

Neutron stars



References:

1. Shapiro & Teukolsky, WD, NS & BHs, Chapters 9 & 10

Plan of the lecture

- History of neutron star (NS) research
- Formation of NS in supernova explosions
- Simple estimates of NS radii
- Internal structure of NS
- Mass-radius relation
- Equation of state for degenerate matter

History of neutron stars

- 1932: Chadwick discovers neutrons
- 1932: Landau suggests the existence of neutron stars
- 1934: Baade & Zwicky suggest that neutron-stars are formed in supernovae
- 1939: Oppenheimer & Volkoff obtain the limiting mass of the neutron star of about $2-3 M_{\odot}$
- 1967: Antony Hewish, Joselyn Bell et al. discover radio pulsars.
- 1968: Gold proposed rotating neutron star model
- 1971: X-ray pulsar Cen X-3 was discovered by Riccardo Giacconi
- 1974: Nobel prize to Hewish *"for his decisive role in the discovery of pulsars"*.
- 1975: R.A. Hulse & J.H. Taylor discover a binary pulsar PSR B1913+16
- 1993: Nobel prize to **Hulse & Taylor** *"for the discovery of a new type of pulsar, a discovery that has opened up new possibilities for the study of gravitation"*

Paper by Landau – before discovery of neutron

On the theory of stars

287

The chemical potential μ is in the extreme-relativistic case equal to $hc \left[\frac{3\pi^2 \rho}{m^4} \right]^{1/3}$ where m is the mass per 1 elektron, that means for most elements two protonic masses. The equation (1) is the $n = 3$ polytropic equation of Emden. With known substitutions it can be brought to the form

$$\frac{1}{r^2} \frac{d}{dr} \left[r^2 \frac{dx}{dr} \right] = -x^3$$

with the boundary condition $-r^2 \frac{dx}{dr} = M \left[\frac{G}{hc} \right]^{1/2} m^2 \left[\frac{4}{3\pi} \right]^{1/2}$ on the outer boundary. As Emden has shown, this equation has a regular solution only in the case $-r^2 \frac{dx}{dr} = 2,015$, in

which case it admits arbitrary radii; that means, we will have an indifferent equilibrium corresponding to $a = 0$ in the former rough treatment. Thus we get an equilibrium state only

for masses greater than a critical mass $M_0 = \frac{3,1}{m^2} \left[\frac{hc}{G} \right]^{3/2} =$
 $= 2,8 \cdot 10^{33}$ gr. or about $1,5 \odot$ (for $m = 2$ protonic masses).

For $M > M_0$ there exists in the whole quantum theory no cause preventing the system from collapsing to a point (the electrostatic forces are by great densities relatively very small). As in reality such masses exist quietly as stars and do not show any such ridiculous tendencies we must conclude that all stars heavier than $1,5 \odot$ certainly possess regions in which the laws of quantum mechanics (and therefore of quantum statistics) are violated.

As we have no reason to believe that stars can be divided into two physically different classes according to the condition $M >$ or $< M_0$, we may with great probability suppose that all stars possess such pathological regions. It does not contradict the above arguments, which prove only that the condition $M > M_0$ is sufficient (but not necessary) for the existence of such regions. It is very natural to think that just the presence of these regions makes stars stars. But if it is so,

L.Landau, On the theory of
Stars,
Physikalische
Zeitschrift der Sowjetunion
Vol. 1, No. 2, 285-188, 1932
Written: Feb. 1931, Zurich
Received: Jan. 7, 1932
Published: Feb. 1932

**One should violate
Quantum Mechanics!**

we have no need to suppose that the radiation of stars is due to some mysterious process of mutual annihilation of protons and electrons, which was never observed and has no special reason to occur in stars. Indeed we have always protons and electrons in atomic nuclei very close together, and they do not annihilate themselves; and it would be very strange if the high temperature did help, only because it does something in chemistry (chain reactions!). Following a beautiful idea of Prof. Niels Bohr's we are able to believe that the stellar radiation is due simply to a violation of the law of energy, which law, as Bohr has first pointed out, is no longer valid in the relativistic quantum theory, when the laws of ordinary quantum mechanics break down (as it is experimentally proved by continuous-rays-spectra and also made probable by theoretical considerations).¹ We expect that this must occur when the density of matter becomes so great that atomic nuclei come in close contact, forming one gigantic nucleus.

This is true!!!

On these general lines we can try to develop a theory of stellar structure. The central region of the star must consist of a core of highly condensed matter, surrounded by matter in ordinary state. If the transition between these two states were a continuous one, a mass $M < M_0$ would never form a star, because the normal equilibrium state (i. e. without pathological regions) would be quite stable. Because, as far as we know, it is not the fact, we must conclude that the condensed and non-condensed states are separated by some unstable states in the same manner as a liquid and its vapour are, a property which could be easily explained by some kind of nuclear attraction. This would lead to the existence of a nearly discontinuous boundary between the two states.

The theory of stellar structure founded on the above considerations is yet to be constructed, and only such a theory can show how far they are true.

February 1931, Zurich.

¹ L. Landau and R. Peierls, ZS. f. Phys. 69, 56, 1931.

Baade ja Zwicky – prediction

**Walter Baade (Mt. Wilson
Observatory)**

Fritz Zwicky (Caltech)

**The meeting of American Physical
Society**

(Stanford, December 15-16, 1933)

**Published in Physical Review
(15.01.1934)**



38. Supernovae and Cosmic Rays. W. BAADE, *Mt. Wilson Observatory*, AND F. ZWICKY, *California Institute of Technology*.—Supernovae flare up in every stellar system (nebula) once in several centuries. The lifetime of a super-

nova is about twenty days and its absolute brightness a maximum may be as high as $M_{\text{vis}} = -14^M$. The visible radiation L_v of a supernova is about 10^8 times the radiation of our sun, that is, $L_v = 3.78 \times 10^{41}$ ergs/sec. Calculations indicate that the total radiation, visible and invisible, is of the order $L_T = 10^7 L_v = 3.78 \times 10^{48}$ ergs/sec. The supernova therefore emits during its life a total energy $E_T \geq 10^5 L_T = 3.78 \times 10^{53}$ ergs. If supernovae initially are quite ordinary stars of mass $M < 10^{34}$ g, E_T/c^2 is of the same order as M itself. In the *supernova* process *mass in bulk is annihilated*. In addition the hypothesis suggests itself that *cosmic rays are produced by supernovae*. Assuming that in every nebula one supernova occurs every thousand years, the intensity of the cosmic rays to be observed on the earth should be of the order $\sigma = 2 \times 10^{-3}$ erg/cm² sec. The observational values are about $\sigma = 3 \times 10^{-3}$ erg/cm² sec. (Millikan, Regener). With all reserve we advance the view that supernovae represent the transitions from ordinary stars into *neutron stars*, which in their final stages consist of extremely closely packed neutrons.

$t_1 = 10^6$ years + 410 seconds for 10^{11} volt electrons.

$t_2 =$ " + 47.6 days " 10^9 " "

$t_3 =$ " + 44 years " 10^{11} " protons.

These time lags $t_i - t$ would tend to smear out the change of intensity caused by the flare-up of individual supernovae. Dr. R. M. Langer in one of our seminars was the first to call attention to the straggling of simultaneously ejected particles.

5. *The super-nova process*

We have tentatively suggested that the super-nova process represents the transition of an ordinary star into a neutron star. If neutrons are produced on the surface of an ordinary star they will "rain" down towards the center if we assume that the light pressure on neutrons is practically zero. This view explains the speed of the star's transformation into a neutron star. We are fully aware that our suggestion carries with it grave implications regarding the ordinary views about the constitution of stars and therefore will require further careful studies.

W. BAADE

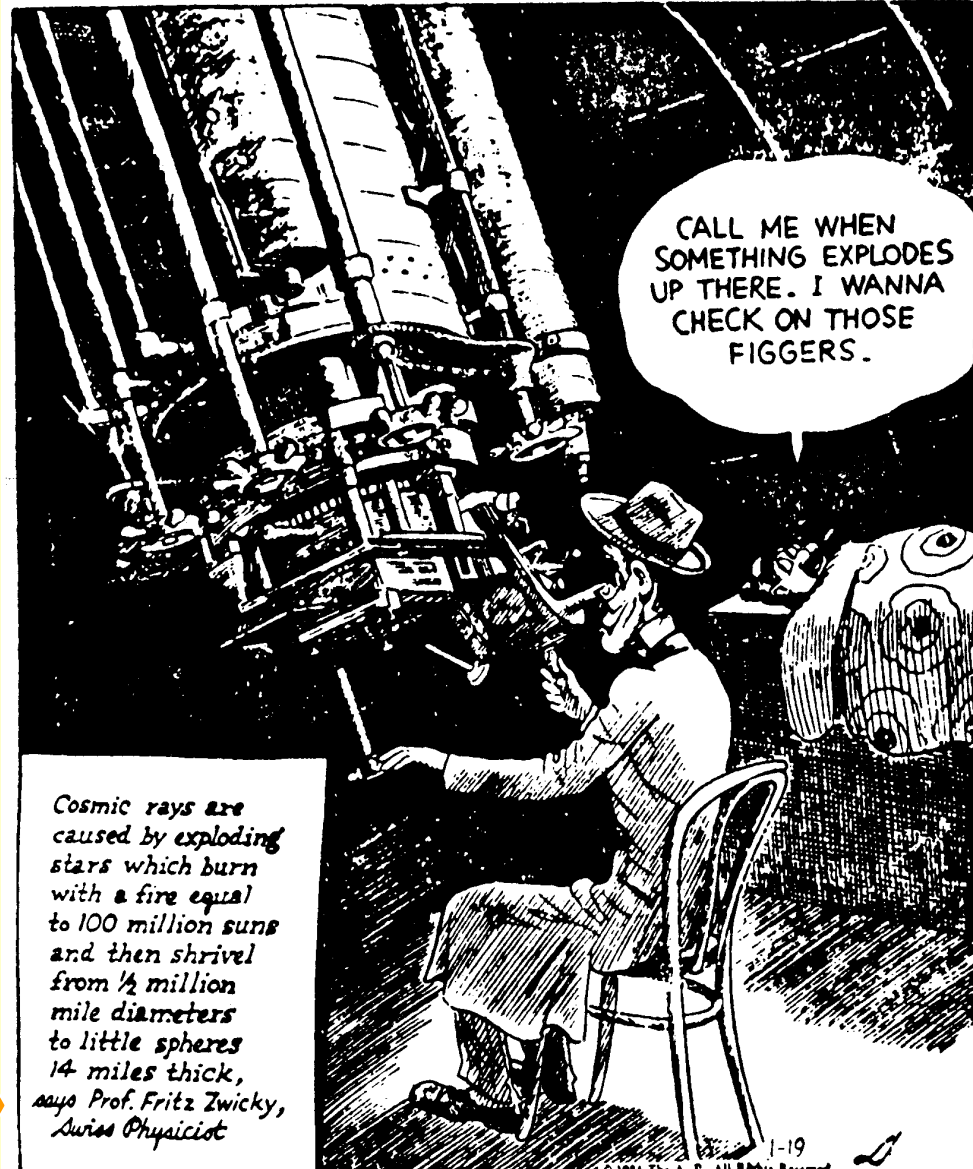
F. ZWICKY

Mt. Wilson Observatory and
California Institute of Technology, Pasadena.
May 28, 1934.

**Phys. Rev. 46, 76, 1934
July 1**

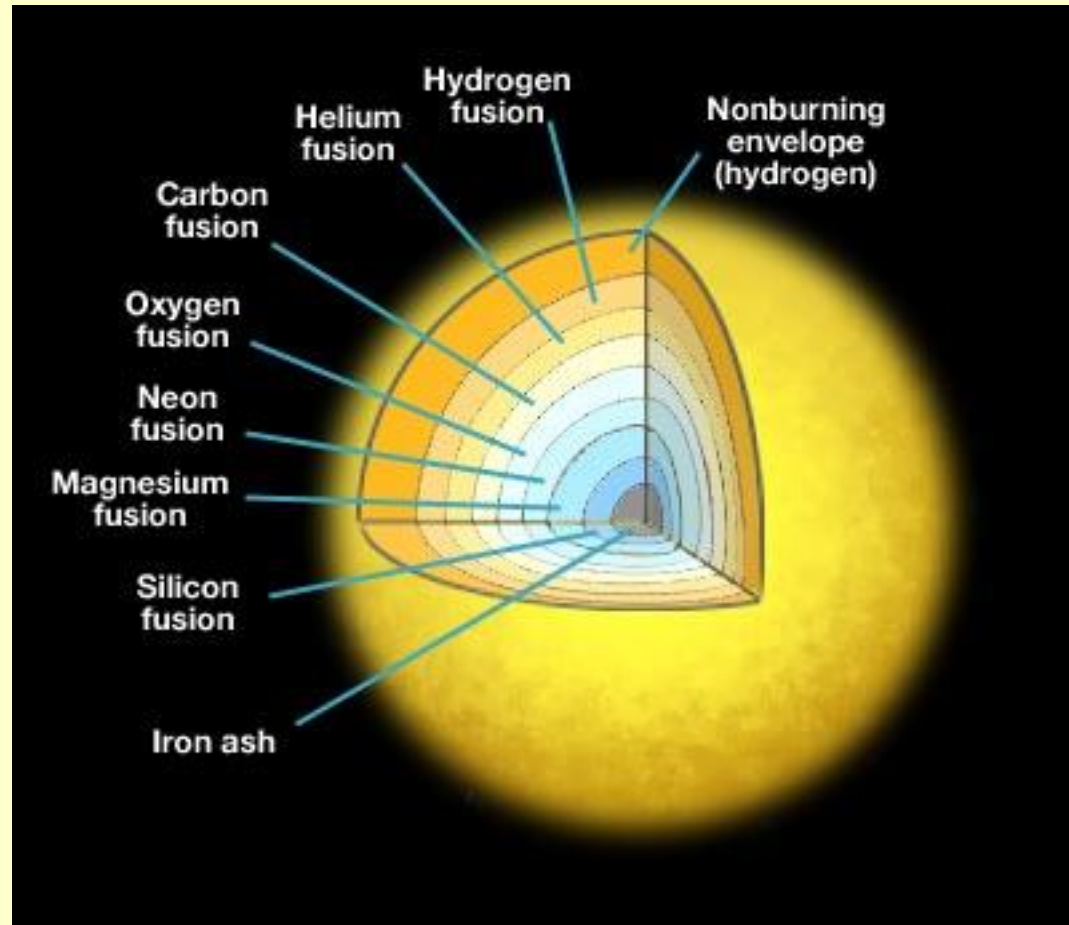
A.G.W. Cameron, recalling his Postdoc academic year 1959-1960 at Caltech reminds (Cameron, 1999): “For years Fritz [Zwicky] had been pushing his ideas about neutron stars to anyone who would listen and had been universally ignored. I believe that the part of the problem was his personality, which implied strongly that people were idiots if they did not believe in neutron stars.”

Los Angeles Times
January 19, 1934

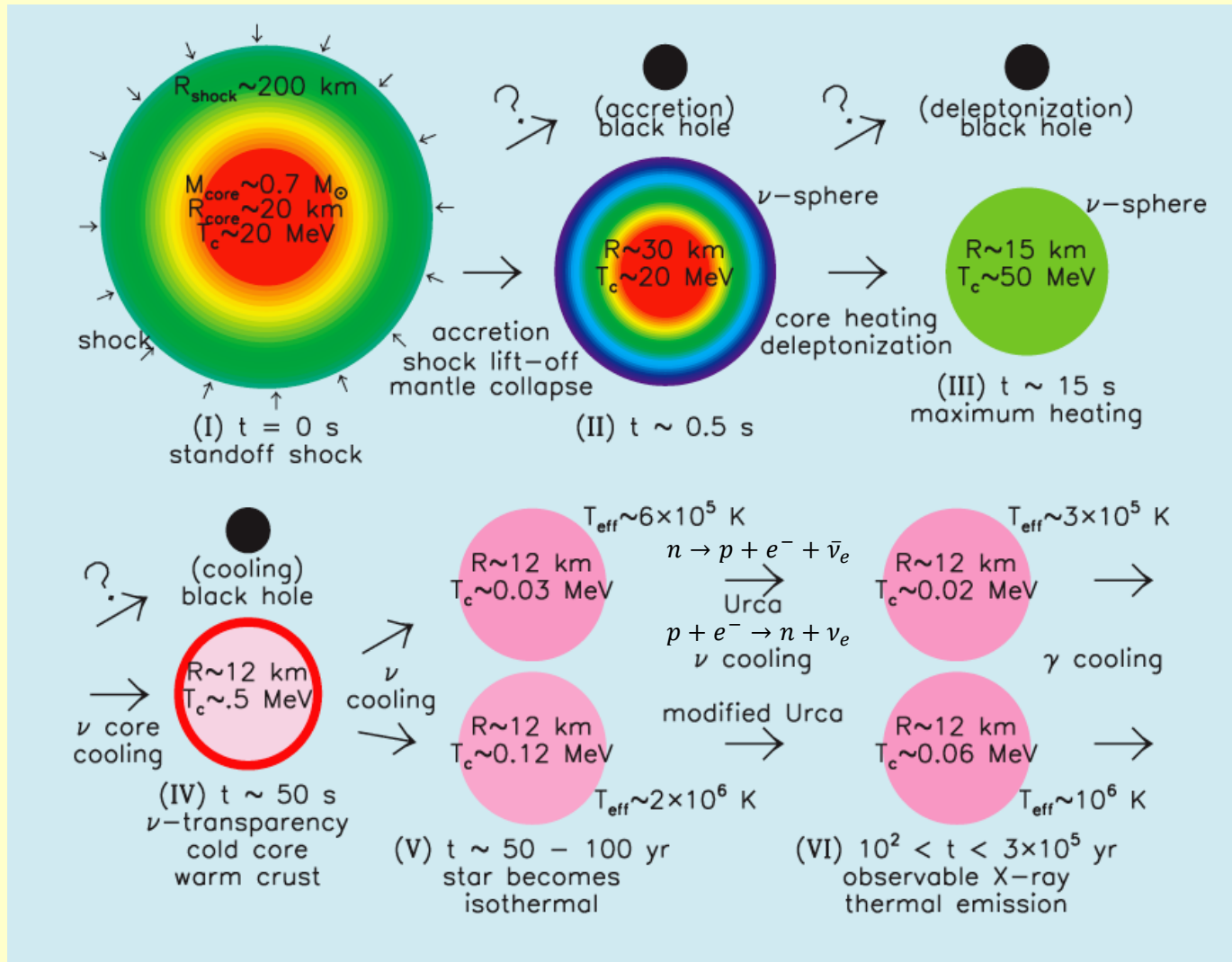


Core collapse in a massive star

- End of a massive star's life ($M > 8M_{\odot}$)
 - Center of star has fused all of the way to iron (iron is the most tightly bound nucleus so it is energetically unfavorable to fuse further)
 - Shells of other elements surround iron core
 - Only takes $\sim$ day to build up “dead” Chandrasekhar mass iron core
 - Core is held up by electron degeneracy pressure

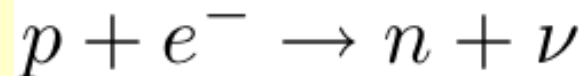


Formation of a neutron star in supernova explosion



From Lattimer & Prakash (2004)

- Once dead core exceeds $1.4M_{\odot}$, electron degeneracy pressure cannot support it.
- Core starts to collapse...
 - $\rho \approx 10^6 \text{ g/cm}^3$ - Density of core when collapse begins (onset of relativistic effects in electron motions)
 - $\rho \approx 10^7 \text{ g/cm}^3$ - Fermi energy exceeds neutron-proton mass difference ($\approx 1.3 \text{ MeV}$)...
 - Inverse beta decay becomes energetically preferable to normal beta decay



- Nuclei become very neutron rich... neutronization

- $\rho \approx 10^8 \text{ g/cm}^3$ - Individual nuclei are so neutron rich that they start to fall apart
 - Remaining nuclei surrounded by sea of free neutrons
 - This is called the neutron drip phase
- $\rho \approx 10^{10} \text{ g/cm}^3$ - Neutron degeneracy pressure starts to become important
- $\rho \approx 10^{12} \text{ g/cm}^3$ - Neutron degeneracy finally halts the collapse provided that $M < 2-3M_{\odot}$
- End up with a neutron star... typical mass of $1.4M_{\odot}$ with a radius of 10km

- From theory of polytropes, for white dwarfs

$$R_{wd} \propto \frac{K}{M^{1/3}}$$

$$R_{wd} = 0.89 \times 10^4 \text{ km} \left(\frac{M}{M_{\text{sun}}} \right)^{-1/3} \left(\frac{\mu_e}{2} \right)^{-5/3}$$

- Where... $P_c = K \rho_c^{5/3}$

$$K = \frac{1}{5} \frac{3}{8\pi} \frac{h^2}{m_e m_p^{5/3} \mu_e^{5/3}}$$

*Mass of particle
producing degeneracy
pressure*

*Number of nucleons per
degenerate particle*

- By analogy, neutron stars have (to a crude approximation)...

$$R_{ns} \propto \frac{K_n}{M^{1/3}}$$

$$P_n = K_n \rho^{5/3}$$

$$K_n = \frac{1}{5} \frac{3}{8\pi} \frac{h^2}{m_n^{8/3}}$$

i.e., degenerate (neutron) particles have mass m_n , and $\mu=1$

- So, we can try to estimate radius of neutron star given what we know about white dwarfs

$$\frac{R_{ns}}{R_{wd}} \approx \frac{m_e}{m_n} 2^{5/3} \left(\frac{M_{wd}}{M_{ns}} \right)^{1/3}$$

- We know that

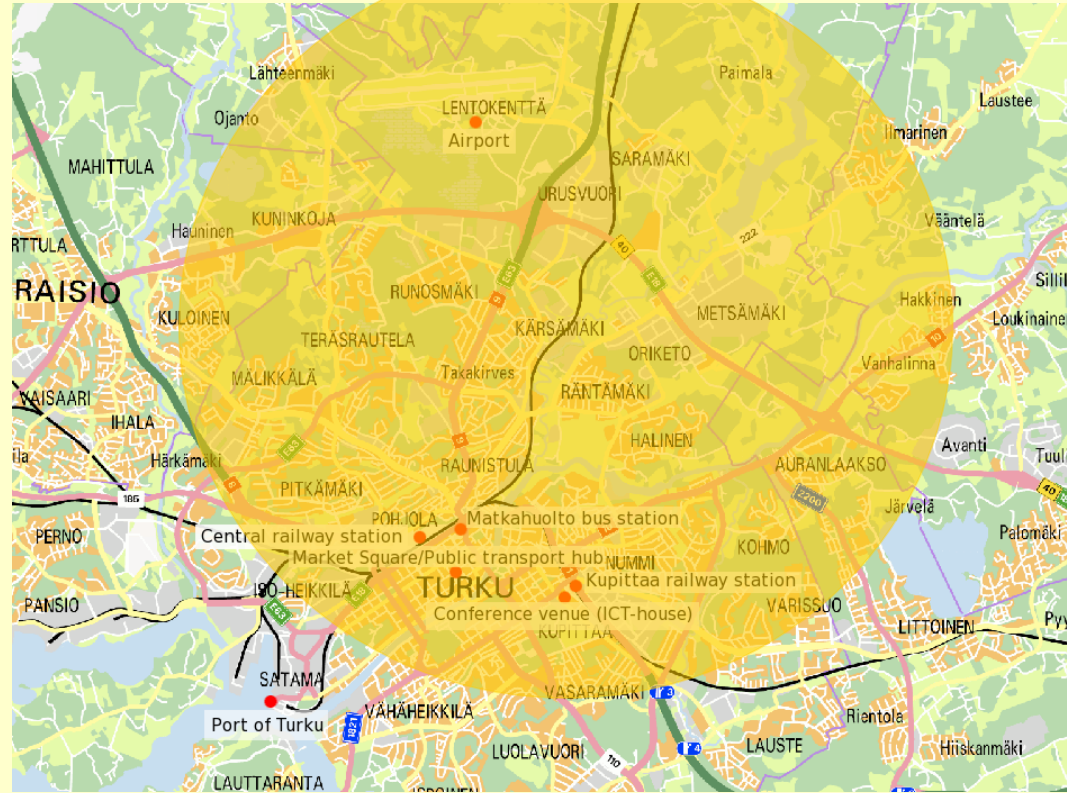
$$R_{wd} \approx 10^4 \text{ km}$$

- Thus, we expect

$$R_{ns} \approx 15 \text{ km}$$

Neutron star

- Size: about 11-13 km.
- Mass: 1.2-2.0 solar masses are accurately measured in pulsars in binary systems
- Consist predominantly of closely packed neutrons which form “one giant nucleus” with $N \sim 10^{57}$
- Magnetic fields 10^8 - 10^{15} G
- Period on 1.4 ms-1000 s.



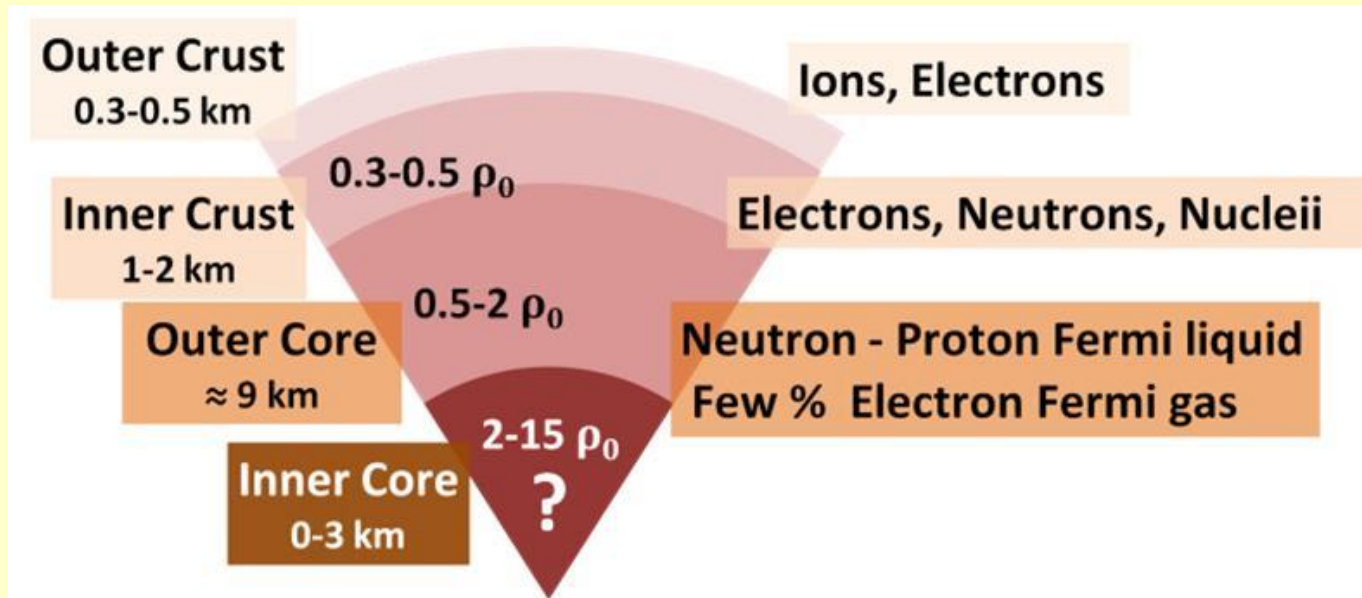
$$U \sim GM^2 / R \sim 5 \times 10^{53} \text{ erg} \sim 0.2 Mc^2$$

$$g \sim GM / R^2 \sim 2 \times 10^{14} \text{ cm/s}^2$$

$$\bar{\rho} \approx 3M / (4\pi R^3) \approx 7 \times 10^{14} \text{ g/cm}^3 \sim (2-3) \rho_0$$

$$\rho_0 = 2.8 \times 10^{14} \text{ g/cm}^3 = \text{standard nuclear density}$$

Internal structure



- Outer crust: degenerate relativistic electrons at density above 10^6g/cc .
- In the inner crust the fraction of free neutrons rises.
- Outer core contains neutrons with admixture of protons, electrons and possibly muons.
- The inner core might contain hyperons (Σ^- , Λ), pion-condensate, kaon-condensates, and also deconfined quarks. The actual composition is unknown.

Mass-radius relation

- Actually more complicated than white dwarf...
 - Have to use General Relativistic form of hydrostatic equilibrium equation
 - Neutrons do not behave like an ideal degenerate gas... strong repulsive interactions in dense matter at supranuclear density are crucial
 - There remain uncertainties about the “equation of state” of neutron stars

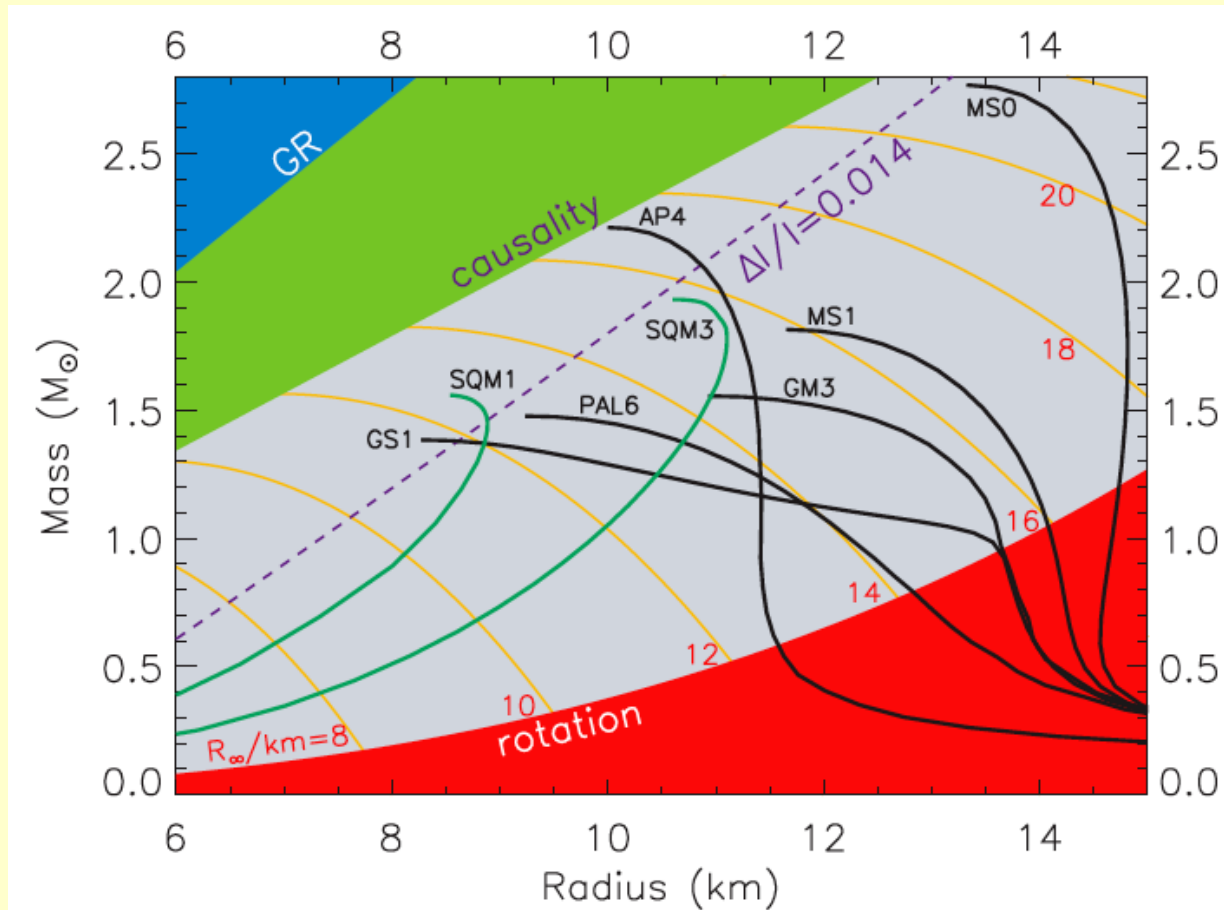


Fig. 2. Mass-radius diagram for neutron stars. Black (green) curves are for normal matter (SQM) equations of state [for definitions of the labels, see (27)]. Regions excluded by general relativity (GR), causality, and rotation constraints are indicated. Contours of radiation radii R_{∞} are given by the orange curves. The dashed line labeled $\Delta/I = 0.014$ is a radius limit estimated from Vela pulsar glitches (27).

From Lattimer & Prakash (2004)

Mass-radius relation

- The mass-radius relation depends on the internal composition of the NS matter and the equation of state (which is unknown for very high densities).
- Measuring M and R of a NS, in principle, can help in determining what is inside a NS.
- Masses can be measured in binary systems, while the radii are difficult to measure accurately.

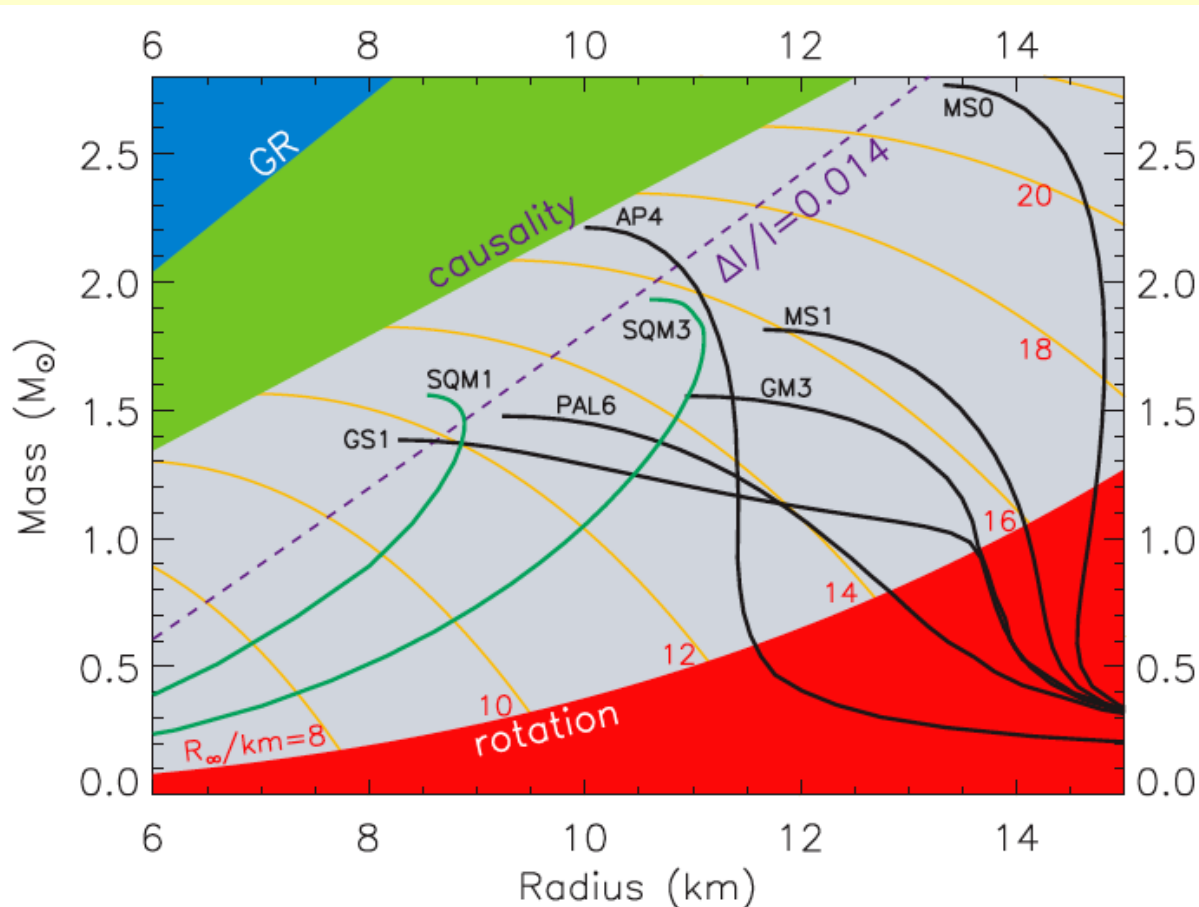
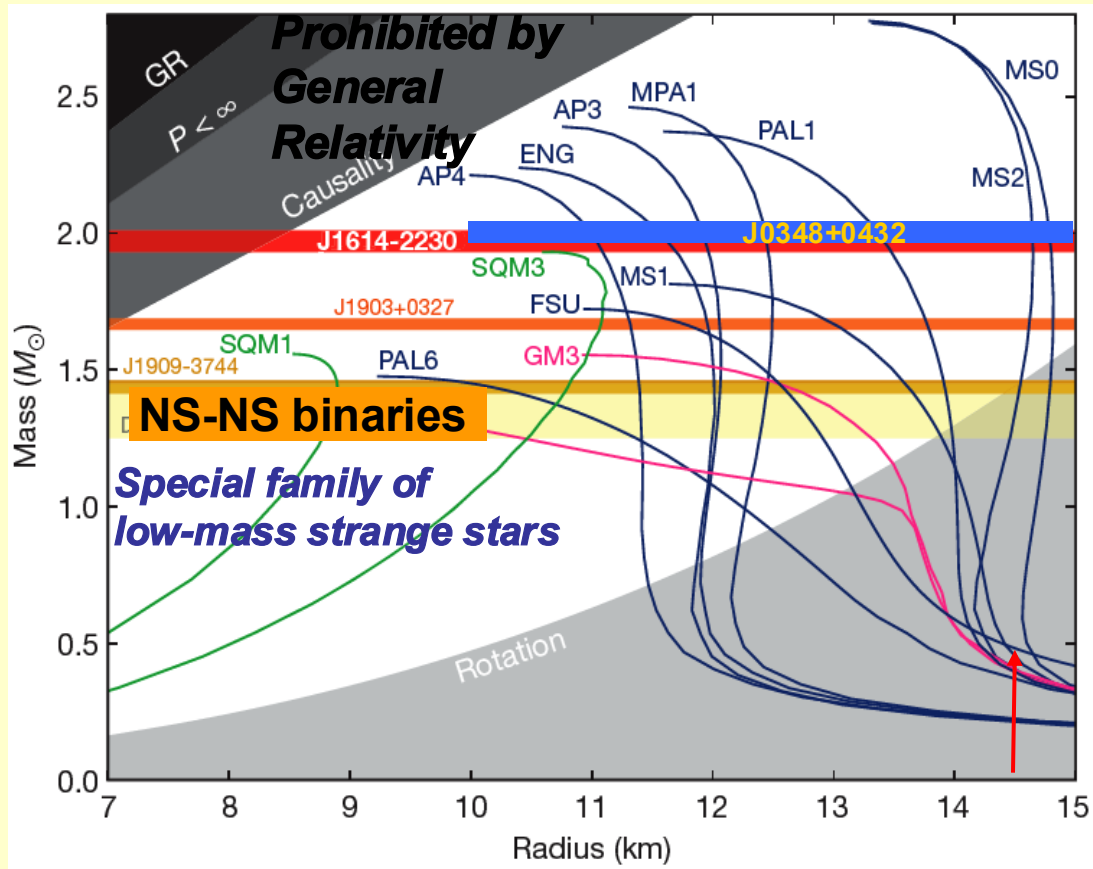


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From Lattimer & Prakash (2004)

Constraints from pulsar masses

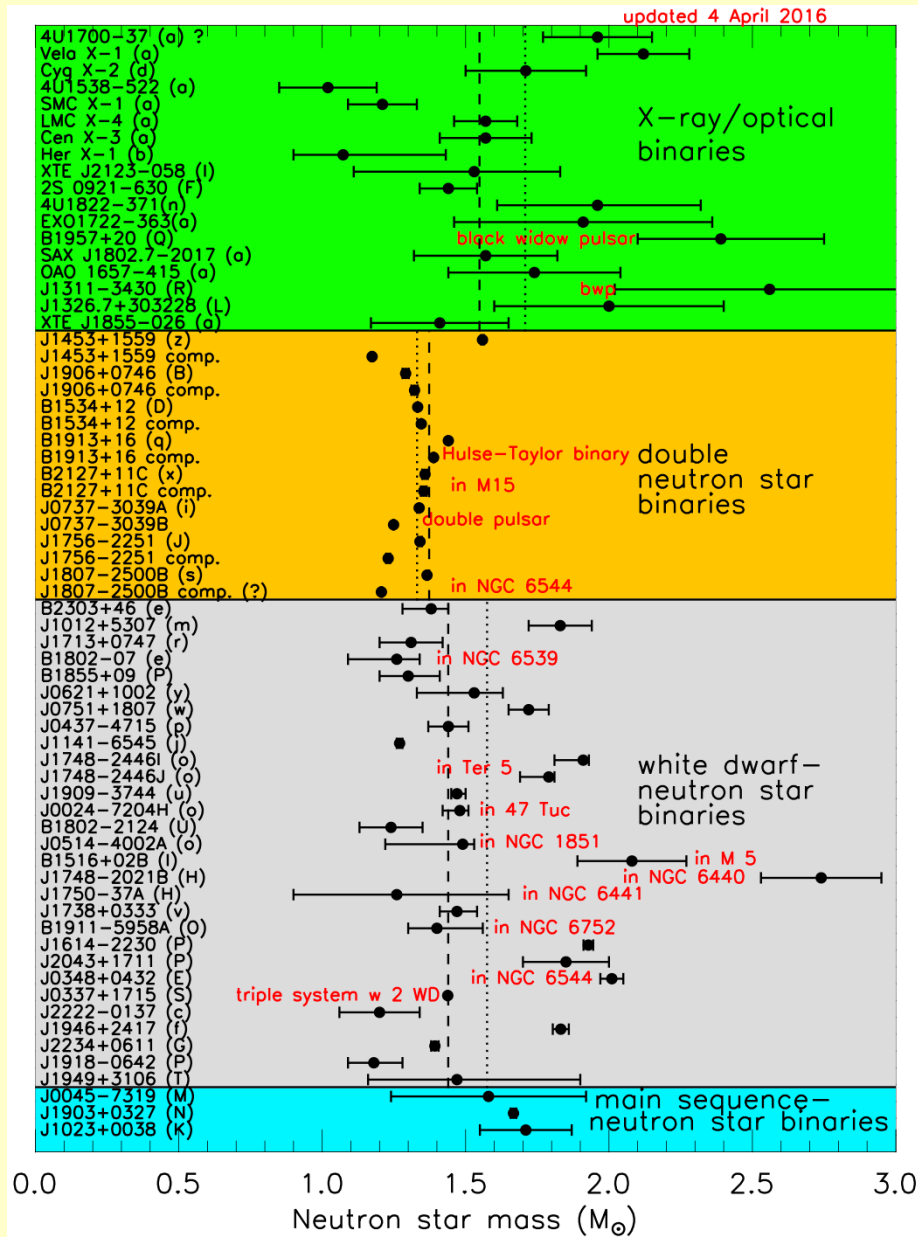


High masses of a few pulsars have been measured with the record holder being J1614–2230, which has $M = (1.93 \pm 0.02) M_{\odot}$ (Demorest et al. 2010, Fonseca et al. 2016)

and in J0348+0432 (Antoniadis et al. 2013) with $M = (2.01 \pm 0.04) M_{\odot}$

This strongly constrains the EoS ruling out the softest ones.

Neutron star masses



Observations :

- 1) Rule out soft EOSs (leave stiff and moderate ones)
- 2) Are favorable for purely nucleonic composition of dense matter
- 3) Radii cannot be determined from binary motions! But constraints can be obtained using astrophysical methods: from X-ray bursts evolutions, pulse profiles of millisecond pulsars, etc.

Equation of state of a degenerate gas

Electrons and neutrons are fermions and obey Fermi-Dirac statistics.

According to the Pauli exclusion principle - **no more than two electrons/neutrons (of opposite spin) can occupy the same quantum cell**

The quantum cell is defined in phase space, and given by 6 values:

x, y, z, p_x, p_y, p_z (three spatial coordinates and momenta in three directions).

The volume of allowed phase space is given by $\Delta x \Delta y \Delta z \Delta p_x \Delta p_y \Delta p_z = h^3$

The number of electrons in this cell must be at most 2.

With increasing density, the e^- become crowded, eventually 2 e^- occupy almost same position. Volume of phase space becomes “full” (from exclusion principle)

It is not possible for another e^- to occupy space, unless p significantly different.

Consider a group of electrons occupying a volume V of position space which have momenta in the range $p \pm \delta p$. The volume of momentum space occupied by these electrons is given by the volume of a spherical shell of radius p , thickness δp :

$$4\pi p^2 \delta p$$

Volume of phase space occupied is ***volume occupied in position space multiplied by volume occupied in momentum space***

$$V_{ph} = 4\pi p^2 V \delta p$$

Number of quantum states in this volume is V_{ph} divided by volume of a quantum state (h^3)

$$\frac{4\pi p^2 V}{h^3} \delta p$$

Define $n(p) \delta p$ = number of electrons in *unit volume* with momenta in the range $p+\delta p$.

Pauli's exclusion principle tells us:

$$n(p) \delta p \leq 2 \frac{4\pi p^2}{h^3} \delta p$$

DEGENERATE GAS of Fermions

Fermi Sea ($T=0$)

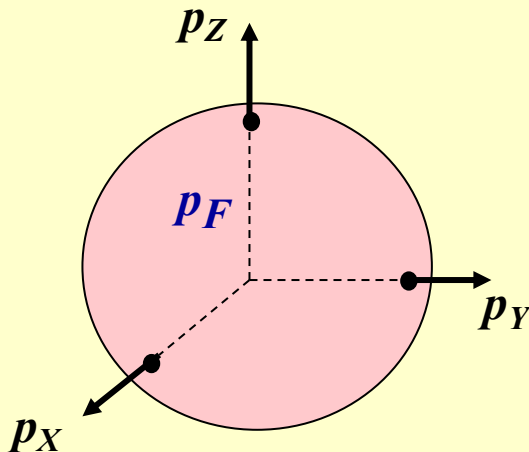
Completely degenerate gas : in which all of momentum states up to some critical Fermi momentum p_F are filled, while the states with momenta greater than p_F are empty.

(Many fully occupied states;
Many “dead” energetic particles)

Important for bulk properties of matter:
particle number density, energy density, pressure

$$\begin{array}{ll} p \leq p_F & n(p) = \frac{8\pi p^2}{h^3} \\ p > p_F & n(p) = 0 \end{array} \Rightarrow$$

$$n_e = \int_0^{p_F} n(p) dp = \frac{8\pi}{h^3} \int_0^{p_F} p^2 dp = \frac{8\pi p_F^3}{3h^3}$$



$$n = \frac{p_F^3}{3\pi^2 \hbar^3} \quad \square \quad p_F = \hbar(3\pi^2 n)^{1/3} = \text{Fermi momentum}$$

**Fermion number density n
is the only parameter!**

DEGENERATE GAS of Fermions

Fermi-Dirac Distribution

Concentration

$$n_e = \frac{2}{h^3} \int d\vec{p} f(\varepsilon),$$

Energy density

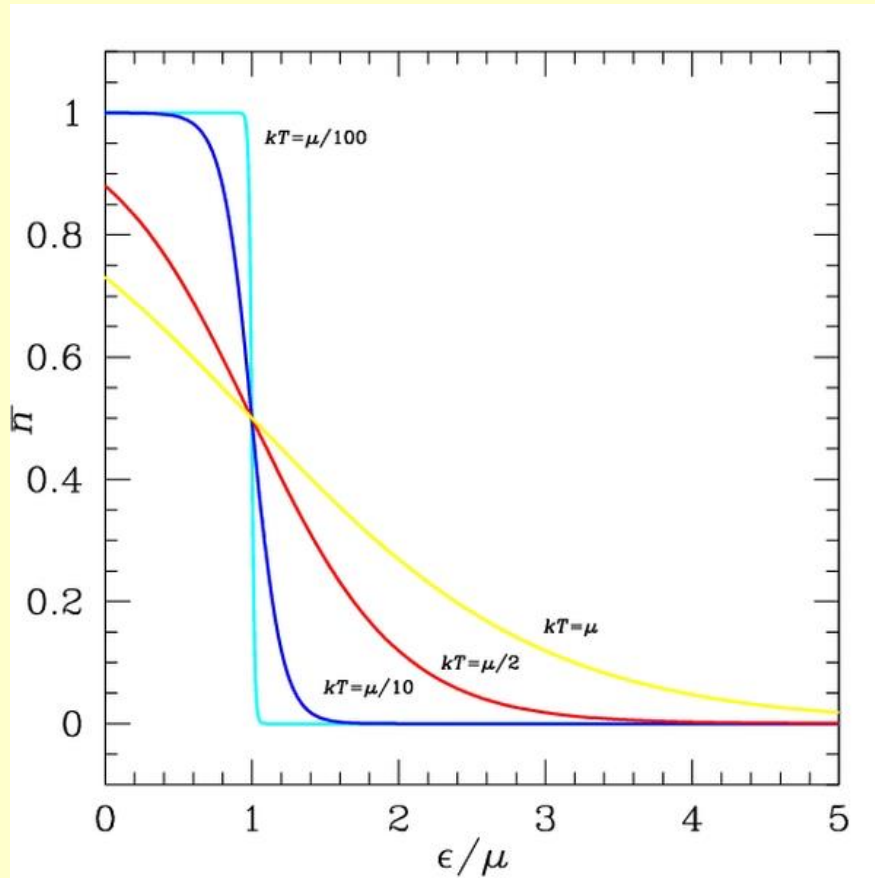
$$E = \frac{2}{h^3} \int d\vec{p} \varepsilon f(\varepsilon)$$

$$\varepsilon = c\sqrt{p^2 + m_e^2 c^2}$$

$$d\vec{p} = 4\pi p^2 dp$$

$$f = \frac{1}{\exp((\varepsilon - \mu) / kT) + 1}$$

μ is chemical potential



Low temperatures: $T \ll \mu$

$$f(\varepsilon) = 1 \quad \text{at } \varepsilon \leq \mu;$$

$$f(\varepsilon) = 0 \quad \text{at } \varepsilon > \mu$$

EXAMPLE: DEGENERATE GAS of ELECTRONS

The gas is *degenerate* if the typical momentum of the thermal distribution

$p = m v \approx \sqrt{3mkT}$ is smaller than the limiting Fermi momentum

$$p_F = \left(\frac{3h^3}{8\pi} n_e \right)^{1/3}$$

i.e.
$$\sqrt{3mkT} \leq \left(\frac{3h^3}{8\pi} n_e \right)^{1/3} \Leftrightarrow \frac{8\pi(3mkT)^{3/2}}{3h^3 n_e} \leq 1$$

Thus, when the electrons are degenerate, the protons do not need to be because of their larger mass.

Now we relate n_e to the mass density ρ . For each mass of H (m_H) there is one e^- . For He and heavier elements there is approximately 1/2 e^- for each m_H .

If X is the hydrogen mass fraction, we get:

$$n_e = \frac{\rho X}{m_H} + \frac{\rho(1-X)}{2m_H} = \frac{\rho(1+X)}{2m_H} \quad \square \quad \rho = n_e \mu_e m_H, \text{ where } \mu_e = 2/(1+X)$$

$$\rho = \mu_e m_H \frac{8\pi p_F^3}{3h^3} = Bx^3, \text{ where } B = \mu_e m_H \frac{8\pi}{3} \left(\frac{m_e c}{h} \right)^3 = 0.97 \times 10^6 \mu_e, \quad x = \frac{p_F}{m_e c}$$

Where m_e = rest mass of e^-

The pressure P is mean rate of transport of momentum across unit area

$$P = \frac{1}{3} \int_0^\infty n(p) p v_p dp$$

where v_p = velocity of e^- with momentum p

Use relation between p and v_p from theory of special relativity

$$p = \frac{m_e v_p}{\sqrt{1 - v_p^2 / c^2}}$$

$$v_p = \frac{p / m_e}{\sqrt{1 + (p / m_e c)^2}}$$

Combining the three expressions for $n(p)$, P , and v_p , we obtain (electron) pressure of a *completely degenerate* electron gas (contribution of ions is negligible as they are not degenerate)

$$P = P_e = \frac{8\pi}{3h^3 m_e} \int_0^{p_F} \frac{p^4}{\sqrt{1 + (p / m_e c)^2}} dp = A f(x),$$

$$\text{where } A = \frac{\pi}{3} \left(\frac{m_e c}{h} \right)^3 m_e c^2 = 6 \times 10^{22} \text{ (cgs) and } A / B = m_e c^2 / (8\mu_e m_H)$$

$$f(x) = 8 \int_0^x \frac{y^4}{\sqrt{1 + y^2}} dy = x \sqrt{1 + x^2} (2x^2 - 3) + 3 \ln(x + \sqrt{1 + x^2})$$

In the two limits of non-relativistic degenerate gas ($p_F \ll m_e c$) and **relativistic degenerate gas** ($p_F \gg m_e c$; when v approaches c and momentum $\rightarrow \infty$) one can get the limiting expressions for function $f(x)$:

$$x \ll 1, f(x) \rightarrow \frac{8}{5} x^5$$

$$x \gg 1, f(x) \rightarrow 2x^4$$

x tells how relativistic the gas is.

Since $\rho \approx 10^6 \mu_e x^3 \approx 2 \times 10^6 x^3$ ($\mu_e = 2$ as there is no H)

$x \ll 1$, non - relativistic, $\rho \ll 2 \times 10^6 \text{ g cm}^{-3}$

$x \gg 1$, relativistic, $\rho \gg 2 \times 10^6 \text{ g cm}^{-3}$

We aim to obtain the equation of state for a degenerate gas. We have

$$P = A f(x) = A \begin{cases} \frac{8}{5} x^5, & \text{if } x \ll 1, \\ 2x^4, & \text{if } x \gg 1, \end{cases}, \quad A = \frac{\pi}{3} \left(\frac{m_e c}{h} \right)^3 m_e c^2$$

$$\rho = B x^3, \quad B = \mu_e m_H \frac{8\pi}{3} \left(\frac{m_e c}{h} \right)^3$$

Express x from the second equation and insert into the first one. We get :

$$P = K_1 (\rho / \mu_e)^{5/3} \quad \text{Equation of state of non-relativistic degenerate gas}$$

$$P = K_2 (\rho / \mu_e)^{4/3} \quad \text{Equation of state of a relativistic degenerate gas}$$

K_1 and K_2 can be expressed via physical constants

In a completely degenerate gas, the pressure depends **only** on the density and chemical composition. It is independent of temperature.

$$K_1 = \frac{1}{5} \frac{3}{8\pi} \frac{h^2}{m_e m_p^{5/3} \mu_e^{5/3}}$$