HIGH ENERGY ASTROPHYSICS
Compulsory Home Exercises. Problem set 5. Solutions.

Problems

5.1: Consider spherical accretion onto a black hole.
(a) Show that the sonic point is

$$r_\mathrm{s} = \frac{GM}{c_\mathrm{s}^2(\infty)}\frac{5-3\gamma}{4}.$$

(b) Show that the accretion rate is

$$\dot{M} = \pi G^2 M^2 \frac{\rho(\infty)}{c_\mathrm{s}^3(\infty)}\left[\frac{2}{5-3\gamma}\right]^{\frac{5-3\gamma}{2(\gamma-1)}}.$$

Compute the limit at $\gamma \to 5/3$. Express

$$\dot{M} = C\left(\frac{M}{M_\odot}\right)^2\left(\frac{\rho(\infty)}{10^{-24}\ \mathrm{g\ cm^{-3}}}\right)\left(\frac{c_\mathrm{s}(\infty)}{10\ \mathrm{km\ s^{-1}}}\right)^{-3}\ \mathrm{g\ s^{-1}}$$

and find constant C.
(c) Show that $\dot{M} = \pi R_\mathrm{acc}^2 c_\mathrm{s}(\infty)\rho(\infty)$ (for $\gamma = 5/3$).
(d) Obtain $\rho(r), T(r)$ for $r \ll r_\mathrm{s}$. Hint: in this region $v(r)$ is a free-fall velocity.

Solution: all the parameters of the solution may be found by considering the sonic point. Let us assume $\gamma < 5/3$. As we found in the lectures, at the sonic surface

$$c_\mathrm{s}(r_\mathrm{s}) = v(r_\mathrm{s}) = \sqrt{\frac{GM}{2r_\mathrm{s}}}.$$

If we substitute these expressions into the general expression for Bernoulli integral at $r = r_\mathrm{s}$

$$B = \frac{v^2}{2} + \frac{c_\mathrm{s}^2}{\gamma-1} - \frac{GM}{r} = \frac{c_{\mathrm{s},\infty}^2}{\gamma-1},$$

we get

$$\frac{GM}{r_\mathrm{s}}\left[\frac{1}{4}+\frac{1}{2(\gamma-1)}-1\right] = \frac{GM}{r_\mathrm{s}}\frac{5-3\gamma}{4(\gamma-1)} = \frac{c_{\mathrm{s},\infty}^2}{\gamma-1},$$

or

$$r_\mathrm{s} = \frac{5-3\gamma}{4}\frac{GM}{c_{\mathrm{s},\infty}^2}, \tag{1}$$

which proves (a). The velocity at the sonic point is

$$c_\mathrm{s}^2(r_\mathrm{s}) = v^2(r_\mathrm{s}) = \frac{2}{5-3\gamma}c_{\mathrm{s},\infty}^2. \tag{2}$$

The mass accretion rate does not depend on radius, hence let us estimate it at r_s, where we already know the velocity. Density is still missing, and this is expectable as we did not use the boundary condition for density at infinity so far. From the equation of state we use,

$$c_{\rm s}^2 = c_{\rm s,\infty}^2 \left(\frac{\rho}{\rho_\infty}\right)^{\gamma-1}.$$

Solving this expression for density at $r = r_{\rm s}$, we get

$$\rho(r_{\rm s}) = \rho_\infty \left(\frac{v(r_{\rm s})}{c_{\rm s,\infty}}\right)^{2/(\gamma-1)} = (5-3\gamma)^{-1/(\gamma-1)}\,\rho_\infty. \tag{3}$$

Now we are ready to collect equations (1)–(3) to calculate the mass accretion rate

$$\dot{M} = 4\pi r^2 v\rho = 4\pi r_{\rm s}^2 v(r_{\rm s})\rho(r_{\rm s}) = \pi \left(\frac{2}{5-3\gamma}\right)^{\frac{5-3\gamma}{2(\gamma-1)}} \frac{G^2M^2\rho_\infty}{c_{\rm s,\infty}^3}. \tag{4}$$

This expression provides the answer to the first question of item (b).

In the limit $\gamma \to 5/3$, the coefficient in equation (4) becomes

$$\lim_{\gamma\to 5/3}\left(\frac{2}{5-3\gamma}\right)^{\frac{5-3\gamma}{2(\gamma-1)}} = e^{-\frac{1}{\gamma-1}\lim_{x\to 0} x\ln x} = 1, \tag{5}$$

where $x = (5-3\gamma)/2$. As $x\ln x$ approaches zero, the coefficient becomes unity, and

$$\dot{M}(\gamma = 5/3) = \pi \frac{G^2M^2\rho_\infty}{c_{\rm s,\infty}^3}. \tag{6}$$

This provides the answer to the second question of item (b) and to (c), as $R_{\rm acc} \simeq \frac{GM}{c_{\rm s,\infty}^2}$, and equation (6) may be re-written in the form

$$\dot{M} \simeq \pi R_{\rm acc}^2 \rho_\infty c_{\rm s,\infty}.$$

In a normalized form, expression (6) becomes

$$\dot{M} \simeq 5.5\times 10^{10} \left(\frac{M}{M_\odot}\right)^2 \left(\frac{\rho(\infty)}{10^{-24}\ {\rm g\ cm^{-3}}}\right) \left(\frac{c_{\rm s}(\infty)}{10\ {\rm km\ s^{-1}}}\right)^{-3}\ {\rm g\ s^{-1}} \tag{7}$$

(d) In the limit $r \ll r_{\rm s}$, $v \gg c_{\rm s} \gg c_\infty$, Bernoulli integral conservation implies

$$\frac{v^2}{2} \simeq \frac{GM}{r},$$

that means that the velocity field is indistinguishable from free fall. Mass conservation allows to express density as

$$\rho(r) = \frac{\dot{M}}{4\pi r^2 v} \simeq \frac{\dot{M}}{4\pi\sqrt{GM}} r^{-3/2}. \tag{8}$$

For the sound speed, we have

$$c_{\rm s}^2(r \ll r_{\rm s}) \simeq c_{\rm s}^2(r_{\rm s}) \left(\frac{r}{r_{\rm s}}\right)^{-\frac{3}{2}(\gamma-1)}.$$

Since the pressure for the ideal gas is

$$P = \frac{\rho}{\mu m_{\mathrm{p}}} kT,$$

where μ is the mean molecular weight, and

$$c_{\mathrm{s}}^2 = \gamma P/\rho = \gamma \frac{kT}{\mu m_{\mathrm{p}}},$$

we get for the temperature

$$kT = \frac{\mu m_{\mathrm{p}}}{\gamma} c_{\mathrm{s}}^2,$$

therefore

$$kT(r \ll r_{\mathrm{s}}) \simeq \frac{\mu m_{\mathrm{p}}}{\gamma} \frac{GM}{2r_{\mathrm{s}}} \left(\frac{r_{\mathrm{s}}}{r}\right)^{\frac{3}{2}(\gamma-1)}.$$

For $\gamma \simeq 5/3$ and pure hydrogen plasma ($\mu = 1/2$), we get temperature proportional to the virial one:

$$kT(r \ll r_{\mathrm{s}}) \simeq \frac{3}{20} \frac{GMm_{\mathrm{p}}}{r}.$$

For $r_{\mathrm{s}} \ll r \ll GM/c_{\infty}^2$, temperature scales with the virial one as well, as one may neglect the dynamical term in Bernoulli integral.

5.2: Prove that the internal energy density ε of a gas with an adiabatic equation of state ($P \propto \rho^\gamma$) satisfies the relation

$$\varepsilon = \frac{P}{\gamma - 1}.$$

Hint: use the 1st law of thermodynamics $\mathrm{d}(\varepsilon V) = -P\ \mathrm{d}V$ and fact $V \propto 1/\rho$.

Solution: According to the first law of thermodynamics, for an adiabatic process,

$$d(\varepsilon V) = -PdV.$$

Volume is inversely proportional to density, therefore we can re-write the equation as

$$d\left(\frac{\varepsilon}{\rho}\right) = -Pd\frac{1}{\rho}.$$

Integrating this equation yields

$$\frac{\varepsilon}{\rho} = -\int_{\rho=0}^{\rho'=\rho} P(\rho')d\frac{1}{\rho'},$$

where we assumed that zero pressure means zero energy density. Substituting $P = K\rho^\gamma$ (where K is constant), we get

$$\frac{\varepsilon}{\rho} = K\int_{\rho=0}^{\rho'=\rho} \rho'^{\gamma-2}d\rho' = \frac{K}{\gamma-1}\rho^{\gamma-1},$$

that implies

$$\varepsilon = \frac{K}{\gamma-1}\rho^\gamma = \frac{P}{\gamma-1}.$$

5.3: Compute γ for ultra-relativistic gas $kT \gg m_{\rm p}c^2$.

Solution: A straightforward way to calculate it is to use the two properties of ultrarelativistic particles: (i) their velocities are close to c and (ii) their energies E and momenta p are related as $E \simeq pc$. Pressure in general may be calculated as the momentum flux

$$P = \int f(E) p_x v_x dE d\cos\theta d\varphi,$$

where $f(E)$ is the particle distribution function over energy per unit solid angle, $p_x = p\cos\theta = \frac{E}{c}\cos\theta$ is the momentum of the particle, $v_x = c\cos\theta$ is its velocity. Momenta and velocities are assumed to be isotropically distributed. Isotropicity allows to integrate over the angles:

$$P = \int f(E)EdE \int \cos^2\theta d\cos\theta d\varphi = \int f(E)EdE \times 2\pi \int_{-1}^{1} \cos^2\theta d\cos\theta = \frac{4\pi}{3}\int f(E)EdE.$$

A similar expression may be written for the energy density

$$\varepsilon = \int f(E)EdEd\cos\theta d\varphi = 4\pi \int f(E)EdE.$$

Finally, we get

$$P = \frac{1}{3}\varepsilon.$$

Together with the result of the previous problem, this implies $\gamma - 1 = 1/3$, or $\gamma = 4/3$.

5.4: Prove

$$\frac{T(r)}{T_*} \approx \left(\frac{r_*}{r}\right)^{2/3}, \quad r \ll r_*,$$

where $kT_* \approx GM/r_* \approx m_{\rm e}c^2$ is the temperature at $r = r_* = R_{\rm g}m_{\rm p}/m_{\rm e}$.
Thus at $R_{\rm g} = 2GM/c^2$, $kT(R_{\rm g}) \approx m_{\rm e}c^2(m_{\rm p}/m_{\rm e})^{2/3} \sim 70$ MeV.

Solution: In this regime, electrons get relativistic, but ions are still non-relativistic. Adiabatic index is equal to $\gamma = 13/9$ (see lecture notes). Density changes (see problem 5.1d) as

$$\rho \propto r^{-3/2},$$

and pressure as

$$P \propto \rho^{\gamma} \propto r^{-\frac{3}{2}\gamma} \propto r^{-13/6},$$

that implies $T \propto P/\rho \propto r^{-2/3}$. As $T(r_*) = T_* \simeq GM/r_*$, $r_* = R_{\rm g}m_{\rm p}/m_{\rm e}$,

$$T(r) \simeq T_* \left(\frac{r_*}{r}\right)^{2/3},$$

and

$$T(R_{\rm g}) \simeq T_* \left(\frac{m_{\rm p}}{m_{\rm e}}\right)^{2/3} \simeq m_{\rm e}c^2 \left(\frac{m_{\rm p}}{m_{\rm e}}\right)^{2/3}.$$

5.5: M87 is a giant elliptical galaxy in the core of the Virgo cluster. It contains a central supermassive black hole with a mass of $\sim 3 \times 10^9 M_\odot$. The nuclear region also contains a diffuse, hot interstellar gas with density $n = 0.5$ cm^{-3} and sound speed 500 km s^{-1}. Some of this gas will accrete onto the black hole. Show that the expected accretion rate is $\sim 0.1 M_\odot$/yr. What fraction of the Eddington accretion rate is this? What would be the produced luminosity if the radiative efficiency η were 0.1?

Solution: Hydrogen number density $n = 0.5$ cm^{-3} corresponds to mass density $\rho \simeq \frac{5}{6} \times 10^{-24}$ g cm^{-3}. Substituting all the numerical values into equation (7), we get

$$\dot{M} \simeq 3.3 \times 10^{24} \mathrm{g\,s^{-1}} \simeq 0.05 M_\odot \mathrm{yr}^{-1}.$$

The Eddington mass accretion rate, at the same time, is

$$\dot{M}_{\rm Edd} = \frac{L_{\rm Edd}}{c^2} = \frac{4\pi G M}{\varkappa c} \simeq 4.2 \times 10^{26} \mathrm{g\,s^{-1}} \simeq 6.6 M_\odot \mathrm{yr}^{-1},$$

where $\varkappa = 0.4\,\mathrm{cm^2\,g^{-1}}$ is the mass absorption coefficient for electron scattering in hydrogen plasma. (Sometimes the Eddington accretion rate is defined as $\dot{M}_{\rm Edd} = L_{\rm Edd}/\eta c^2$, resulting in ten times larger value.) This means that the accretion rate is $\sim$0.8% of the Eddington one. The estimated luminosity for $\eta = 0.1$ is

$$L = \eta \dot{M} c^2 \simeq 3.0 \times 10^{44} \mathrm{erg\,s^{-1}}.$$