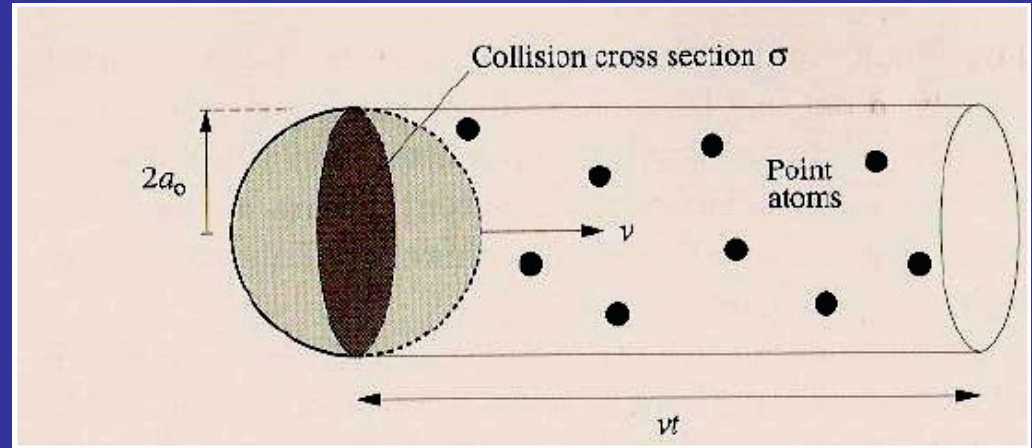
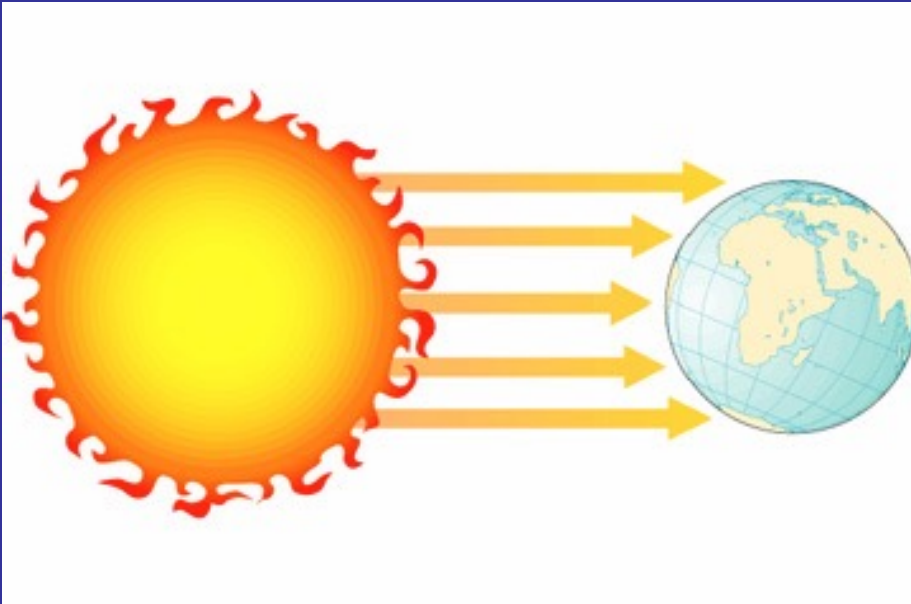




BASICS of RADIATIVE TRANSFER

BASICS of RADIATIVE TRANSFER

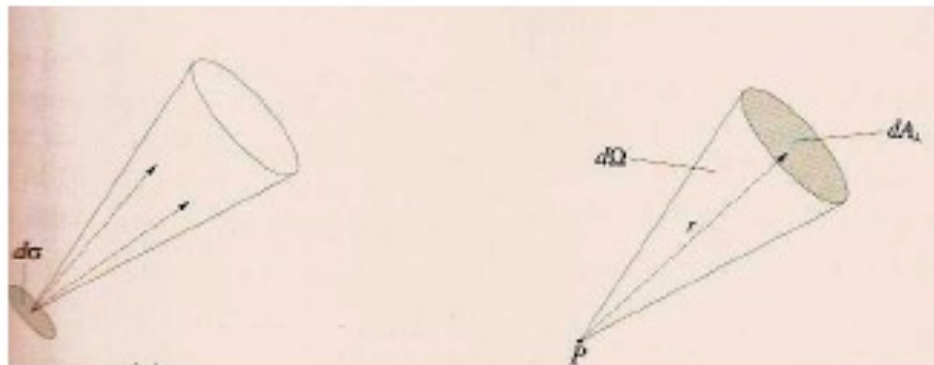
Definitions

Macroscopic Description of Radiation

describe the energy flux associated with electro-magnetic radiation.

The relationship between intensity and the energy flux, momentum flux, radiation pressure and energy density.

Solid Angle



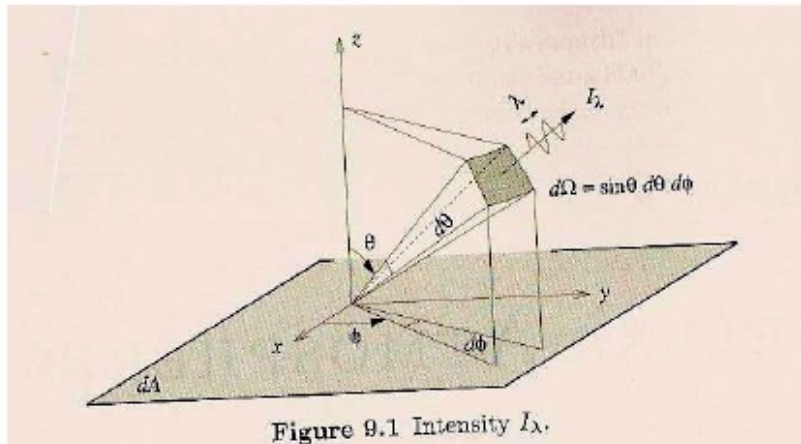
$$d\Omega \equiv dA_{\perp} / r^2$$

$$d\Omega_{tot} = \oint d\Omega = 4\pi \text{ sr}$$

BASICS of RADIATIVE TRANSFER

Definitions

Description of a Radiation Field



dE

The amount of energy that the rays carry into cone in time dt

Average intensity of rays is defined:

$$\bar{I}_\lambda = \frac{dE}{d\lambda dt dA \cos\theta d\Omega}$$

$$dE = I_\lambda d\lambda dt dA \cos\theta d\Omega = I_\lambda d\lambda dt dA \cos\theta \sin\theta d\theta d\phi$$

$$\langle I_\lambda \rangle \equiv \frac{1}{4\pi} \int I_\lambda d\Omega = \frac{1}{4\pi} \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} I_\lambda \sin\theta d\theta d\phi$$

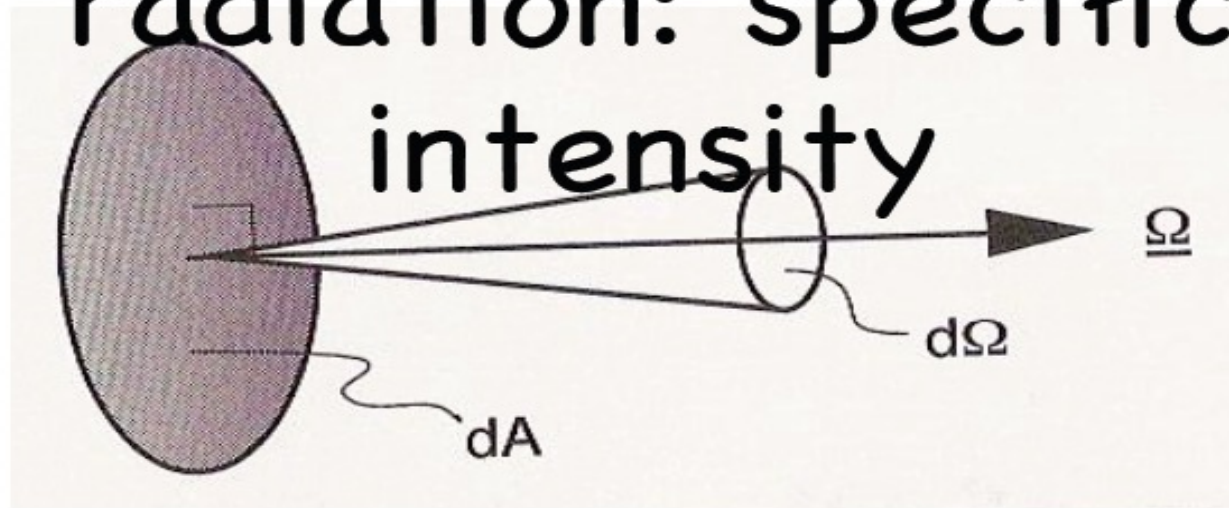
For isotropic radiation field $\langle I_\lambda \rangle = I_\lambda$

Solid angle in spherical coordinates $d\Omega = \sin\theta d\theta d\phi$

BASICS of RADIATIVE TRANSFER

Definitions

transport of energy via
radiation: specific
intensity



$$dE = I_{\nu}(\Omega) dA dt d\Omega d\nu$$

$$\text{erg/cm}^2/\text{s/steradian/Hz}$$

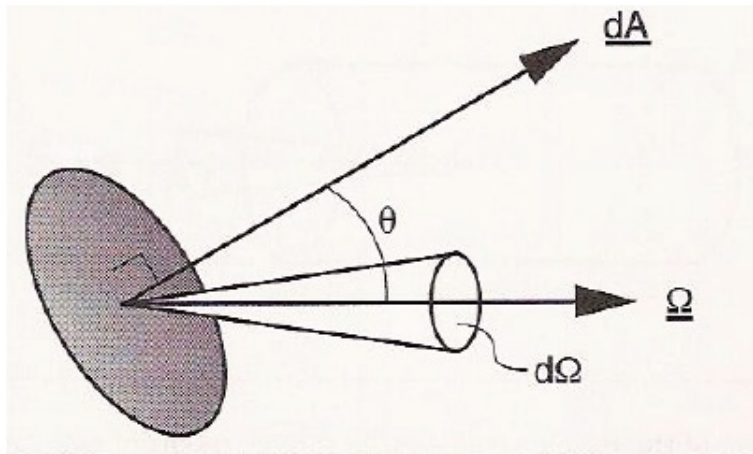
The intensity provides a fairly complete description of the transport of energy via radiation.

BASICS of RADIATIVE TRANSFER

Moments of intensity

Energy Flux

energy carried by a set of rays



$$dA_{\perp} = dA \cos \theta$$

$$dF_{\nu} = I_{\nu} \cos \theta d\Omega$$

differential flux

$$F_{\nu} = \int I_{\nu} \cos \theta d\Omega$$

net flux

($\text{erg s}^{-1} \text{cm}^{-2} \text{Hz}^{-1}$)

Multiplying a function of direction by n -th power of $\cos \theta$ and integrating is often called "taking the n -th moment".

Net flux is the first moment of the intensity

Energy flux not truly intrinsic since it depends on the orientation of the surface element, energy propagating downwards is negative energy flux. If I_{ν} is isotropic (not a fn. of angle) then net flux = 0.

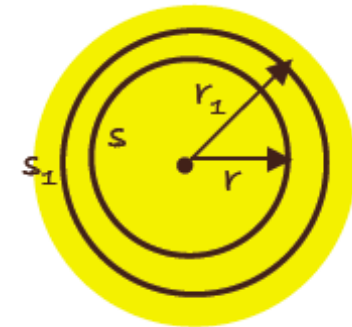
BASICS of RADIATIVE TRANSFER

Moments of intensity

Inverse square law for energy flux

Isotropic radiation:

emits energy equally in all directions (e.g., a star)



CONSERVATION OF ENERGY

$$F(r_1) \cdot 4\pi r_1^2 = F(r) \cdot 4\pi r^2$$

$$F(r) = \frac{\text{const}}{r^2}$$

If we can regard
sphere s_1 as fixed

$$F \propto 1/r^2$$

BASICS of RADIATIVE TRANSFER

Moments of intensity

Momentum Flux

Photons carry momentum

$$\mathbf{p} = \hat{\mathbf{n}}E/c$$

$\hat{\mathbf{n}}$, unit vector in direction of photon motion

component of momentum in the direction normal to the surface element is $p_{\perp}$

$$p_{\perp} = |p| \cos\theta$$

Differential momentum flux is $dF_{\nu}\cos\theta/c$ and integrating over all directions gives

$$p_{\nu} = \frac{1}{c} \int I_{\nu} \cos^2\theta d\Omega$$

the momentum flux is the second moment of the intensity

BASICS of RADIATIVE TRANSFER

Moments of intensity

Specific Energy Density

$u_\nu(\Omega)$ = energy/volume/solid angle/frequency
 $\propto$ intensity

To determine constant of proportionality
 the energy enclosed:

$$dE = u_\nu(\Omega) d\Omega \, dl \, dA \, d\nu$$

will pass out of cylinder in time, $dt = dl/c$

From our definition of specific intensity: $I_\nu = dE/dt \, dA \, d\Omega \, d\nu$

$$u_\nu(\Omega) = I_\nu(\Omega)/c$$

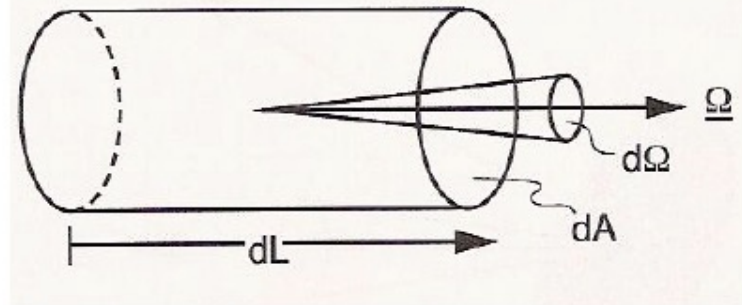
Integrating over direction: $u_\nu = \int u_\nu(\Omega) d\Omega = \frac{1}{c} \int I_\nu(\Omega) d\Omega$

Energy density is proportional to the zeroth moment of intensity

$$u_\nu = \frac{4\pi}{c} J_\nu \quad J_\nu \equiv \frac{1}{4\pi} \int I_\nu(\Omega) d\Omega$$

$$u = \int u_\nu d\nu = \frac{4\pi}{c} \int J_\nu d\nu$$

is the mean intensity



Black body radiation

The black body intensity is defined (following discovery by Max Planck in 1900) as either

$$B_{\lambda}(T) = \frac{2hc^2}{\lambda^5} \frac{1}{e^{hc/\lambda kT} - 1} \quad \text{or} \quad B_{\nu}(T) = \frac{2h\nu^3}{c^2} \frac{1}{e^{h\nu/kT} - 1}$$

where $c=2.99 \times 10^{10}$ cm, $h=6.63 \times 10^{-27}$ erg s, $k=1.38 \times 10^{-16}$ erg/K

Note that: $B_{\nu}(T)d\nu = B_{\lambda}(T)d\lambda \Rightarrow B_{\lambda} = B_{\nu} \left| \frac{d\nu}{d\lambda} \right| = B_{\nu} \frac{c}{\lambda^2}$

Let us compute the bolometric flux:

Stefan-Boltzmann law

$$F = \pi \int_0^{\infty} B_{\nu}(T) d\nu = \pi \int_0^{\infty} \frac{2h\nu^3}{c^2} \frac{1}{e^{h\nu/kT} - 1} d\nu = \pi \frac{2h}{c^2} \left(\frac{kT}{h} \right)^4 \int_0^{\infty} \frac{x^3}{e^x - 1} dx = \pi \frac{2h}{c^2} \left(\frac{kT}{h} \right)^4 \frac{\pi^4}{15} = \sigma T^4$$

$$\sigma = 2 \frac{\pi^5 k^4}{15 c^2 h^3} = 5.67 \cdot 10^{-5} \text{ erg cm}^{-2} \text{ s}^{-1} \text{ K}^{-4} - \text{Stefan - Boltzmann constant} \quad x = h\nu/kT$$

Planck function is monotonic with temperature: $\frac{\partial B_{\nu}(T)}{\partial T} = \frac{2h^2\nu^4}{c^2 k T^2} \frac{e^{h\nu/kT}}{(e^{h\nu/kT} - 1)^2} > 0$

At any ν , $T \uparrow$, $B_{\nu} \uparrow$. $T \rightarrow 0$, $B_{\nu} \rightarrow 0$. $T \rightarrow \infty$, $B_{\nu} \rightarrow \infty$.

Black body radiation

The energy density of the black body radiation $u(T) = aT^4$

$a = 7.56 \times 10^{-15} \text{ erg cm}^{-3} \text{ K}^{-4}$ is the radiation constant.

Flux of the black body radiation from the surface $F = \sigma_{\text{SB}} T^4$

$$\sigma_{\text{SB}} = ac/4 = 5.67 \times 10^{-5} \text{ erg cm}^{-2} \text{ K}^{-4} \text{ s}^{-1}$$

Since $F = \pi B$ and $u = \frac{4\pi}{c} B$

Properties of the Planck law

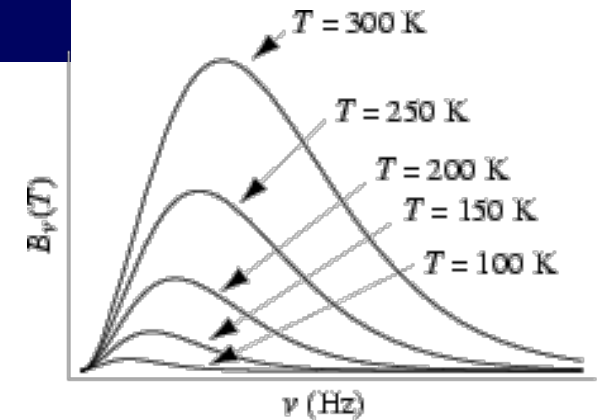
Maximum of the Planck function $B_\nu(T) = \frac{2h}{c^2} \left(\frac{kT}{h} \right)^3 \frac{x^3}{e^x - 1}$

$$\frac{\partial}{\partial x} \left(\frac{x^3}{e^x - 1} \right) = \frac{x^2}{(e^x - 1)^2} [(3 - x)e^x - 3] = 0 \Rightarrow x = 3(1 - e^{-x})$$

guess

$$x_0 = 3, \quad x_1 = 3(1 - e^{-3}) = 2.85, \quad x_2 = 2.82 \dots$$

$$x_{\max} \approx 2.82 \Rightarrow h\nu_{\max} = x_{\max} kT \approx 2.82 kT$$



Wien displacement law

$$\nu_{\max} \approx 5.88 \times 10^{10} T$$

Do the same with B_λ : $x_{\max} = 4.97, \lambda_{\max} = \frac{hc}{kT} x_{\max}$

$$\lambda_{\max} \neq c / \nu_{\max} !!!$$

$$\lambda_{\max} T \approx 0.29 \text{ cm K}$$

For the Sun $\lambda_{\max} = 5175 \text{ \AA}$, but $\lambda = c/\nu_{\max} = 8800 \text{ \AA}$.

At **long wavelengths** $\lambda \gg \lambda_{\max}$ (small frequencies $\nu \ll \nu_{\max}$)

the Planck formulae can be approximated by the **Rayleigh-Jeans law**

$$B_\nu(T) \approx 2 \frac{\nu^2}{c^2} kT, \quad B_\lambda(T) \approx 2ckT\lambda^{-4}$$

At **short wavelengths** $\lambda \leq \lambda_{\max}$ (large frequencies $\nu \geq \nu_{\max}$), the **Wien law** is a good approximation

$$B_\nu(T) \approx 2 \frac{h\nu^3}{c^2} e^{-\frac{h\nu}{kT}}, \quad B_\lambda(T) \approx 2 \frac{hc^2}{\lambda^5} e^{-\frac{hc}{\lambda kT}}$$

