac{2k}{m_p c^2} \left(\frac{2\pi m_p c^3}{\sigma_T \sigma_{SB} r_g}\right)^{1/4} \approx 7.35 \cdot 10^{-6}$.

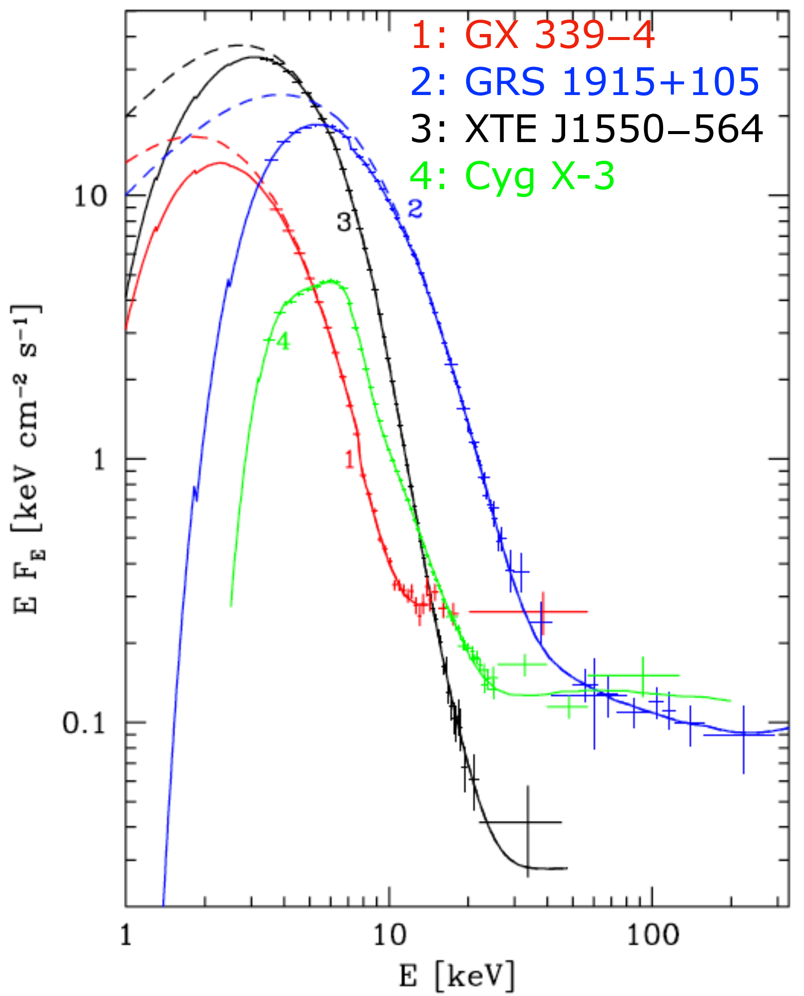
Black holes (Cyg X-1)



Black holes

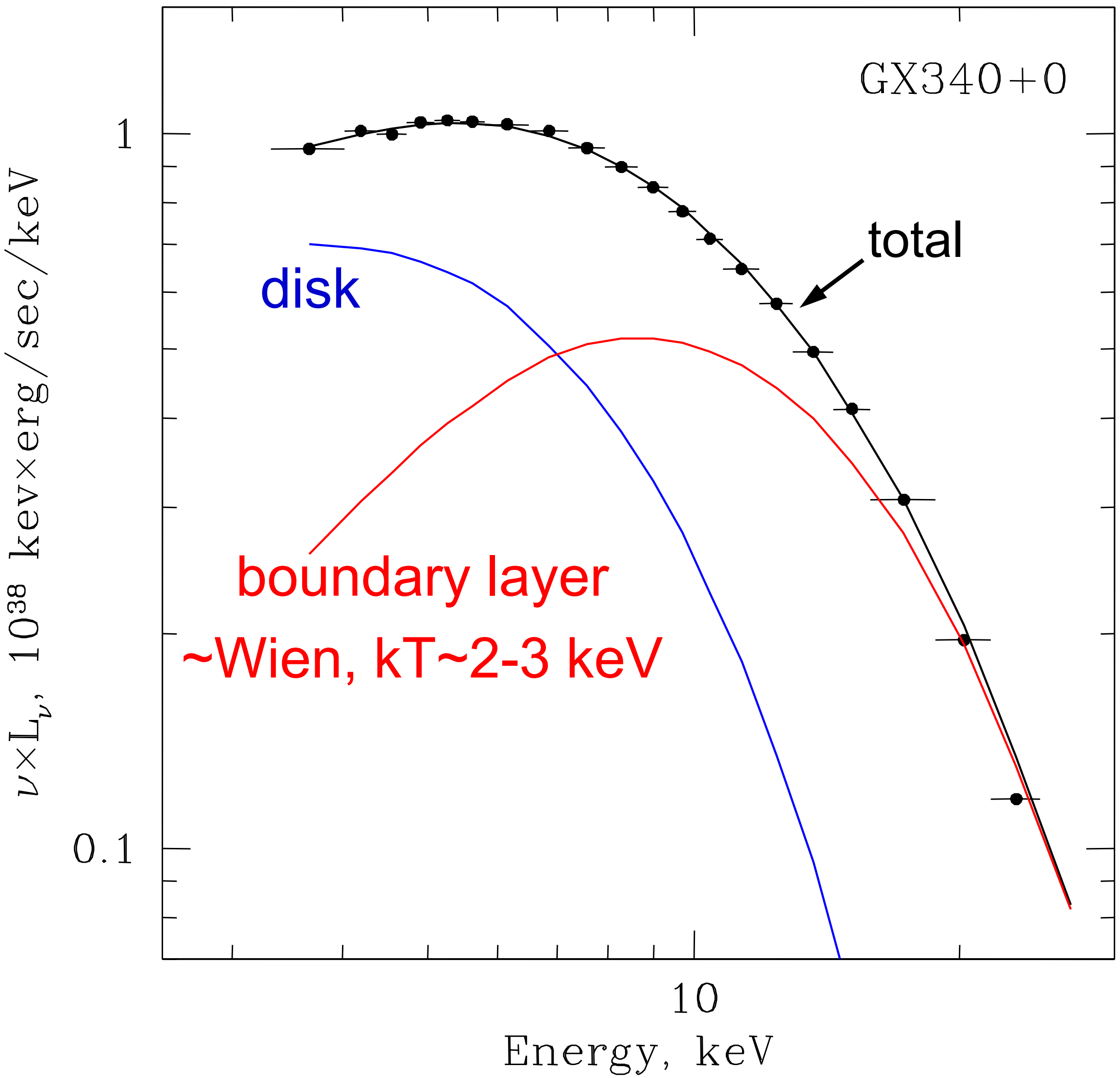


(a)
GX 339–4
4 (×5)
3
2
1
10
1
1
10
100
10³
E [keV]
E F_E [keV cm⁻² s⁻¹]



1: GX 339–4
2: GRS 1915+105
3: XTE J1550–564
4: Cyg X-3
10
1
0.1
1
10
100
E [keV]
E F_E [keV cm⁻² s⁻¹]

Boundary layer and disk in NS

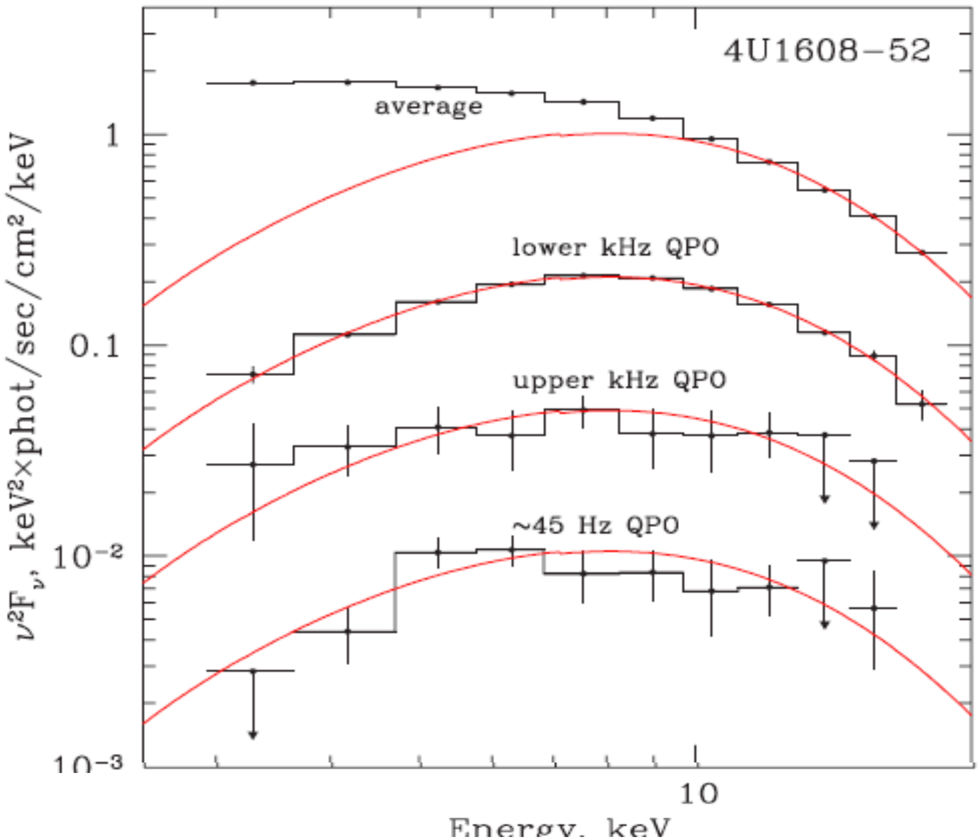


$$\dot{M} > 5 - 10\% \dot{M}_{\rm Edd}$$

Boundary layer:

- spectral component
- variability component

Boundary layer and disk in NS



$$R = N_\gamma / T$$

$$P_j = 2|a_j|^2 / N_\gamma R$$

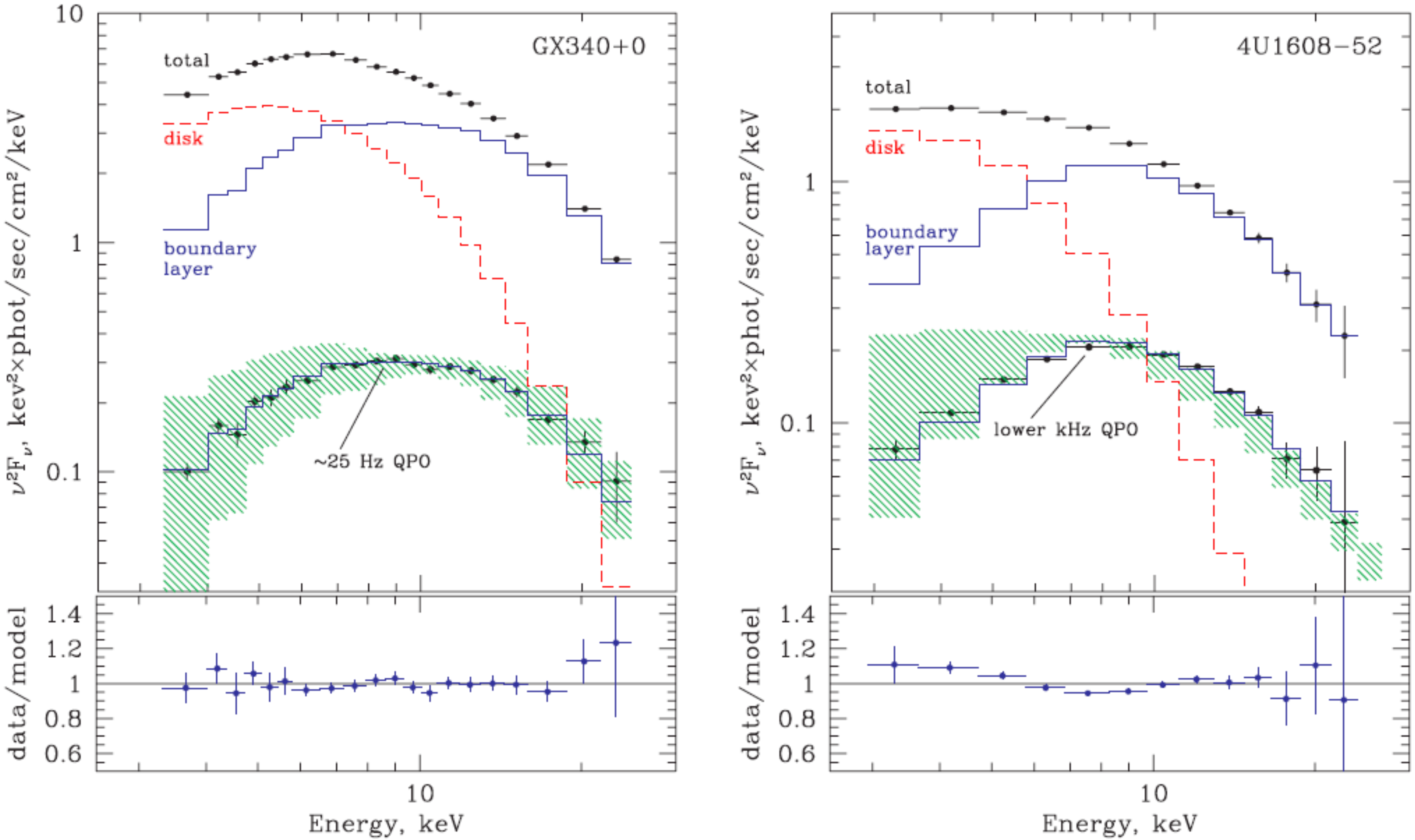
$$a_j = \sum_{k=1}^{2^m} x_k e^{i\omega_j t_k}$$

$S(E_i, f_j)$ is the countrate of the spectrum at frequency f_j in the energy chanel E_i

Fourier frequency resolved spectra

$$S(E_i, f_j) = R_i \sqrt{P_i(f_j)\Delta f_j} = \sqrt{\frac{2|a_{ij}|^2}{T}\Delta f_j}.$$

Boundary layer and disk in NS



Fourier frequency resolved spectra

Gilfanov et al. 2003

$$S(E_i, f_j) = R_i\sqrt{P_i(f_j)\Delta f_j} = \sqrt{\frac{2|a_{ij}|^2}{T}\Delta f_j}.$$