HIGH ENERGY ASTROPHYSICS

Compulsary Home Exercises. Problem set 6. Solutions.

Problems

6.1: The X-ray spectrum of an accreting black hole GX 339–4 is shown in Fig. 1. Estimate the photon spectral index Γ of the Comptonized component (shown with dashed line) in the standard X-ray band 2–10 keV. Estimate the electron temperature that is needed to produce the observed spectrum by Comptonization. Compute the X-ray luminosity of the object, assuming the distance of 5 kpc.

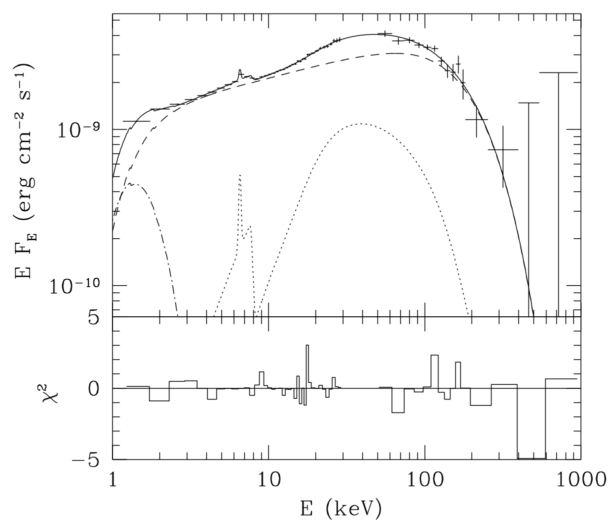

Figure 1: Broad-band spectrum of GX 339–4 as observed by *Ginga* and OSSE/*CGRO* in 1991 (from Zdziarski et al. 1998).

Solution:

Let us first estimate the photon index:

$$\begin{aligned}\Gamma &= 1 - \frac{d\log F_E}{d\log E} = 2 - \frac{d\log(EF_E)}{d\log E} \approx 2 - \frac{\log[EF_E(10\,\mathrm{keV})] - \log[EF_E(2\,\mathrm{keV})]}{\log 10 - \log 2} \\ &= 2 - \frac{\log[EF_E(10\,\mathrm{keV})/EF_E(2\,\mathrm{keV})]}{\log(10/2)} \approx 2 - \frac{0.3}{0.7} \approx 1.57. \qquad (1)\end{aligned}$$

The electron temperature can be estimated from the characteristic energy of the cut-off in the spectrum, i.e. approximately 100 keV (or 10^9 K).

The flux can be calculated by integrating over the flux energy distribution, $F_E = F_0 E^{1-\Gamma}$, where E is in keV and $F_0 \approx 10^{-9}$ erg cm^{-2} s^{-1}:

$$F = \int_0^{100} F_E dE = \int_0^{100} F_0 E^{1-\Gamma} dE = F_0 \left.\frac{E^{2-\Gamma}}{2-\Gamma}\right|_0^{100} \approx 17 \times F_0 = 1.7 \times 10^{-8} \mathrm{erg\ cm^{-2}\,s^{-1}}. \qquad (2)$$

The X-ray luminosity (D is distance, 1 kpc $\approx 3 \times 10^{21}$ cm):

$$L = 4\pi D^2 F = 4\pi (15 \times 10^{21})^2\, 1.7 \times 10^{-8} = 4.8 \times 10^{37}\,\mathrm{erg\ s^{-1}}. \qquad (3)$$

6.2: Consider photon gas with the intensity given by the Planck (blackbody) distribution of temperature of $kT_{\rm BB} = 0.33$ keV. The photons are penetrating into a hot medium with electron temperature $kT_{\rm e} = 100$ keV and are being Compton up-scattered. Compute how many scatterings are needed for a typical photon to achieve the final energy $E_{\rm f} = 100$ keV.

Solution:

Number of scatterings $N_{\rm sca}$ can be estimated assuming the energy in each scattering grows as $\Delta E/E = 4kT_e/(m_e c^2)$, so to get to the final energy we need

$$E_{\rm f} = E_{\rm seed}(1 + \Delta E/E)^{N_{\rm sca}}. \tag{4}$$

Peak of the blackbody spectrum with seed photons is at $\sim 3kT_{\rm BB} = 1$ keV. To get to 100 keV, we need

$$N_{\rm sca} = \log(E_{\rm f}/E_{\rm seed})/\log(1 + \Delta E/E) \approx \log(100)/\log(1 + 4 \times 100/511) \approx 8. \tag{5}$$

6.3: Consider an accretion disc illuminated by an isotropic X-ray source located 30 km above the centre of the disc. The disc has a hole in the centre with radius of 100 km, but otherwise is flat and extends to infinity. Assuming flat space, calculate the reflection factor $R = \Omega/2\pi$ from such a disc, where Ω is the solid angle occupied by the disc as viewed from the X-ray source.

Solution:

Let us introduce spherical coordinate system with the z-axis directed from the X-ray source towards the disc centre. The solid angle of the disc with the hole is given by the integral:

$$\Omega = \int_0^{2\pi} d\phi \int_0^{\cos\theta_0} d\cos\theta = 2\pi\cos\theta_0, \tag{6}$$

where θ_0 is the polar angle at which the disc edge is seen from the X-ray source with

$$\tan\theta_0 = \frac{100}{30} \approx 3.3. \tag{7}$$

Thus the reflection factor is

$$R = \frac{\Omega}{2\pi} = \frac{2\pi\cos\theta_0}{2\pi} = \cos\theta_0 = \sqrt{\frac{1}{1+\tan^2\theta_0}} \approx 0.3. \tag{8}$$

6.4: Consider a light curve with the counts per bin $s_k,\ k = 1, ..., N.$ Show the relation

$$\mathrm{rms}^2 \equiv \frac{\overline{s^2} - \overline{s}^2}{\overline{s}^2} = \Delta f \sum_{j>0} P(f_j), \tag{9}$$

where $P(f_j) = 2|S_j|^2/(R^2 T)$, $R = \overline{s}N/T$ - mean count rate per second, $f_j = j/T$, $\Delta f = 1/T$,

$$S_j = \sum_{k=0}^{N-1} s_k e^{2\pi i jk/N}, \quad j = -N/2, ..., N/2 - 1, \tag{10}$$

is the discrete Fourier transform of the count rate and

$$\overline{s} = \frac{1}{N}\sum_{k=1}^{N} s_k, \quad \overline{s^2} = \frac{1}{N}\sum_{k=1}^{N} s_k^2. \tag{11}$$

Solution:

First, let's expand and simplify the right-hand side:

$$\Delta f \sum_{j>0} P(f_j) = \Delta f \frac{2}{R^2 T} \sum_{j>0} |S_j|^2 = \Delta f \frac{2T^2}{\overline{s}^2 N^2 T} \sum_{j>0} |S_j|^2 = \frac{2}{\overline{s}^2 N^2} \sum_{j>0} |S_j|^2. \tag{12}$$

Now, from the condition that the all counts in the light curve are real numbers, we deduce that the Fourier amplitudes with subscripts j and $-j$ obey the condition $S^*_{-j} = S_j$ (where * denotes complex conjugation). Hence, the summation over positive frequencies can be rewritten as

$$2\sum_{j>0} |S_j|^2 = \sum_{j>0} |S_j|^2 + \sum_{j<0} |S_j|^2 = \sum_{j} |S_j|^2 - S_0^2. \tag{13}$$

Using Parseval's theorem $\sum_k s_k^2 = \frac{1}{N}\sum_j |S_j|^2$ and substituting the expression for S_0, we get

$$\Delta f \sum_{j>0} P(f_j) = \frac{N\sum_k s_k^2 - (\sum_k s_k)^2}{\overline{s}^2 N^2} = \frac{\overline{s^2} - \overline{s}^2}{\overline{s}^2}. \tag{14}$$

6.5: Prove a relation between the discrete autocorrelation function and the power-density spectrum:

$$A_p = \frac{1}{N} \sum_{j=-N/2}^{N/2-1} |S_j|^2 e^{-2\pi i j p/N} \tag{15}$$

using the formal definition

$$A_p \equiv \sum_k s_k s_{k-p} \tag{16}$$

and the orthogonality condition

$$\sum_{j=-N/2}^{N/2-1} e^{2\pi i j(m-n)/N} = N\delta_{mn}. \tag{17}$$

Solution:

Using the inverse Fourier transforms

$$s_k = \frac{1}{N} \sum_j S_j e^{-2\pi i j k/N}, \tag{18}$$

we get

$$A_p = \frac{1}{N} \sum_k \sum_j S_j e^{-2\pi i j k/N} \sum_{j'} S_{j'} e^{-2\pi i j'(k-p)/N} = \frac{1}{N^2} \sum_k \sum_j \sum_{j'} S_j S_{j'} e^{-2\pi i (kj+kj'-pj')/N}. \tag{19}$$

From orthogonality condition we have

$$\sum_k e^{2\pi i k(j+j')/N} = N\delta_{j,-j'}. \tag{20}$$

Using the Kronecker symbol, we get:

$$A_p = \frac{1}{N} \sum_j \sum_{j'} S_j S_{j'} e^{-2\pi i(-pj')/N} \delta_{j,-j'}, \tag{21}$$

the summation $\sum_{j'}$ becomes trivial, everywhere instead of j' we need to substitute $-j$. Using the property $S_{-j} = S_j^*$, we get

$$A_p = \frac{1}{N} \sum_j S_j S_j^* e^{-2\pi i p j/N} = \frac{1}{N} \sum_j |S_j|^2 e^{-2\pi i j p/N}. \tag{22}$$

6.6: Consider the shot noise model with the shot profile at soft energies described by

$$g_s(t) = e^{-t/\tau_s}, t \geq 0. \tag{23}$$

(a) Compute the PDS of the light curve.
(b) Let the hard photons have a similar shot profile with time-constant $\tau_h = \gamma\tau_s$. Assume that the start time of the shots in both energies coincide. Compute the phase and time lags, $\Delta\phi(f)$ and $\Delta t(f)$. Are they positive or negative? Explain.
(c) Compute the low ($f \ll 1/2\pi\tau_s$) and the high ($f \gg 1/2\pi\tau_s$) frequency limits for $\Delta\phi(f)$ and $\Delta t(f)$.

Solution:

(a) The continuous Fourier transform for soft photon light curve is

$$G_s(f) = \int_0^\infty e^{-t/\tau_s} e^{2\pi i f t} dt = \left. \frac{e^{2\pi i f t - t/\tau_s}}{2\pi i f - 1/\tau_s} \right|_0^\infty = \frac{\tau_s}{1 - 2\pi i f \tau_s}. \tag{24}$$

The PDS is

$$PDS = G_s^* G_s = \frac{\tau_s}{1 + 2\pi i f \tau_s} \frac{\tau_s}{1 - 2\pi i f \tau_s} = \frac{\tau_s^2}{1 + (2\pi f \tau_s)^2}. \tag{25}$$

(b) For the hard photons, the Fourier transform is

$$G_h(f) = \frac{\tau_h}{1 - 2\pi i f \tau_h} = \frac{\gamma\tau_s}{1 - 2\pi i \gamma f \tau_s}. \tag{26}$$

The phase lags are computed from the cross spectrum. For the *hard* lags:

$$CS = G_s^* G_h = \frac{\tau_s}{1 + 2\pi i f \tau_s} \frac{\gamma\tau_s}{1 - 2\pi i f \gamma\tau_s} = \frac{\gamma\tau_s^2(1 + \gamma(2\pi f \tau_s)^2 + i(\gamma - 1)2\pi f \tau_s)}{[1 + (2\pi f \tau_s)^2][1 + (2\pi f \gamma\tau_s)^2]}. \tag{27}$$

The phase lags can be computed as

$$\tan \Delta\phi = \frac{Im(CS)}{Re(CS)} = \frac{(\gamma - 1)2\pi f \tau_s}{1 + \gamma(2\pi f \tau_s)^2}. \tag{28}$$

For $\gamma > 1$, the lag is positive, i.e. the hard band lags behind the soft band. Time lags are $\Delta t = \Delta\phi/(2\pi f)$. At low frequencies, $f \ll 1/(2\pi\tau_s)$, $\Delta\phi(f) \sim (\gamma - 1)2\pi f \tau_s \ll 1$ and $\Delta t(f) \sim (\gamma - 1)\tau_s$=const. At high frequencies, $f \gg 1/(2\pi\tau_s)$, $\Delta\phi(f) \sim (\gamma - 1)/(\gamma 2\pi f \tau_s) \ll 1$ and $\Delta t(f) \sim (\gamma - 1)\tau_s/\gamma(2\pi f \tau_s)^2 \propto 1/f^2$. The absolute value of phase lag reaches maximum when $2\pi f \tau_s = 1/\sqrt{\gamma}$ and it is $\tan \Delta\phi_{\max} = (\gamma - 1)/2\sqrt{\gamma}$.