

## HIGH ENERGY ASTROPHYSICS

Compulsary Home Exercises. Problem set 1. Solutions.

**1.1:** Consider two model mass density profiles in a neutron star of radius  $R$ : (1)  $\rho(r) = \rho_c$ ; (2)  $\rho(r) = \rho_c [1 - (r/R)^2]$ , where  $r$  is radial coordinate within the star and  $\rho_c$  is the central density. Take the canonical neutron star model with  $M = 1.4 M_\odot$  and  $R = 10$  km. Find  $\rho_c$  (expressed in units of standard nuclear matter density  $\rho_0 = 2.8 \times 10^{14} \text{ g cm}^{-3}$ ).

Solution:

$$M = \int_0^R 4\pi r^2 \rho(r) dr = \frac{4\pi}{3} R^3 \bar{\rho} = V \bar{\rho}$$

where  $\bar{\rho} = M/V = 6.65 \times 10^{14} \text{ g cm}^{-3} = 2.38 \rho_0$  is the average density.

Case 1:

$$M = \frac{4\pi}{3} R^3 \rho_c = V \rho_c,$$

as thus

$$\rho_c = \bar{\rho} = 2.38 \rho_0.$$

Case 2:

$$M = \int_0^R 4\pi r^2 \rho_c [1 - (r/R)^2] dr = 4\pi R^3 \rho_c \int_0^1 x^2 (1 - x^2) dx = 4\pi R^3 \rho_c \frac{2}{15} = \frac{2}{5} V \rho_c,$$

where  $x = r/R$  and  $\int_0^1 x^2 (1 - x^2) dx = 1/3 - 1/5 = 2/15$ . Thus we get

$$\rho_c = \frac{5}{2} \bar{\rho} = 5.95 \rho_0.$$

**1.2:** Calculate moment of inertia  $I$  of a neutron star of  $M$  and radius  $R$  for the two model density profiles (from problem 1.1, neglecting the effects of General Relativity). Evaluate  $I$  for a canonical neutron star with  $M = 1.4 M_\odot$  and  $R = 10$  km.

Solution:

Let us first compute the product  $MR^2 = 1.4 \times 2 \times 10^{33} \times 10^{12} = 2.79 \times 10^{45} \text{ g cm}^2$

(note that  $10 \text{ km} = 10^6 \text{ cm}$ ).

The general expression for the moment of inertia relative to the rotation axis can be written as a volume integral (using spherical coordinates  $r, \theta, \phi$ ) as:

$$I = \int \varpi^2 (\rho dV),$$

where  $\varpi = r \sin \theta$  is the cylindrical radius and the volume element is  $dV = r^2 dr \sin \theta d\theta d\phi$ . Thus

$$I = \int_0^R r^2 dr \int_0^\pi \sin \theta d\theta \int_0^{2\pi} d\phi \rho(r) (r^2 \sin^2 \theta) = 2\pi \int_{-1}^1 (1 - \cos^2 \theta) d \cos \theta \int_0^R \rho(r) r^4 dr = \frac{8\pi}{3} R^5 \int_0^1 \rho(x) x^4 dx.$$

Case 1:  $\rho_c = M/V = 3M/4\pi R^3$ ,

$$I = \frac{8\pi}{3} R^5 \frac{\rho_c}{5} = \frac{8\pi}{15} R^5 \frac{3M}{4\pi R^3} = \frac{2}{5} MR^2 = 1.11 \times 10^{45} \text{ g cm}^2.$$

Case 2:  $\rho_c = 5M/2V = 15M/8\pi R^3$ ,  $\int_0^1 x^4 (1 - x^2) dx = 1/5 - 1/7 = 2/35$ .

$$I = \frac{8\pi}{3} R^5 \rho_c \frac{2}{35} = \frac{8\pi}{3} R^5 \frac{2}{35} \frac{15M}{8\pi R^3} = \frac{2}{7} MR^2 = 0.78 \times 10^{45} \text{ g cm}^2.$$

**1.3:** Estimate at what density inside a neutron star neutrons become degenerate. Assume temperature of the neutron star of  $10^9$  K. At what density neutrons become relativistic?

Solution: The neutron concentration when they become degenerate is

$$n_n = \frac{8\pi (3m_n kT)^{3/2}}{3 h^3} = 1.66 \times 10^{34} \text{ cm}^{-3},$$

this corresponds to the density

$$\rho = n_n m_n = 2.78 \times 10^{10} \text{ g cm}^{-3},$$

They become relativistic when the Fermi momentum

$$p_F = \left( \frac{3h^3 n_n}{8\pi} \right)^{1/3}$$

becomes comparable to  $m_n c$ . This happens at

$$n_n = \frac{8\pi}{3} \left( \frac{m_n c}{h} \right)^3 = 3.6 \times 10^{39} \text{ cm}^{-3},$$

corresponding to the density

$$\rho = n_n m_n = 6.1 \times 10^{15} \text{ g cm}^{-3},$$

which is not achieved in neutron star cores.

**1.4:** Suppose a spherical body of a radius  $R_1$  is spinning originally at a rate  $\nu_1$  Hz collapses to a body of radius  $R_2$  conserving its mass  $M$  and angular momentum  $L$ . Express the ratio of the new and old spin rates  $\nu_2/\nu_1$  and the new and old rotational energies  $E_2/E_1$  in terms of the ratio  $R_2/R_1$ . You may assume the moment of inertia for a homogeneous sphere. By what factor would the star spin faster if it were to collapse from a radius typical of a white dwarf to the dimensions typical of a neutron star? By what factor would the rotational energy increase in such a collapse? Where ultimately does this energy come from?

Solution: The angular momentum is

$$L = I\Omega,$$

where  $\Omega = 2\pi\nu$ ,  $I = (2/5)MR^2$ . Thus

$$\nu_1 R_1^2 = \nu_2 R_2^2 \Rightarrow \frac{\nu_2}{\nu_1} = \frac{R_1^2}{R_2^2}.$$

Rotational energy  $E = I\Omega^2/2$ , thus

$$\frac{E_2}{E_1} = \frac{I_2 \nu_2^2}{I_1 \nu_1^2} = \frac{R_2^2 R_1^4}{R_1^2 R_2^4} = \frac{R_1^2}{R_2^2}.$$

For WD to become NS, radius changes by a factor  $\sim 10^3$ , so the frequency and the energy increase by a factor  $\sim 10^6$ . Energy comes from gravitational energy.

**1.5:** A neutron star cannot spin with less than a certain period or it will start to shed mass from its equator due to centrifugal force. Consider a neutron star of mass  $M$  and radius  $R$ . Show that this critical period is

$$P_{\min} = K \left( \frac{1.4M_{\odot}}{M} \right)^{1/2} \left( \frac{R}{10 \text{ km}} \right)^{3/2} \text{ ms},$$

where  $K$  is a constant. Compute  $K$  using Newtonian gravity (neglect also any deformation of neutron star due to rotation). Calculations using general relativity give  $K = 0.77$ . Compute the limit on the radius of the neutron star which has a period  $P = 1.4$  ms.

Solution: The neutron star angular frequency should be smaller than the Keplerian one

$$\Omega_K = \sqrt{\frac{GM}{R^3}}.$$

The limiting period is then

$$P_{\min} = \frac{2\pi}{\Omega_K} = \frac{2\pi}{\sqrt{GM}} M^{-1/2} R^{3/2} = 0.46 (1.4M_{\odot}/M)^{1/2} (R/10 \text{ km})^{3/2} \text{ ms}.$$

The upper limit on the radius for  $P = 1.4$  ms is (taking  $K = 0.77$ )

$$R = 14.9 (M/1.4M_{\odot})^{1/3} \text{ km}.$$

**1.6:** Crab pulsar has period of  $P = 0.033$  s and period derivative  $\dot{P} = 4.21 \times 10^{-13}$  s/s. Estimate the age of the pulsar. Compare it with the true age (supernova in year 1054). Estimate the pulsar magnetic field using magnetic dipole radiation formula.

Solution: The characteristic age is

$$\tau = \frac{P}{2\dot{P}} = 3.9 \times 10^{10} \text{ s} = 1250 \text{ yr}.$$

The actual age is about 970 yr, i.e. slightly smaller. The magnetic field is

$$B = 6 \times 10^{19} \sqrt{P\dot{P}} \text{ G} = 7 \times 10^{12} \text{ G}.$$