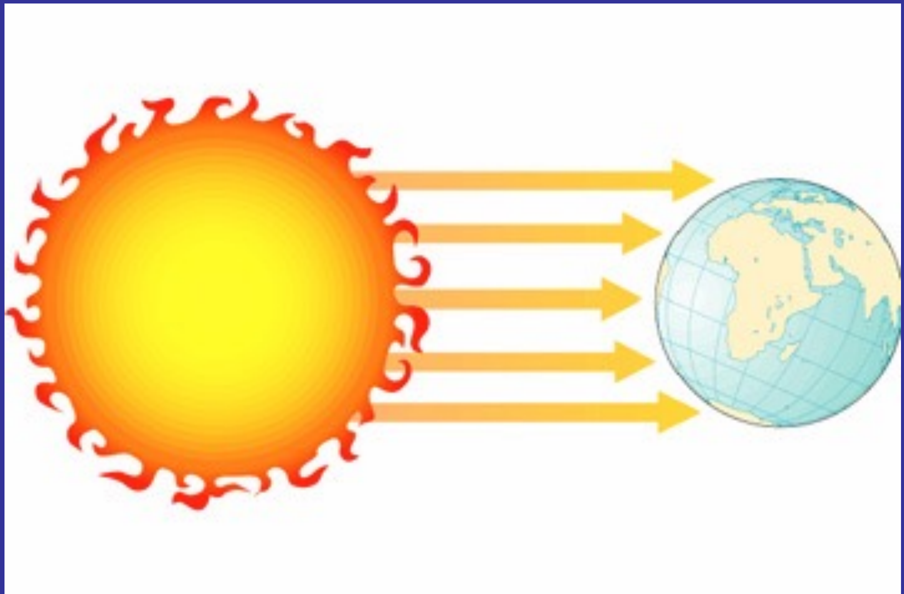
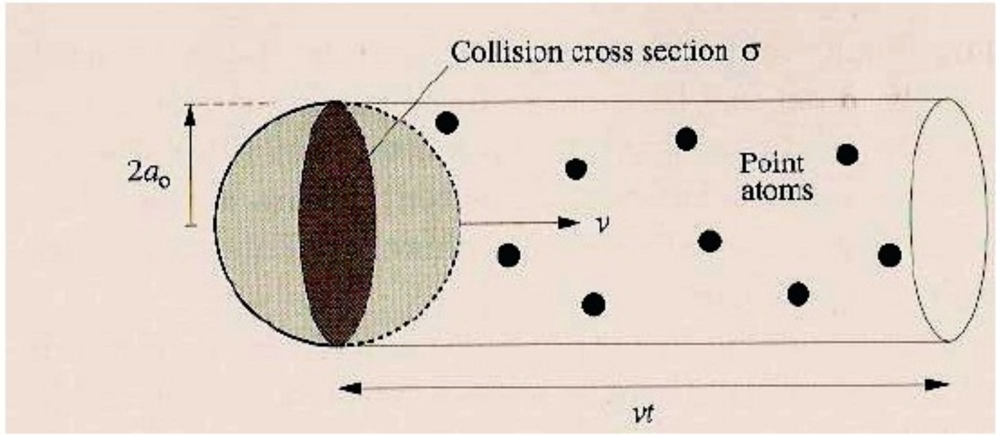




RADIATIVE PROCESSES in ASTROPHYSICS

1. Introduction
2. Basics of radiative transfer
3. Radiation fields
4. Radiation from moving charges
5. Relativistic covariance and kinematics
6. Bremsstrahlung
7. Synchrotron radiation
8. Compton scattering
9. Pair production
10. Cherenkov radiation

TURKU 2026

BASICS of RADIATIVE TRANSFER
Definitions

Electromagnetic Spectrum

refraction, diffraction & interference indicate em-radiation behaves as waves:

$$\lambda = c/\nu$$

photo-electric effect shows energy is given to or taken from radiation field in discrete quanta, or photons, with energy:

$$E = h\nu$$

For thermal energy emitted by matter in thermodynamic equilibrium, the characteristic photon energy is related to the temperature of the emitting material:

$$T = E/k$$

gamma-rays	$T > 10^9$ K
X-rays	$10^6 < T < 10^9$ K
UV	$10^4 < T < 10^6$ K
Optical	$3\ 10^3 < T < 10^4$ K
IR	$100 < T < 3\ 10^3$ K
Radio	$3 < T < 10$ K

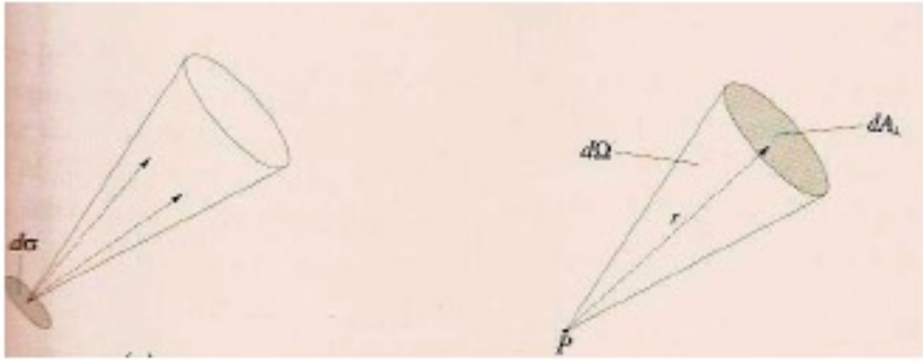
BASICS of RADIATIVE TRANSFER
Definitions

Macroscopic Description of Radiation

describe the energy flux associated with electro-magnetic radiation.

The relationship between intensity and the energy flux, momentum flux, radiation pressure and energy density.

Solid Angle



$$d\Omega \equiv dA_{\perp}/r^2$$

$$d\Omega_{tot} = \oint d\Omega = 4\pi \text{ sr}$$

BASICS of RADIATIVE TRANSFER
Definitions

Description of a Radiation Field

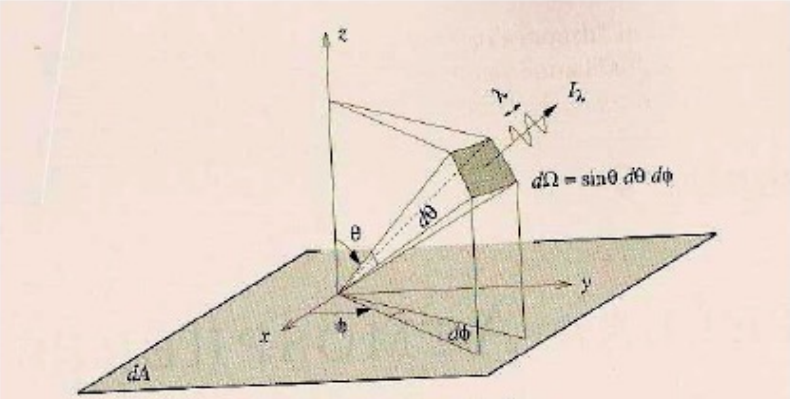


Figure 9.1 Intensity I_λ.

dE

The amount of energy that the rays carry into cone in time dt

Average intensity of rays is defined:

$$\bar{I}_\lambda = \frac{dE}{d\lambda\ dt\ dA\ cos\theta\ d\Omega}$$

$$dE = I_\lambda d\lambda\ dt\ dA\ cos\theta\ d\Omega = I_\lambda d\lambda dt dA cos\theta sin\theta d\theta d\phi$$

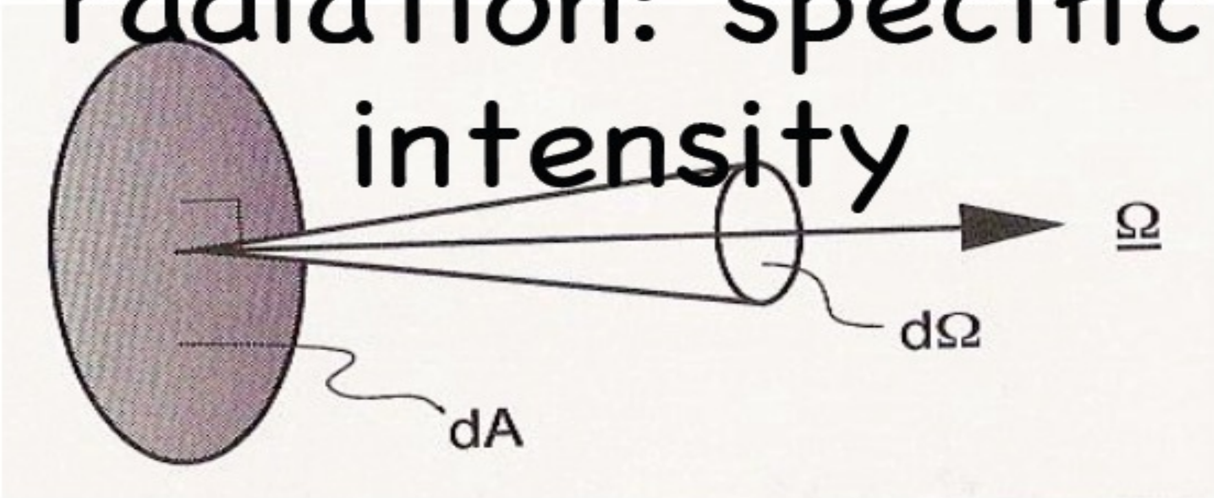
$$\langle I_\lambda \rangle \equiv \frac{1}{4\pi}\int I_\lambda d\Omega = \frac{1}{4\pi}\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi} I_\lambda sin\theta d\theta d\phi$$

For isotropic radiation field $\langle I_\lambda \rangle = I_\lambda$

Solid angle in spherical coordinates $d\Omega = \sin\theta\, d\theta\, d\phi$

BASICS of RADIATIVE TRANSFER
Definitions

transport of energy via radiation: specific intensity



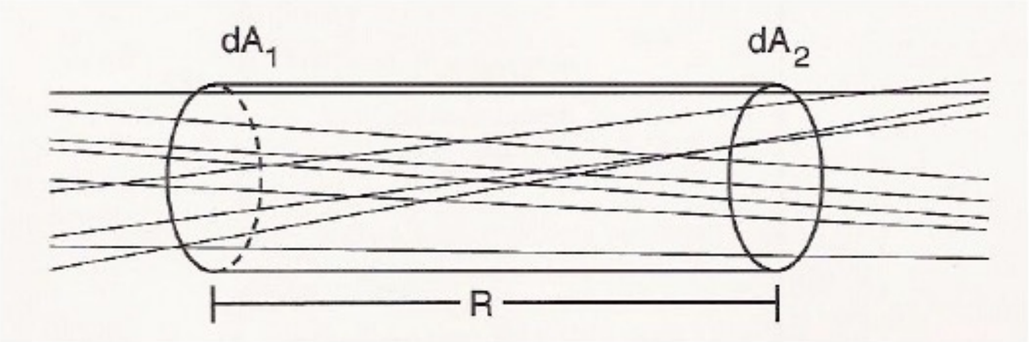
$$dE = I_\nu(\Omega)\ dA\ dt\ d\Omega\ d\nu$$

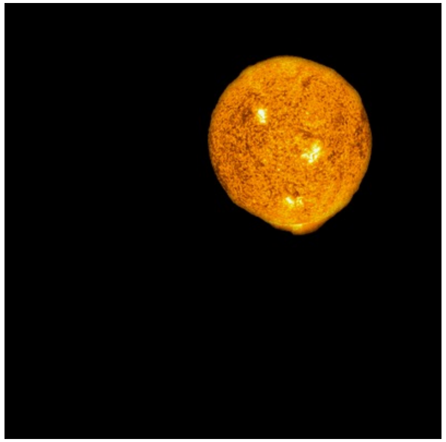
$$erg/cm^2/s/steradian/Hz$$

The intensity provides a fairly complete description of the transport of energy via radiation.

Constancy of Specific Intensity in vacuum

Consider the energy carried by that set of rays passing through both dA_1 dA_2





$$dE_1 = I_{\nu_1} dA_1 dt\ d\Omega_1 d\nu_1 = dE_2 = I_{\nu_2} dA_2 dt\ d\Omega_2 d\nu_2$$

$d\Omega_1 = dA_2/R^2$, $d\Omega_2 = dA_1/R^2$

$\nu_1 = \nu_2 \Rightarrow$

$$I_{\nu_1} = I_{\nu_2}$$

thus the intensity is constant along a ray

$$I_\nu = \text{constant}$$

Another way of stating this is by the differential relation

$$\frac{dI_\nu}{ds} = 0$$

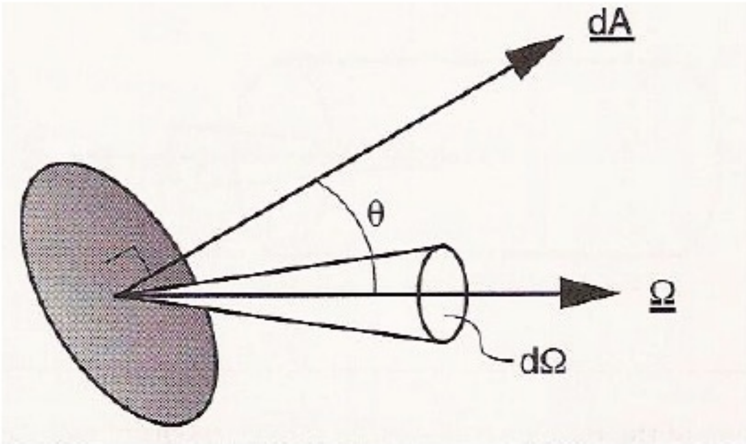
where ds is a differential element along the ray

Question in class: how brightness of the Sun changes with the distance?
Read about Olbers' paradox.

BASICS of RADIATIVE TRANSFER
Moments of intensity

Energy Flux

energy carried by a set of rays



$$dA_\perp = dA cos\theta$$

$$dF_\nu = I_\nu cos\theta d\Omega$$

differential flux

$$F_\nu = \int I_\nu cos\theta d\Omega$$

net flux

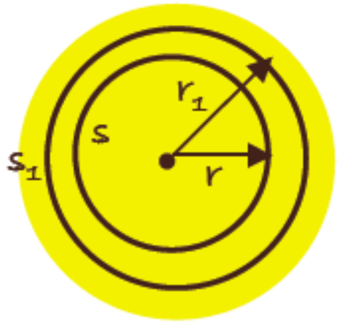
(erg s^{-1} cm^{-2} Hz^{-1})

Multiplying a function of direction by n-th power of cos θ and integrating is often called "taking the n-th moment".

Net flux is the first moment of the intensity

Energy flux not truly intrinsic since it depends on the orientation of the surface element, energy propagating downwards is negative energy flux. If I_ν is isotropic (not a fn. of angle) then net flux = 0.

Inverse square law for energy flux



Isotropic radiation:

emits energy equally in all directions (e.g., a star)

CONSERVATION OF ENERGY

$$F(r_1) \cdot 4\pi r_1^2 = F(r) \cdot 4\pi r^2$$

$$F(r) = \frac{\text{const}}{r^2}$$

If we can regard sphere s_1 as fixed

$$F \propto 1/r^2$$

BASICS of RADIATIVE TRANSFER
Moments of intensity

Momentum Flux

Photons carry momentum

$$\mathbf{p} = \hat{\mathbf{n}} E/c$$

$\hat{\mathbf{n}}$, unit vector in direction of photon motion

component of momentum in the direction normal to the surface element is $p_\perp$

$$p_\perp = | p | \, cos\theta$$

Differential momentum flux is $dF_\nu \cos\theta/c$ and integrating over all directions gives

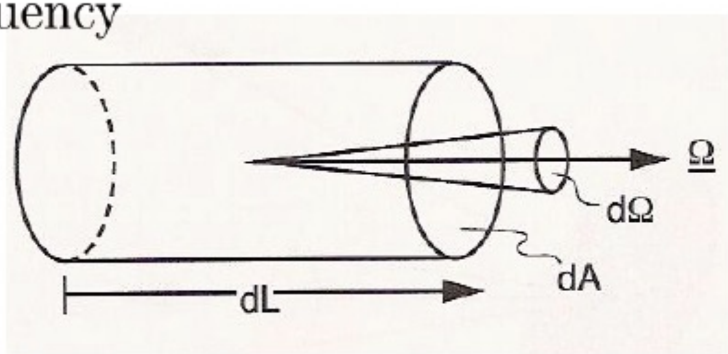
$$p_\nu = \frac{1}{c} \int I_\nu cos^2\theta d\Omega$$

the momentum flux is the second moment of the intensity

BASICS of RADIATIVE TRANSFER
Moments of intensity

Specific Energy Density

$u_\nu(\Omega) = \text{energy/volume/solid angle/frequency}$
$\quad\quad \propto \text{intensity}$



To determine constant of proportionality the energy enclosed:

$$dE = u_\nu(\Omega) d\Omega \; dl \; dA \; d\nu$$

will pass out of cylinder in time, $dt = dl/c$

From our definition of specific intensity: $I_\nu = dE/dt \; dA \; d\Omega \; d\nu$

$$u_\nu(\Omega) = I_\nu(\Omega)/c$$

Integrating over direction: $u_\nu = \int u_\nu(\Omega) d\Omega = \frac{1}{c} \int I_\nu(\Omega) d\Omega$

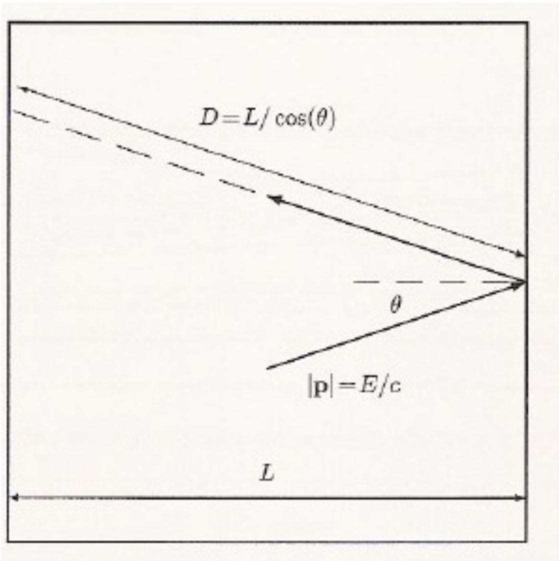
Energy density is proportional to the zeroth moment of intensity

$$u_\nu = \frac{4\pi}{c} J_\nu \qquad J_\nu \equiv \frac{1}{4\pi} \int I_\nu(\Omega) d\Omega$$

$$u = \int u_\nu d\nu = \frac{4\pi}{c} \int J_\nu d\nu$$

is the mean intensity

Radiation Pressure



transfer of momentum: $\Delta P = 2(E/c)cos\theta$

time between reflections: $\Delta t = 2L/(c\ cos\theta)$

Rate of transfer of momentum to the wall per unit area, or radiation pressure

$$\Delta P/(L^2\Delta t) = (E/L^3)cos^2\theta = u\ cos^2\theta$$

for isotropic radiation: $\langle cos^2\theta\rangle = 1/3$

For isotropic radiation, the radiation pressure is: $P = u/3$

Each photon transfers twice the normal component of momentum

$$P_\nu = \frac{2}{c}\int I_\nu cos^2\theta d\Omega$$

Integrating over 2π steradians, $I_\nu = J_\nu$

$$P = \frac{2}{c}\int J_\nu d\nu \int cos^2\theta d\Omega = \frac{u}{3}$$

Radiation pressure of an isotropic radiation field is 1/3 the energy density

Radiative Transfer

Now we consider radiation passing through matter, which may absorb, emit and/or scatter radiation into or out of our beam. We will derive the equation governing the evolution of the intensity

Radiative Transfer Equation

Intensity is conserved along a ray $\frac{dI_\nu}{ds} = 0$

Unless there is emission or absorption

$$\frac{dI_\nu}{ds} = -\alpha_\nu I_\nu + j_\nu$$

The Equation of Radiative Transfer

Emission

We define an emission coefficient such that matter in a volume element, dV adds to the radiation field an amount of energy, dE.

$$dE = j_\nu dV\ dt\ d\Omega\ d\nu$$

emission coefficient j_ν (erg s^{-1}cm^{-3}str^{-1}Hz^{-1})

The (angle averaged) EMISSIVITY ε_ν is the energy per unit mass per unit time per unit frequency, and is defined, such that:

$$dE = \epsilon_\nu\ \rho\ dV\ \ dt\ d\nu\ (d\Omega/4\pi)$$

with ρ the mass density, from which follows the relation, for isotropic emission

$$j_\nu = \frac{\epsilon_\nu \rho}{4\pi}$$

In going a distance ds, a beam of cross section dA travels through a volume dV=dAds, thus the intensity added to the beam by spontaneous emission is:

$$dI_\nu = j_\nu\ ds$$

BASICS of RADIATIVE TRANSFER
Radiative transfer equation

Absorption

absorption will remove from the beam an amount of intensity proportional to the incident intensity and the path length, ds.

$$dI_\nu = -\alpha_\nu \; I_\nu \; ds$$

where the absorption coefficient, α, has units (length)$^{-1}$

For a simple model of randomly placed absorbing spheres with a cross-section σ and number density n the mean coverina factor for objects in a tube of area A and length ds is $dA/A = n\sigma dV/A = n\sigma ds$ so the attenuation of intensity:

$$dI = -n \; \sigma \; I \; ds$$

$$\alpha_\nu = \rho \kappa_\nu$$

Here κ_ν is the cross-section per unit mass [cm^2 g^{-1}]

Radiation Force

If a medium absorbs radiation then the radiation exerts a force on the medium, because radiation carries momentum.

radiation flux vector $$\mathbf{F}_\nu = \int I_\nu \, \mathbf{n} \, d\Omega$$

A photon has momentum E/c, so the vector momentum per unit area per unit time per unit path length absorbed by medium is:

$$\frac{\text{momentum}}{\text{volume} \times \text{time}} = \frac{\text{force}}{\text{volume}} = \frac{1}{c} \int \alpha_\nu \mathbf{F}_\nu \, d\nu$$

$d\tau_\nu$ is the probability for photon to be absorbed within ds. Momentum absorbed per unit length is $F_\nu \, d\tau_\nu /(c \text{ ds}) = F_\nu \, \alpha_\nu \, \text{ds} /(c \text{ ds})$

$$\frac{\text{force}}{\text{mass}} = \text{acceleration} = \frac{1}{c} \int \kappa_\nu \, \mathbf{F}_\nu \, d\nu$$

Solutions to two simple limiting cases:

emission only $\alpha_\nu = 0$

$$\frac{dI_\nu}{ds} = j_\nu \rightarrow I_\nu(s) = I_\nu(0) + \int j_\nu \, ds$$

increase in brightness is equal to emission coeff integrated along los

absoprtion only $j_\nu = 0$

$$\frac{dI_\nu}{ds} = -\alpha_\nu \, I_\nu \rightarrow I_\nu(s) = I_\nu(0) e^{-\int ds \alpha_\nu}$$

brightness decreases by exponential of absorption coeff integrated along los

BASICS of RADIATIVE TRANSFER
Radiative transfer equation

Optical Depth & Source Function

The RTE takes a particularly simple form if we replace path length, s by optical depth, τ_ν

$$d\tau_\nu = \alpha_\nu ds \qquad \tau_\nu = \int \alpha_\nu ds$$

In terms of pure absorption: $I_\nu(s) = I_\nu(0)e^{-\tau}$

A medium is said to be <u>optically thick</u>, or opaque when τ_ν integrated along a typical path through the medium > 1

When τ_ν < 1 then the medium is said to be <u>optically thin</u> or transparent

$$\frac{dI_\nu}{d\tau_\nu} = -I_\nu + S_\nu \qquad S_\nu \equiv \frac{j_\nu}{\alpha_\nu}$$

Thus the formal solution of the RTE is:

$$I_\nu(\tau) = I_\nu(0)e^{-\tau} + \int_0^\tau e^{-(\tau-\tau')} S_\nu(\tau')d\tau'$$

Constant source function

$$I_\nu(\tau) = I_\nu(0)e^{-\tau} + \int_0^\tau e^{-(\tau-\tau')} S_\nu(\tau') d\tau'$$

Constant source function

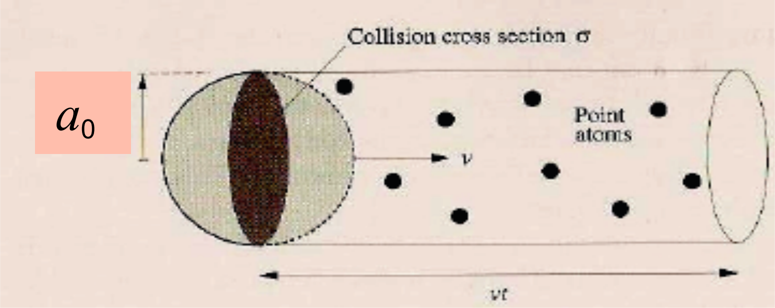
$$I_\nu(\tau_\nu) = I_\nu(0)e^{-\tau_\nu} + S_\nu(1 - e^{-\tau_\nu})$$
$$= S_\nu + e^{-\tau_\nu}(I_\nu(0) - S_\nu)$$

as $\tau \rightarrow \infty$ then $I_\nu \rightarrow S_\nu$

Mean Free Path

$$\langle \tau_\nu \rangle \equiv \int_0^\infty \tau_\nu e^{-\tau_\nu} d\tau_\nu = 1$$

Consider a single atom of radius a_0 moving with speed υ through a collection of stationary points that represent the centres of other atoms:



$$V = \pi a_0^2 \upsilon t = \sigma \upsilon t$$

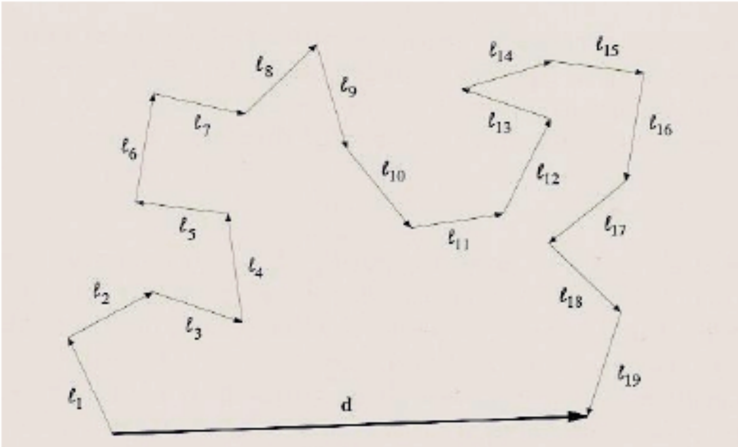
Number of collisions by the atom within volume, V, are $nV = n\sigma\upsilon t$ point-like atoms with which the moving atom has collided, thus the average distance between collisions:

$$l = \frac{v\ t}{n\ \sigma\ v\ t} = \frac{1}{n\ \sigma}$$

The mean free path is the reciprocal of the absorption coeff, α_ν for a homogeneous material.

BASICS of RADIATIVE TRANSFER
Scattering

Random walks



Net displacement of photon after N free paths: $\mathbf{d} = l_1 + l_2 + l_3 + ... + l_N$

Mean vector displacement vanishes

$$\langle \mathbf{d} \rangle = N \langle l_i \rangle = 0$$

but mean square displacement traveled by photon:

$$l_\star^2 \equiv \langle \mathrm{d}^2 \rangle = \langle l_1^2 \rangle + \langle l_2^2 \rangle + ... + \langle l_N^2 \rangle + 2\langle l_1 \cdot l_2 \rangle + 2\langle l_1 \cdot l_3 \rangle + ...$$

Now all the cross terms like $\langle l_1 \cdot l_2 \rangle$ vanish since the directions of different path segments are presumed uncorrelated.

$$\langle d^2 \rangle = N \ \langle l^2 \rangle$$

As an example, consider the escape of a photon from a cloud of size R If the mean free path is $l \ll L$ then the optical depth of the cloud is $\tau \sim L/l$. In N steps the photon will travel a distance $l_\star \sim \sqrt{N} l$. Equating this with the size of the cloud L yields the required number of steps for escape, $N \sim \tau^2$ for optically thick region, for optically thin, $1 - e^{-\tau} \sim \tau$, so $N \sim \tau$

$$N \approx \tau^2 + \tau \qquad \text{or} \qquad N \approx \max(\tau^2, \tau)$$

BASICS of RADIATIVE TRANSFER
Scattering

Escape time.
This is the time it takes for a photon to diffuse from the medium:

$$t_{\rm esc} = \frac{Nl}{c} = \frac{NR/\tau}{c} = \begin{cases} \frac{R}{c}\tau, & \tau \gg 1, \\ \frac{R}{c}, & \tau \ll 1. \end{cases}$$

$(R=L)$

Note that it grows as τ, not as the number of scatterings $\propto \tau^2$, because for large τ the distance travelled between scatterings decreases as $1/\tau$.

Scattering

When scattering is present, solution of RTE becomes more difficult because emission into $d\Omega$ depends on I_ν in solid angles $d\Omega'$ integrated over $d\Omega$ (ie, scattering from $d\Omega'$ into $d\Omega$).

RTE becomes an integro-differential equation which must generally be solved by numerical techniques

BASICS of RADIATIVE TRANSFER
Scattering

(Isotropic) Scattering

Emission coefficient for coherent, isotropic scattering can be found simply by equating the power absorbed per unit volume and frequency ranges to corresponding power emitted:

$$j_\nu = \alpha_{sc} J_\nu$$

where α_{sc} is the SCATTERING COEFFICIENT

Dividing by α_{sc} we find the source function for scattering equals mean intensity of the emitting material:

$$S_\nu = J_\nu = \frac{1}{4\pi}\int I_\nu d\Omega$$

Transfer equation for pure scattering:

$$\frac{dI_\nu}{ds} = -\alpha_{sc}(I_\nu - J_\nu)$$

Cannot use formal soln to RTE since source function is not known a priori and depends on solution to I_ν at all directions through a given point. It is now an INTEGRO-DIFFERENTIAL EQUATION, which is difficult to solve, need to find approximate method of treating scattering problems.

BASICS of RADIATIVE TRANSFER
Scattering and absorption

- Combined scattering and absorption (in a thermal medium)

$$\frac{dI}{ds} = -(\alpha_\nu + \alpha_{sc})(I_\nu - S_\nu)$$

with

$$S_\nu = \frac{\alpha_\nu B_\nu + \alpha_{sc} J_\nu}{\alpha_\nu + \alpha_{sc}}$$

- Mean free path (for absorption and scattering).
  The average distance a photon can travel without being absorbed or scattered. The extinction coefficient $\alpha_\nu + \alpha_{sc}$ and the optical depth for both processes is $d\tau_\nu = (\alpha_\nu + \alpha_{sc})ds$. The mean free path is then

$$l_\nu = \frac{\langle \tau_\nu \rangle}{\alpha_\nu + \alpha_{sc}} = \frac{1}{\alpha_\nu + \alpha_{sc}}.$$

In (local)thermodynamic equilibrium (where collisions dominate), the source function is $S_\nu = j_\nu / \alpha_\nu = B_\nu$ Planck function

BASICS of RADIATIVE TRANSFER
Scattering and absorption

- A chance that after the free path the photon will be absorbed is $= \epsilon_\nu = \alpha_\nu/(\alpha_\nu + \alpha_{sc})$; chance that it will be scattered $= 1 - \epsilon_\nu = \alpha_{sc}/(\alpha_\nu + \alpha_{sc})$. The quantity $1 - \epsilon_\nu$ is called the single-scattering albedo. Source function is then

$$S_\nu = (1 - \epsilon_\nu)J_\nu + \epsilon_\nu B_\nu$$

- Thermalization length.
  A photon is created by thermal emission of an atom. It scatters many times, but at some point it can get absorbed by some other atom. The total path between creation and absorption is called thermalization length. Because the probability of getting absorbed in each interaction act (i.e. in the end of each free path) is ϵ, a photon on average has $N = 1/\epsilon$ scatterings before absorption. Thus we have

$$l_*^2 = \frac{l^2}{\epsilon}, \quad l_* = \frac{l}{\sqrt{\epsilon}}, \qquad \text{or} \qquad l_* = \frac{1}{\sqrt{\alpha_\nu(\alpha_\nu + \alpha_{sc})}}$$

BASICS of RADIATIVE TRANSFER
Scattering and absorption

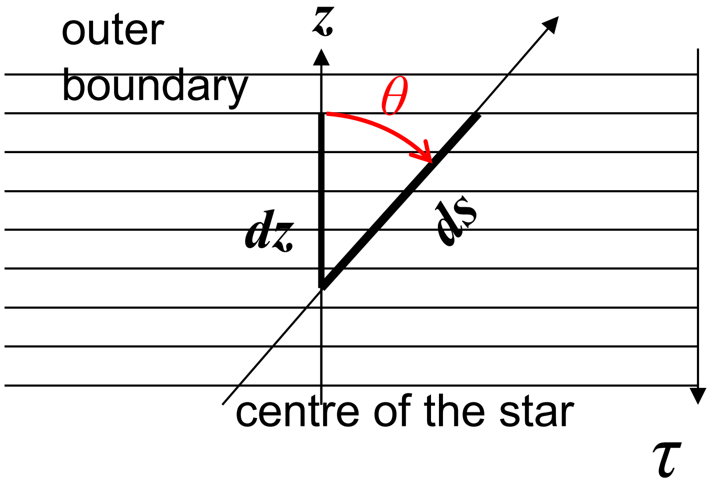
- Effective optical thickness of the medium.

$$\tau_* = \sqrt{\tau_a(\tau_a + \tau_s)}$$

where $\tau_a = \alpha_\nu R$ and $\tau_s = \alpha_{\rm sc} R$ are the optical thickness of the medium of size R for absorption and scattering separately. If $\tau_* \gg 1$, the medium is effectively optically thick. The radiation field is then close to thermalization with the matter and $I_\nu \approx B_\nu, S_\nu \approx B_\nu$.

BASICS of RADIATIVE TRANSFER
Solution of RTE: Eddington approximation

In plane-parallel atmosphere $ds = dz/\mu$, $\mu = \cos\theta$:



$$\mu \frac{dI_\nu}{dz} = -(\alpha_\nu + \alpha_{sc})(I_\nu - S_\nu)$$

$$S_\nu = \frac{\alpha_\nu B_\nu + \alpha_{sc} J_\nu}{\alpha_\nu + \alpha_{sc}}$$

- Eddington approximation.
  Now assume that in a near-homogenous medium the intensity is almost isotropic, but no longer assume that total opacity is large. Expanding the intensity into first-order terms of μ:

$$I_\nu(\tau,\mu) = a_\nu(\tau) + b_\nu(\tau)\mu.$$

three moments of the intensity

$$J \equiv \frac{1}{2}\int_{-1}^{1} I \, d\mu = a,$$

$$H \equiv \frac{1}{2}\int_{-1}^{1} I\mu \, d\mu = b/3,$$

$$K \equiv \frac{1}{2}\int_{-1}^{1} I\mu^2 \, d\mu = a/3$$

$I = J + 3H\,\mu$

BASICS of RADIATIVE TRANSFER
Solution of RTE: Eddington approximation

$$\mu \frac{\mathrm{d}I}{\mathrm{d}\tau} = I - S \qquad \mathrm{d}\tau = -(\alpha + \alpha_{\mathrm{sc}})\mathrm{d}z \qquad S = (1 - \epsilon)J + \epsilon B$$

. Integrating over μ we get: $$\frac{\mathrm{d}H}{\mathrm{d}\tau} = J - S.$$

Multiplying RTE by μ before integrating $$\frac{\mathrm{d}K}{\mathrm{d}\tau} = H = \frac{1}{3}\frac{\mathrm{d}J}{\mathrm{d}\tau}.$$

$$\frac{1}{3}\frac{\partial^2 J}{\partial \tau^2} = \epsilon(J - B).$$

$I = J + \mu\, \mathrm{d}J/\mathrm{d}\tau$

Introducing optical depth

$$\tau_* = \sqrt{3\epsilon}\,\tau = \sqrt{3\tau_a(\tau_a + \tau_s)},$$

We get 2nd order differential equation for J : $$\frac{\partial^2 J}{\partial \tau_*^2} = J - B$$

General solution : $J = c_1 \exp(\tau_*) + c_2 \exp(-\tau_*) + B$

BASICS of RADIATIVE TRANSFER
Solution of RTE: Eddington approximation

Two-stream approximation.

$$I^+ = I(\tau, \mu = 1/\sqrt{3}),$$
$$I^- = I(\tau, \mu = -1/\sqrt{3}).$$

$$J = \frac{1}{2}(I^+ + I^-),$$
$$H = \frac{1}{2\sqrt{3}}(I^+ - I^-),$$
$$K = \frac{1}{6}(I^+ + I^-) = J/3,$$

We can find a solution

$$I^+ = J + \frac{1}{\sqrt{3}}\frac{\partial J}{\partial \tau},$$
$$I^- = J - \frac{1}{\sqrt{3}}\frac{\partial J}{\partial \tau}.$$

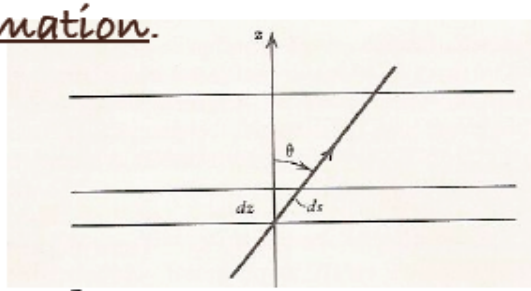
Need two boundary conditions

For example

$$I^-(\tau = \tau_1) = 0, \quad I^+(\tau = \tau_2) = 0.$$

Radiative Diffusion: Rosseland Approx

Derive a simple expression for the energy flux, relating it to the local temperature gradient - called Rosseland Approximation.



First assume that the material (temperature, absorption coeff etc.) depend on depth in the medium - called Plane-Parallel assumption.

Convenient to use $\mu = \cos\theta$

$$ds = \frac{dz}{cos\theta} = \frac{dz}{\mu}$$

Therefore, transfer Eqn:

$$\mu \frac{\partial I_\nu(z,\mu)}{\partial z} = -(\alpha_\nu + \alpha_{sc})(I_\nu - B_\nu)$$

$$I_\nu(z,\mu) \approx B_\nu(T) - \frac{\mu}{\alpha_\nu + \alpha_{sc}} \frac{\partial B_\nu}{\partial z}$$

$$F_\nu(z) = \int I_\nu(z,\mu) cos\theta d\Omega = 2\pi \int_{-1}^{+1} I_\nu(z,\mu)\mu d\mu$$

$$\pi B = \sigma T^4$$

$$\frac{1}{\alpha_R} \equiv \frac{\int \frac{dB_\nu}{d\bar{\tau}} \frac{1}{\alpha_\nu + \alpha_{sc}} d\nu}{\int \frac{dB_\nu}{d\bar{\tau}} d\nu} = \frac{\int \frac{dB_\nu}{dT} \frac{1}{\alpha_\nu + \alpha_{sc}} d\nu}{\int \frac{dB_\nu}{dT} d\nu}$$

$$F(z) = -\frac{16\sigma T^3}{3\alpha_R} \frac{\partial T}{\partial z}$$

Black body radiation

The black body intensity is defined (following discovery by Max Planck in 1900) as either

$$B_\lambda(T) = \frac{2hc^2}{\lambda^5}\frac{1}{e^{hc/\lambda kT}-1} \quad \text{or} \quad B_\nu(T) = \frac{2h\nu^3}{c^2}\frac{1}{e^{h\nu/kT}-1}$$

where c=2.99x10^{10} cm, h=6.63x10^{-27} erg s, k=1.38x10^{-16} erg/K

Note that: $$B_\nu(T)\mathrm{d}\nu = B_\lambda(T)\mathrm{d}\lambda \Rightarrow B_\lambda = B_\nu\left|\frac{\mathrm{d}\nu}{\mathrm{d}\lambda}\right| = B_\nu\frac{c}{\lambda^2}$$

Let us compute the bolometric flux:

Stefan-Boltzmann law

$$F = \pi\int_0^\infty B_\nu(T)\mathrm{d}\nu = \pi\int_0^\infty \frac{2h\nu^3}{c^2}\frac{1}{e^{h\nu/kT}-1}\mathrm{d}\nu = \pi\frac{2h}{c^2}\left(\frac{kT}{h}\right)^4\int_0^\infty\frac{x^3}{e^x-1}\mathrm{d}x = \pi\frac{2h}{c^2}\left(\frac{kT}{h}\right)^4\frac{\pi^4}{15} = \sigma T^4$$

$$\sigma = 2\frac{\pi^5k^4}{15c^2h^3} = 5.67\ 10^{-5}\ \text{erg cm}^{-2}\text{s}^{-1}\ \text{K}^{-4} - \text{Stefan-Boltzmann constant}$$

$x = h\nu/kT$

Planck function is monotonic with temperature: $$\frac{\partial B_\nu(T)}{\partial T} = \frac{2h^2\nu^4}{c^2kT^2}\frac{e^{h\nu/kT}}{\left(e^{h\nu/kT}-1\right)^2} > 0$$

At any ν, $T\uparrow$, $B_\nu\uparrow$. $T\to 0$, $B_\nu\to 0$. $T\to\infty$, $B_\nu\to\infty$.

Black body radiation

The source function for thermal radiation

Kirchhoff's Law: material emitting thermal radiation has

$$S_\nu = B_\nu(T)$$

and therefore

$$j_\nu = \alpha_\nu B_\nu(T).$$

The energy density of the black body radiation

$$u(T) = aT^4$$

$a = 7.56\times10^{-15}$ erg cm^{-3} K^{-4} is the radiation constant.

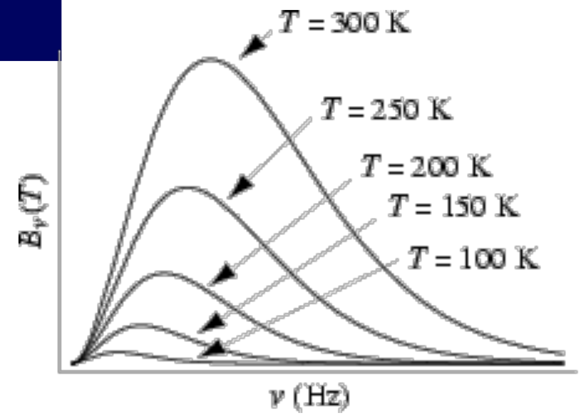
Flux of the black body radiation from the surface

$$F = \sigma_{\rm SB} T^4$$

$$\sigma_{\rm SB} = ac/4 = 5.67 \times 10^{-5} \text{ erg cm}^{-2}\text{ K}^{-4}\text{ s}^{-1}$$

Since $F = \pi B$ and $u = \frac{4\pi}{c} B$

Properties of the Planck law



Maximum of the Planck function $B_\nu(T) = \frac{2h}{c^2}\left(\frac{kT}{h}\right)^3 \frac{x^3}{e^x - 1}$

$$\frac{\partial}{\partial x}\left(\frac{x^3}{e^x - 1}\right) = \frac{x^2}{\left(e^x - 1\right)^2}\left[(3 - x)e^x - 3\right] = 0 \Rightarrow x = 3\left(1 - e^{-x}\right)$$

guess

$$x_0 = 3, \quad x_1 = 3\left(1 - e^{-3}\right) = 2.85, \; x_2 = 2.82...$$

$$x_{max} \approx 2.82 \Rightarrow h\nu_{max} = x_{max} kT \approx 2.82kT$$

Wien displacement law

$$\nu_{max} \approx 5.88 \times 10^{10} T$$

Do the same with B_λ: $x_{max} = 4.97, \; \lambda_{max} = \frac{hc}{kT} x_{max}$

$$\lambda_{max} T \approx 0.29 \text{ cm K}$$

$$\lambda_{max} \neq c / \nu_{max} \text{ !!!}$$

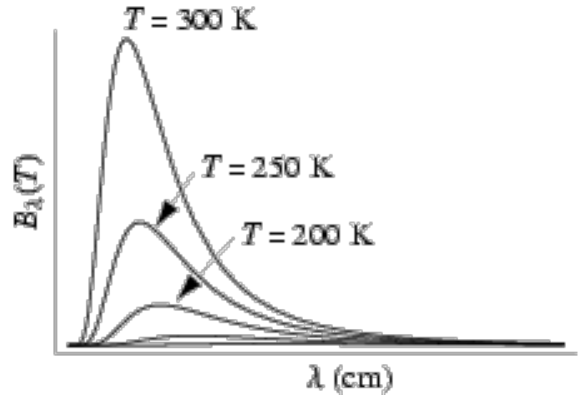
For the Sun λ_{max} =5175 Å, but $\lambda = c/\nu_{max}$ =8800Å.

At long wavelengths $\lambda >> \lambda_{max}$ (small frequencies $\nu << \nu_{max}$)
the Planck formulae can be approximated by the Rayleigh-Jeans law

$$B_\nu(T) \approx 2\frac{\nu^2}{c^2} kT, \quad B_\lambda(T) \approx 2ckT\lambda^{-4}$$

At short wavelengths $\lambda \leq \lambda_{max}$ (large frequencies $\nu \geq \nu_{max}$),
the Wien law is a good approximation

$$B_\nu(T) \approx 2\frac{h\nu^3}{c^2} e^{-\frac{h\nu}{kT}}, \quad B_\lambda(T) \approx 2\frac{hc^2}{\lambda^5} e^{-\frac{hc}{\lambda kT}}$$



Colour and brightness temperatures

Define brightness temperature as $I_\nu = B_\nu(T_b)$

In radio band we get

$$I_\nu = 2\frac{\nu^2}{c^2}kT_b \quad \text{, so that } T_b = \frac{c^2}{2\nu^2 k} I_\nu \text{ for } h\nu << kT$$

Colour temperature T_c is obtained by "fitting" the observed spectrum with the Planck function ignoring normalization. It gives correctly the temperature of the black body source of unknown absolute scale of the intensity.

Effective temperature $F = \sigma_{SB} T_{\mathrm{eff}}^4$