HIGH ENERGY ASTROPHYSICS

Compulsary Home Exercises. Problem set 7. Solutions.

Problems

7.1: Derive formulae for rotational velocity, angular velocity and specific angular momentum in pseudo-Newtonian potential. Show that $\mathrm{d}l/\mathrm{d}r = 0$ at $r = 3R_\mathrm{S}$. Here $R_\mathrm{S} = 2GM/c^2$ is the Schwarzschild radius.

Solution:

Paczynski-Wiita potential has the form

$$\Phi = -\frac{GM}{r - R_\mathrm{S}}, \tag{1}$$

where $R_\mathrm{S} = 2GM/c^2$. Force per unit mass

$$f = -\frac{d\Phi}{dr} = \frac{GM}{(r - R_\mathrm{S})^2}.$$

Circular velocity may be obtained by balancing this force with centrifugal force of inertia v^2/r. This yields

$$v = \sqrt{rf} = \frac{\sqrt{GMr}}{r - R_\mathrm{S}}. \tag{2}$$

Rotation frequency is

$$\Omega = v/r = \frac{\sqrt{GM}}{\sqrt{r}(r - R_\mathrm{S})}, \tag{3}$$

and net angular momentum

$$l = rv = \frac{\sqrt{GMr^3}}{r - R_\mathrm{S}}. \tag{4}$$

To calculate the position of the minimum of Keplerian momentum, let us differentiate l.

$$\frac{dl}{dr} = \frac{r - 3R_\mathrm{S}}{2(r - R_\mathrm{S})^2}\sqrt{GMr}. \tag{5}$$

The derivative becomes zero when $r = 3R_\mathrm{S}$.

7.2: Compute the radiative efficiency

$$\epsilon \equiv \frac{L}{\dot{M}c^2} = -\frac{v_\varphi^2(r_*)}{2c^2} - \frac{\phi(r_*)}{c^2}$$

of the accretion disc around a black hole for Newtonian $\phi = \phi_N$ and pseudo-Newtonian (Paczynski-Wiita) $\phi = \phi_{PN}$ potentials. Here $r_* = 3R_\mathrm{S}$. At what accretion rate $\dot{M}$ the black hole should accrete to produce Eddington luminosity? Compute the numerical value of this $\dot{M}$ (in g/s and $M_\odot$/year) for 10 $M_\odot$ and $10^8 M_\odot$ black holes.

What is the luminosity of a 10 $M_\odot$ black hole accreting at a rate $\dot{M} = L_\mathrm{Edd}/c^2$? What is the corresponding luminosity of a quasar of $10^8 M_\odot$?

<u>Solution:</u>

For Newtonian potential $v_\varphi^2(r) = GM/r$ and $\phi(r) = -GM/r$. Thus the Newtonian efficiency for $r_* = 3R_\mathrm{S}$ is

$$\epsilon_\mathrm{N} = -\frac{GM}{2r_*c^2} + \frac{GM}{r_*c^2} = \frac{GM}{2r_*c^2} = \frac{1}{12}. \tag{6}$$

For Paczynski-Wiita's potential, the potential according to equation (1) is $\phi(r_*) = -GM/(r - R_\mathrm{S}) = -c^2/4$ and, according to equation (2), the velocity $v_\varphi(3R_\mathrm{S}) = \frac{\sqrt{6}}{4}c$. Hence

$$\epsilon_\mathrm{PW} = -\frac{3}{16} + \frac{1}{4} = \frac{1}{16}. \tag{7}$$

The Eddington accretion rate can be defined as

$$\epsilon \dot{M}_\mathrm{Edd} c^2 = L_\mathrm{Edd} = \frac{4\pi GMc}{\varkappa}, \tag{8}$$

or

$$\dot{M}_\mathrm{Edd} = \frac{4\pi GM}{\varkappa c}\frac{1}{\epsilon}. \tag{9}$$

For $M = 10M_\odot$ (and taking opacity $\varkappa = 0.34$), for the Paczynski-Wiita potential we get:

$$\dot{M}_\mathrm{Edd} \simeq 2.6 \times 10^{19} \mathrm{g\,s^{-1}} \simeq 4.1 \times 10^{-7} M_\odot\, yr^{-1}. \tag{10}$$

Using Newtonian potential results in a 30% smaller value, as Newtonian efficiency is 4/3 times larger. For a supermassive black hole with $M = 10^8 M_\odot$,

$$\dot{M}_\mathrm{Edd} \simeq 2.6 \times 10^{27} \mathrm{g\,s^{-1}} \simeq 4.1 M_\odot\, yr^{-1}. \tag{11}$$

The luminosity of a black hole accreting at a rate $\dot{M} = L_\mathrm{Edd}/c^2$ is smaller than the Eddington limit by the efficiency factor ϵ (say 1/16). Thus for a 10 $M_\odot$, the luminosity is $10 \times 1.3 \times 10^{38}/16 = 0.8 \times 10^{38}$ erg/s. For a quasar of $10^8 M_\odot$, the corresponding luminosity is 0.8×10^{45} erg/s.

7.3: Compute the radius (in units of R_S) where the effective temperature of standard accretion disc reaches the maximum for Newtonian and pseudo-Newtonian potentials. Compute the numerical value of the maximum temperature (in K and keV) for a 10 $M_\odot$ black hole accreting at a rate $\dot{M} = L_{\rm Edd}/c^2$. How the maximum temperature scales with the mass of the central object? What would be the corresponding temperature for a $1.4 M_\odot$ neutron star and a $10^8 M_\odot$ quasar? How the maximum temperature scales with the accretion rate?

Solution:

Effective temperature is defined by the local energy balance

$$2\sigma_{\rm SB} T_{\rm eff}^4 = Q_+ = -\frac{1}{2\pi r}\frac{d\Omega}{dr}\left[l(r) - l_{\rm in}\right]\dot{M}. \qquad (12)$$

In Newtonian case, rotation is Keplerian, $\Omega_{\rm N} = \sqrt{GM/r^3}$. In the pseudo-Newtonian case, rotation profile is given by equation (3). In Newtonian case, the specific angular momentum $l = \sqrt{GMr}$, and for the Paczynski-Wiita potential l is given by equation (4). Zero-torque boundary condition at the last stable orbit is assumed, hence $l_{\rm in} = l(r_{\rm in} = 3R_{\rm S})$.

In Newtonian case, we get

$$2\sigma_{\rm SB} T_{\rm N}^4(r) = Q_+ = \frac{3}{2}\sqrt{\frac{GM}{r^3}}\frac{\dot{M}}{2\pi r^2}\left(\sqrt{GMr} - \sqrt{GMr_{\rm in}}\right) = \frac{3}{4\pi}\frac{GM\dot{M}}{r^3}\left(1 - \sqrt{\frac{r_{\rm in}}{r}}\right). \qquad (13)$$

Effective temperature in Newtonian case

$$T_{\rm eff,\ N}(r) = \left(\frac{3}{8\pi\sigma_{\rm SB}}\frac{GM\dot{M}}{r^3}\left(1 - \sqrt{\frac{r_{\rm in}}{r}}\right)\right)^{1/4}. \qquad (14)$$

By differentiating $T_{\rm eff}^4$, one can find the position of the temperature maximum:

$$\frac{d}{dr}\left(\frac{1}{r^3}\left(1 - \sqrt{\frac{r_{\rm in}}{r}}\right)\right) = \frac{1}{r^4}\left(-3 + \frac{7}{2}\sqrt{\frac{r_{\rm in}}{r}}\right), \qquad (15)$$

that suggests an extremum at

$$r_{\rm max,N} = \frac{49}{36} r_{\rm in} = \frac{49}{12} R_{\rm S} = \frac{49}{6}\frac{GM}{c^2}. \qquad (16)$$

Maximal temperature is

$$T_{\rm max,\ N}(r_{\rm max,N}) = \left(\frac{3}{8\pi\sigma_{\rm SB}}\frac{GM\dot{M}}{r^3}\left(1 - \sqrt{\frac{r_{\rm in}}{r}}\right)\right)^{1/4} = 0.488 T_*, \qquad (17)$$

where

$$T_* = \left(\frac{3}{8\pi\sigma_{\rm SB}}\frac{GM\dot{M}}{r_{\rm in}^3}\right)^{1/4} = \left(\frac{3}{8\pi\sigma_{\rm SB}}\frac{GM\dot{M}c^6}{6^3 G^3 M^3}\right)^{1/4} = \left(\frac{1}{9\times 64\pi\sigma_{\rm SB}}\frac{\dot{M}c^6}{G^2 M^2}\right)^{1/4}. \qquad (18)$$

For the mass accretion rate $\dot{M} = L_{\rm Edd}/c^2$,

$$T_* = \frac{1}{2\sqrt{3}} \left(\frac{c^5}{GM\varkappa\sigma_{\rm SB}} \right)^{1/4} = 1.6 \times 10^7 {\rm K} \left(\frac{M}{M_\odot} \frac{\varkappa}{0.34} \right)^{-1/4}. \tag{19}$$

and thus

$$T_{\rm max,\ eff}(r_{\rm max,N}) = 7.8 \times 10^6 {\rm K} \left(\frac{M}{M_\odot} \frac{\varkappa}{0.34} \right)^{-1/4} = 0.67 \ {\rm keV} \left(\frac{M}{M_\odot} \frac{\varkappa}{0.34} \right)^{-1/4}. \tag{20}$$

As we can see, $T_{\rm max,\ N} \propto M^{-1/4}$. For $M = 1.4 M_\odot$ (and $\varkappa = 0.34$), $T_{\rm max,\ N} \simeq 0.62$ keV, for a $10M_\odot$ black hole, $T_{\rm max,\ N} \simeq 0.38$ keV, and for a supermassive black hole with $M = 10^8 M_\odot$, $T_{\rm max,\ N} \simeq 7.8 \times 10^4$K. Note, that for a mass accretion rate that would give the Eddington luminosity, i.e. $\dot{M} = L_{\rm Edd}/(\epsilon\ c^2)$, the maximum temperature is larger by a factor $\epsilon^{-1/4} \sim 2$.

For the Paczynski-Wiita case,

$$\frac{d\Omega}{dr} = -\frac{3r - R_{\rm S}}{2(r - R_{\rm S})^2} \sqrt{\frac{GM}{r^3}} = -\sqrt{\frac{GM}{R_S^5}} \frac{3x - 1}{2(x-1)^2 x^{3/2}}. \tag{21}$$

where $x = r/R_{\rm S}$. The specific angular momentum term is

$$l(r) - l_* = \frac{\sqrt{GMr^3}}{r - R_{\rm S}} - \frac{\sqrt{GM3^3R_S^2}}{2R_{\rm S}} = \sqrt{GMR_{\rm S}} \left[\frac{x^{3/2}}{x-1} - \frac{3\sqrt{3}}{2} \right]. \tag{22}$$

To estimate the position of the temperature maximum, let us differentiate

$$Q^+(x) \propto \frac{1}{r}\frac{d\Omega}{dr}(l(r) - l_*) \propto \frac{(3x-1)}{x(x-1)^3} \left(1 - \frac{3^{3/2}}{2} \frac{x-1}{x^{3/2}} \right) \tag{23}$$

as a function of x, ignoring the constant multiplier. Numerical differentiation of this expression yields $x_{\rm max} \simeq 4.75$. This is somewhat larger than the value $x_{\rm max,N} = 49/12 \simeq 4.08$ we got for the Newtonian case.

7.4: Show that the prescription of the viscous stress $t_{r\phi} = \alpha P$ is equivalent to the prescription

$$\nu = \frac{2}{3}\alpha c_s H.$$

Solution:

The viscous stress for pure Keplerian rotation is

$$t_{r\varphi} = -\nu\rho r\frac{d\Omega}{dr} = \nu\rho\frac{3}{2}\Omega, \tag{24}$$

This implies for viscosity

$$\nu = \frac{2}{3}\frac{t_{r\varphi}}{\rho\Omega}. \tag{25}$$

According to (local) α-prescription, we can replace $t_{r\varphi} = \alpha P$, and

$$\nu = \frac{2}{3}\alpha\frac{P}{\rho}\frac{1}{\Omega}. \tag{26}$$

Hydrostatics in vertical direction

$$\frac{dP}{dz} = -\Omega^2\rho z, \tag{27}$$

that approximately gives

$$P \simeq \Omega^2 H^2 \rho, \tag{28}$$

or $c_{\rm s}^2 \simeq P/\rho \simeq H^2\Omega^2$. That allows to re-write equation (26) as

$$\nu \simeq \frac{2}{3}\alpha H(H\Omega) \simeq \frac{2}{3}\alpha c_{\rm s} H. \tag{29}$$

7.5: (a) How long (in years) a stellar mass black hole (of say $M = 7M_\odot$) has to accrete matter at the Eddington limit (i.e. producing Eddington luminosity) in order to reach a luminosity $L = 10^{47}$ erg s^{-1}? To determine this, write down and solve a simple differential equation for how the mass changes with time due to accretion. Assume the radiative efficiency of 0.1. Compare the time scale you get to the Hubble time (the age of the Universe).
(b) If a galaxy with 10^{11} stars contains a dead quasar that grew as in a previous part (a) until reaching $10^8 M_\odot$, compare its total gravitational energy release to the energy release due to thermonuclear burning (in stars) during the time it took for the black hole to grow. You may take all stars to have $M = 1M_\odot$ (and $L = L_\odot$).
(c) A quasar has luminosity $L = 10^{47}$ erg s^{-1} and varies on the time-scale of a day. Deduce a mass and a radius for the emitting region using Eddington limit and the light crossing time arguments. How does the implied density compare with that of the Earth? What is the mass accretion rate assuming the radiative efficiency of 10%? How does the amount of mass accreted per second compare with the mass of the Earth?

Solution:

The differential equation in question may be written as an equation for mass

$$\frac{dM}{dt} = \dot{M}, \tag{30}$$

where $\dot{M}$ should trace the instantaneous Eddington limit for the black hole (see eq. 9)

$$\dot{M} = \frac{L_{\rm Edd}}{\epsilon c^2} = \frac{4\pi GM}{\epsilon \varkappa c}, \tag{31}$$

where $\epsilon = 0.1$ is the accretion (radiative) efficiency and we can take $\varkappa = 0.34$. This is an equation linear in $\ln M$, and its solution has the form

$$M(t) = M_0 e^{t/t_{\rm Edd}}, \tag{32}$$

where

$$t_{\rm Edd} = \frac{\epsilon \varkappa c}{4\pi G} \simeq 1.2 \times 10^{15}\,{\rm s} \simeq 4 \times 10^7 {\rm yr} \tag{33}$$

is Eddington-limited e-fold time for mass (and luminosity). The accretion luminosity will grow proportionally to M. $L = 10^{47} {\rm erg\,s^{-1}}$ corresponds to a mass of

$$M_{\rm final} = \frac{L\varkappa}{4\pi Gc\epsilon} \simeq 7 \times 10^8 M_\odot. \tag{34}$$

The time required to gain a factor of 10^8 in mass is

$$t_8 \simeq t_{\rm Edd} \ln 10^8 \simeq 18.4\, t_{\rm Edd} \simeq 7.4 \times 10^8 {\rm yr}. \tag{35}$$

This is still about a factor of 20 smaller than Hubble time.

(b) Growing to $10^8 M_\odot$ would require $\Delta t \simeq 6.5 \times 10^8$yr. The energy released by accretion is

$$\Delta E_{\rm a} = \epsilon \left(M - M_0\right) c^2 \simeq \epsilon M c^2 \sim 2 \times 10^{61} {\rm erg}. \tag{36}$$

The total output of the stellar light is, on the other hand,

$$\Delta E_* = N L_\odot \Delta t \simeq 8 \times 10^{60} \text{erg}, \quad (37)$$

that is roughly the same order of magnitude.

(c) As we have seen, the mass of the quasar having the Eddington limit equal to $10^{47} \text{erg}\,\text{s}^{-1}$ is about $7 \times 10^8 M_\odot$.

The variability time scale, if interpreted as a light-travel time, allows to estimate the radius of the emitting region as

$$R \simeq c\Delta t \simeq 2.6 \times 10^{15} \text{cm}. \quad (38)$$

At the same time, Schwarzschild radius for the considered mass is about 10^{14}cm. The mean density inside the estimated radius is $M/R^3 \sim 8 \times 10^{-5} \text{g}\,\text{cm}^{-3}$, which is orders of magnitude below Earth's density.

The mass accretion rate

$$\dot{M} = \frac{L}{\epsilon c^2} \simeq 10^{27} \text{g}\,\text{s}^{-1}, \quad (39)$$

that is about 15% of the mass of the Earth per second.