ANOTHER NOTE ON SMOOTH NUMBERS IN SHORT INTERVALS

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Abstract. We prove that, for any positive constants δ and ε and every large enough x, the interval $[x, x+\sqrt{x}(\log x)^{7/3+\delta}]$ contains numbers whose all prime factors are smaller than x^ε.

1. Introduction

Let $\Psi(x, y)$ denote the number of integers below x whose all prime factors are at most y. As usual we call such numbers y-smooth. It is well known that, for a wide range of x and $y = x^{1/u}$, we have $\Psi(x, y) \sim \rho(u)x$, where $\rho(u)$ is the Dickman function which is defined through a differential-difference equation and which satisfies $\rho(u) = u^{-u(1+o(1))}$ (see for instance the survey [9]). Besides being of theoretical interest, smooth numbers play an important role in computational number theory. For such applications, see for instance the survey [4].

In this note we are interested in y-smooth numbers in short intervals. One expects that, for a wide range of variables x, y and z,

$$\Psi(x + z, y) - \Psi(x, y) \sim \frac{z}{x}\Psi(x, y), \tag{1}$$

and Friedlander and Granville [3] have established this for

$$\exp((\log x)^{5/6+o(1)}) \leq y \leq x \quad \text{and} \quad \sqrt{x}y^2 \exp((\log x)^{1/6}) \leq z \leq x.$$

It is also interesting to prove just the existence of smooth numbers in a given short interval instead of establishing the asymptotic formula. Furthermore, intervals with length around $\sqrt{x}$ are of special interest from applications point of view but also because it is a breaking point for Dirichlet polynomial techniques. Indeed Granville [4, Section 4.1] writes that he believes that the outstanding problen in the whole area of smooth numbers is to show that, for any $\varepsilon > 0$ and large enough x, one has

$$\Psi(x + \sqrt{x}, x^\varepsilon) - \Psi(x, x^\varepsilon) > 0.$$

This is currently only known, by work of Harman [7], for $\varepsilon = 1/(4\sqrt{e})$. This value can be somewhat improved for slightly longer intervals: the author [13] has shown, refining an idea of Croot [2], that, for every $\varepsilon > 0$, there is a positive constant $C = C(\varepsilon)$ such that the interval $[x, x + C\sqrt{x}]$ contains $x^{1/(5\sqrt{e})+\varepsilon}$-smooth numbers. Assuming the Riemann hypothesis, Soundararajan [14] has remarkably shown that such intervals contain x^ε-smooth numbers for every $\varepsilon > 0$ (with C again depending on ε, see Theorem 1.2 below for detailed statement).

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In this note we investigate how much longer intervals one must take to guarantee existence of x^{ε}-smooth numbers unconditionally. By work of Lenstra, Pila and Pomerance [12] which extends Balog's [1] and Harman's [6] earlier works one knows that the interval $[x, x+\sqrt{x}\exp(C(\log x)^{3/4}(\log\log x)^{1/4})]$ contains x^{ε}-smooth numbers (actually $\exp(C'(\log x)^{3/4}(\log\log x)^{1/4})$-smooth numbers). Granville [4, Section 1.5] speculates that perhaps pushing several known methods to their extreme leads to a better result, possibly for intervals $[x, x+\sqrt{x}c(x)]$ with $c(x)$ a power of a logarithm. In this note we confirm this intuition by proving the following theorem.

Theorem 1.1. *Let $\varepsilon > 0$ and let $x \geq y \geq \exp((\log x)^{2/3}(\log\log x)^{4/3+\varepsilon})$ be large, and write $y = x^{1/u}$. Then, with*
$$z = \sqrt{x}\exp((7/3+\varepsilon)(\log\log x + 4u\log u)),$$
one has
$$\Psi(x+z, y) - \Psi(x, y) \gg zx^{-\varepsilon}.$$
In particular, for any $\alpha > 0$ and large enough x,
$$\Psi(x+\sqrt{x}(\log x)^{7/3+\varepsilon}, x^{\alpha}) - \Psi(x, x^{\alpha}) \gg x^{1/2-\varepsilon}.$$

It is clear from the proof that if, for some $\lambda \leq 1/2$, the zero density conjecture for the Riemann zeta function can be beated in the strip $\Re s \in (1-\lambda, 1]$ (see Lemma 2.3(i) below for an exact statement with the best known value $\lambda = 3/14$), then $7/3$ above can be replaced by $1/(2\lambda)$.

Therorem 1.1 should be compared with Soundararajan's conditional result

Theorem 1.2 (Soundararajan [14])**.** *Assume the Riemann Hypothesis. Let $\varepsilon > 0$ and let $x \geq y \geq \exp(\sqrt{\log x \log\log x})$ be large, and write $y = x^{1/u}$. There is an absolute positive constant B such that with $z = Bu\sqrt{x}/\rho(u/2)$ one has*
$$\Psi(x+z, y) - \Psi(x, y) \gg zx^{-\varepsilon}.$$
In particular, for every $\alpha > 0$, there is a constant $C = C(\alpha)$ such that
$$\Psi(x, x+C\sqrt{x}, x^{\alpha}) - \Psi(x, x^{\alpha}) \gg x^{1/2-\varepsilon}.$$

Let us now give an outline of the proof of Theorem 1.1. As in the previous works, we study a carefully chosen weighted sum over short intervals, in our case something like

$$\sum_{\substack{x \leq n_1 \cdots n_{2k} r_1 r_2 q \leq x+z \\ \text{conditions on ranges of } n_j \text{ and } r_j}} \Lambda(n_1)\cdots\Lambda(n_{2k})\Lambda(q)\cdot(\text{nice weight}), \tag{2}$$

where $\Lambda(n)$ is the von Mangoldt function. The ranges of the variables are chosen so that $n_j, r_j \leq y$ and $q \asymp z^2/x \leq y$, so that only y-smooth numbers are counted. As in previous works we use Perron's formula to relate this sum to an integral, in our case the sum equals
$$\frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} -\frac{\zeta'}{\zeta}(s)M(s)^2 x^s \frac{(1+(z/x))^s - 1)^2}{s^2}ds,$$
where $M(s)$ is certain Dirichlet polynomial of length $\asymp x/z$.

In previous works the integration path is next moved to the left but only so little that it stays in the zero-free region of the Riemann zeta function. The main term comes from the pole of $\zeta(s)$ at $s = 1$. An important new ingredient in our proof is that we move the integration path further to the left, following works of Wolke [15] and Harman [5] on E_2s (numbers with exactly two prime factors) in almost all short intervals $[x, x + (\log x)^C]$.

More precisely we move the integration path close to the line $\sigma = 11/14 + \eta$, where η is a very small positive constant (depending on δ). The new integration path is chosen carefully so that it avoids going too close to zeros of the zeta function. Thanks to this the integral on the new path can be handled using a good estimate for $\frac{\zeta'}{\zeta}(s)$ and the mean value theorem for Dirichlet polynomials. Since we go beyond the zero-free region, we encounter several poles and so need to handle a sum of $x^\beta|M(\beta+i\gamma)|^2$, where $\beta+i\gamma$ runs over zeros of the zeta function to the right of the line $\sigma = 11/14 + \eta$. Handling this sum is the most technical part of the argument and it can be handled thanks to the special form of the polynomial $M(s)$ and since the zero density conjecture is known in this region.

Our techniques can naturally be adapted to get corresponding results concerning smooth numbers in almost all very short intervals. Here and later we say that a claim holds for almost all $x \sim X$ (i.e. $x \in [X, 2X]$) if the measure of the exceptional set is $o(X)$.

Theorem 1.3. *Let $\varepsilon > 0$ and let $x \geq y \geq \exp((\log x)^{2/3}(\log x \log x)^{4/3+\varepsilon})$ be large, and write $y = x^{1/u}$. Then, with*

$$z = \exp((14/3 + \varepsilon)(\log\log x + 4u \log u)),$$

one has

$$\Psi(x + z, y) - \Psi(x, y) > 0$$

for almost all $x \in [X, 2X]$. In particular, for any $\alpha > 0$ and large enough X,

$$\Psi(x + (\log x)^{14/3+\varepsilon}, x^\alpha) - \Psi(x, x^\alpha) > 0.$$

for almost all $x \in [X, 2X]$.

Again this theorem covers shorter intervals than have been covered before, see [9, Section 5] for earlier results.

Remark 1.4. After completion of the present paper, the author and M. Radziwiłł have investigated a different way to study multiplicative problems in very short intervals. This new method seems to, among other things, give improvements to the logarithmic powers 7/3 and 14/3 in Theorems 1.1 and 1.3. The new work will appear later.

2. Auxiliary results

As indicated in the introduction, in the proofs of our theorems we will use Perron's formula to relate a weighted sum (2) of smooth numbers to an integral of a Dirichlet polynomial. This Dirichlet polynomial will be closely related to a product of polynomials of types $P(s) = \sum_n \Lambda(n)n^{-s}$ and $R(s) = \sum_r r^{-s}$ where summations run over all integers in an interval. To study those we need some auxiliary results. Following lemma follows immediately from the weighted truncated Poisson formula (see for example [11, Lemma 8.8]).

Lemma 2.1. *Let $|t| \leq N_1 \leq N_2$ and $s = \sigma + it$. Then*

$$\sum_{N_1 \leq n \leq N_2} n^{-s} = \frac{N_2^{1-s} - N_1^{1-s}}{1-s} + O(N_1^{-\sigma}).$$

We will need to relate $P(s)$ to a sum of zeros of the Riemann zeta function.

Lemma 2.2. *Let $s = \sigma + it$ and $P \geq 1$. One has*

$$\sum_{n \sim P} \frac{\Lambda(n)}{n^s} \ll P_1(s) + P_2(s) + P_3(s), \tag{3}$$

where

$$P_1(s) = \frac{P^{1-\sigma}}{1+|t|}, \quad P_2(s) = \sum_{\substack{\varrho = \beta + i\gamma \\ |\gamma - t| \leq P}} \frac{P^{\beta - \sigma}}{1 + |\gamma - t|}, \quad \text{and} \quad P_3(s) = P^{-\sigma} \log^2 P,$$

and the sum in $P_2(s)$ is over non-trivial zeros of the Riemann ζ-function.

Proof. This is a standard consequence of Perron's formula and move of the integration path. See also Wolke [15, Hilfssatz 2]. □

The following lemma collects the information we will need about the zeros of the zeta-function. We write $\varrho = \beta + i\gamma$ for non-trivial zeros and, as usual, denote by $N(\sigma, T)$ the number of zeros with $\beta \geq \sigma$ and $|\gamma| \leq T$.

Lemma 2.3. *Let $V \geq 1$.*

(i) For any $\varepsilon > 0$ there exists a positive constant $\theta = \theta(\varepsilon) < 2$ such that

$$N(\sigma, V) \ll V^{\theta(1-\sigma)} (\log V)^{15} \quad \text{for } \sigma \geq 11/14 + \varepsilon;$$

(ii) One has

$$N(\sigma, V+1) - N(\sigma, V) \ll \log V;$$

(iii) There exists a small positive constant c such that

$$N(\sigma, V) = 0 \quad \text{for } \sigma \geq 1 - c(\log V)^{-2/3} (\log \log V)^{-1/3}.$$

Proof. See [10, formulas (11.83) and (11.33), Theorem 1.7, and Theorem 6.1] respectively. □

To estimate $\frac{\zeta'}{\zeta}(s)$ we use the following lemma which is [10, formula (1.52)].

Lemma 2.4. *In the strip $\sigma \in [-1, 2]$ one has*

$$\frac{\zeta'}{\zeta}(\sigma + it) = \sum_{\substack{\varrho = \beta + i\gamma \\ |t - \gamma| < 1}} \frac{1}{s - \varrho} + O(\log(|t| + 2)),$$

where the summation runs over the zeros ϱ of the Riemann zeta function.

We will use the following quick consequence of the mean value theorem for Dirichlet polynomials.

Lemma 2.5. *Let $\sigma \in \mathbb{R}, \delta > 0$ and $M, T \geq 1$. Let $M(s) = \sum_{m \sim M} a_m m^{-s}$. Then*

$$\int_{\sigma - iT}^{\sigma + iT} |M(s)|^2 x^\sigma \min\left\{\delta^2, \frac{1}{|s|^2}\right\} dt \ll \delta^2 x \left(\frac{x}{M^2}\right)^{\sigma - 1} \left(1 + \frac{1}{\delta M}\right) \sum_{m \sim M} \frac{|a_m|^2}{m}.$$

Proof. Write I for the integral. Splitting it into intervals of length $1/\delta$,

$$I \ll \sum_{|r|\leq \delta T} \frac{1}{(1+|r|)^2} \cdot \delta^2 x^\sigma \int_{r/\delta}^{(r+1)/\delta} |M(\sigma+it)|^2 dt.$$

Applying the mean value theorem for Dirichlet polynomials (see for example [11, Theorem 9.1]) to the last integral we get

$$\begin{aligned} I &\ll \sum_{|r|\leq \delta T} \frac{1}{(1+|r|)^2} \cdot \delta^2 x^\sigma \cdot \left(\frac{1}{\delta}+M\right) \sum_{m\sim M} \left(\frac{|a_m|}{m^\sigma}\right)^2 \\ &\ll \delta^2 x \left(\frac{x}{M^2}\right)^{\sigma-1} \left(1+\frac{1}{\delta M}\right) \sum_{m\sim M} \frac{|a_m|^2}{m}. \end{aligned}$$

□

The following simple lemma ([5, Lemma 3] rescaled) is used when we select the integration contour which avoids going too close to zeros of the zeta function.

Lemma 2.6. *Let $\beta > \alpha$ be real numbers. For every set $\{\lambda_1, \dots, \lambda_N\}$ of N real numbers there is a set $\mathcal{S} \subseteq (\alpha, \beta)$ of measure $\geq (\beta-\alpha)/2$ such that*

$$\sum_{j=1}^{N} \frac{1}{|t-\lambda_j|} \ll \frac{N \log N}{\beta - \alpha}$$

for every $t \in \mathcal{S}$.

3. Sums over zeros of the ζ-function

As explained in the introduction, we will need to handle a sum of certain Dirichlet polynomaial over zeros of the zeta function. The following proposition allows us to do this. The proposition is rather tailored to our needs, but similar arguments could be used to show more general results.

Proposition 3.1. *Let $\eta > 0, \beta_0 = 11/14+\eta$ and let $\theta = \theta(\eta)$ be as in Lemma 2.3*(i)*. Let $P_2 = 2^\ell P_1 \geq 2$, $T \geq Q \geq 1$, $R \geq 1$, and $M = RP_2^k$, where k and ℓ are positive integers, be such that, for some $\varepsilon > 0$,*

$$k \ll \frac{(\log T)^{1/3}}{(\log\log T)^{4/3+\varepsilon}}, \quad P_1^{2k} \geq T^{\theta+\varepsilon}, \quad T \geq \max\{M^\varepsilon, P_2\}, \quad \text{and} \quad R \geq (\log T)^{20k}.$$

Then, with

$$M(s) = \sum_{\substack{n_1 \cdots n_k r \sim M \\ P_1 \leq n_j \leq P_2}} \frac{\Lambda(n_1)\cdots\Lambda(n_k)}{(n_1 \cdots n_k r)^s},$$

one has

$$\sum_{\substack{\varrho=\beta+i\gamma \\ \beta\geq\beta_0 \\ Q\leq|\gamma|\leq T}} |M(\varrho)|^2 M^{2(\beta-1)} = o(M(1)^2+1)$$

when $P_1, Q \to \infty$.

Let us first study a somewhat simpler sum. The arguments in the proof of the following lemma go back to works of Wolke [15] and Harman [5] who study similar sums with $k = 1$.

Lemma 3.2. *Let $P, T \geq 1$ and $T \geq T_0 \geq 0$, let $t_0 \in \mathbb{R}, k \in \mathbb{N}, \beta_0 \in [0,1)$, and $P(s) = \sum_{n \sim P} \Lambda(n) n^{-s}$. Then there is an absolute positive constant C such that*

$$\sum_{\substack{\varrho=\beta+i\gamma \\ \beta \geq \beta_0, |\gamma| \leq T \\ |\gamma - t_0| \geq T_0}} |P(\varrho - it_0)|^{2k} P^{2k(\beta-1)}$$
$$\ll C^k \left(\frac{\log T}{(T_0+1)^{2k-1}} + (\log PT(|t_0|+1))^{4k+1} \max_{\beta_0 \leq \sigma \leq 1} N(\sigma, P+T+|t_0|) P^{2k(\sigma-1)} \right).$$

Proof. Let C' be the implied constant in (3). By Lemma 2.2 we have

$$|P(\varrho - it_0)|^{2k} \leq (3C')^{2k} (|P_1(\varrho - it_0)|^{2k} + |P_2(\varrho - it_0)|^{2k} + |P_3(\varrho - it_0)|^{2k}),$$

where $P_j(\varrho)$ are as in that lemma. Hence our claim follows once we have shown that, for $j = 1, 2, 3$, and a certain positive constant C,

$$S_j := \sum_{\substack{\varrho=\beta+i\gamma \\ \beta \geq \beta_0, |\gamma| \leq T \\ |\gamma - t_0| \geq T_0}} |P_j(\varrho - it_0)|^{2k} P^{2k(\beta-1)}$$
$$\ll C^k \left(\frac{\log T}{(T_0+1)^{2k-1}} + (\log PT(|t_0|+1))^{4k+1} \max_{\beta_0 \leq \sigma \leq 1} N(\sigma, P+T+|t_0|) P^{2k(\sigma-1)} \right).$$

By Lemma 2.3(ii)

$$S_1 = \sum_{\substack{\varrho=\beta+i\gamma \\ \beta \geq \beta_0, |\gamma| \leq T \\ |\gamma - t_0| \geq T_0}} \frac{1}{(1+|\gamma - t_0|)^{2k}} \ll \sum_{r \geq T_0+1} \frac{\log T}{r^{2k}} \ll \frac{\log T}{(T_0+1)^{2k-1}}.$$

and

$$S_3 \leq (\log P)^{4k} \sum_{\substack{\varrho=\beta+i\gamma \\ \beta \geq \beta_0, |\gamma| \leq T}} P^{2k(\beta-1)} \ll k(\log P)^{4k+1} \max_{\beta_0 \leq \sigma \leq 1} N(\sigma, T) P^{2k(\sigma-1)}.$$

From now on we can concentrate to the most difficult sum S_2. Expanding out the $2k$th power, we have

$$S_2 = \sum_{\substack{\varrho=\beta+i\gamma \\ \beta \geq \beta_0, |\gamma| \leq T}} \sum_{\substack{\varrho_1, \dots, \varrho_{2k} \\ |\gamma_j - \gamma + t_0| \leq P}} \frac{P^{\beta_1 + \cdots + \beta_{2k} - 2k\beta}}{(1+|\gamma_1 - \gamma + t_0|) \cdots (1+|\gamma_{2k} - \gamma + t_0|)} P^{2k(\beta-1)}.$$

We change the order of summation so that we first sum over the zero with the largest real part. Notice also that the summands are increasing in β_j, so that we can replace all β_j by $\max\{\beta, \beta_1, \dots, \beta_j\}$ in the summand. Since the situation is symmetric in ϱ_j, we get

$$S_2 \leq \sum_{\substack{\varrho=\beta+i\gamma \\ \beta \geq \beta_0, |\gamma| \leq T}} \sum_{\substack{\varrho_1, \dots, \varrho_{2k} \\ \beta_j \leq \beta \\ |\gamma_j - \gamma + t_0| \leq P}} \frac{P^{2k(\beta-1)}}{(1+|\gamma_1 - \gamma + t_0|) \cdots (1+|\gamma_{2k} - \gamma + t_0|)}$$
$$+ 2k \cdot \sum_{\substack{\varrho_1=\beta_1+i\gamma_1 \\ |\gamma_1| \leq T+P+|t_0|}} \sum_{\substack{\varrho=\beta+i\gamma \\ \beta_0 \leq \beta \leq \beta_1 \\ |\gamma - \gamma_1 - t_0| \leq P}} \sum_{\substack{\varrho_2, \dots, \varrho_{2k} \\ \beta_j \leq \beta_1 \\ |\gamma_j - \gamma + t_0| \leq P}} \frac{P^{2k(\beta_1-1)}}{(1+|\gamma_1 - \gamma + t_0|) \cdots (1+|\gamma_{2k} - \gamma + t_0|)}.$$

Performing then all but the outmost sum using Lemma 2.3(ii) and noticing that in the second line $\beta_1 \geq \beta \geq \beta_0$, we obtain, for some large enough constant C,

$$\begin{aligned} S_2 &\ll (C \log^2(P+T+|t_0|))^{2k} \sum_{\substack{\varrho=\beta+i\gamma \\ \beta\geq\beta_0, |\gamma|\leq T+P+|t_0|}} P^{2k(\beta-1)} \\ &\ll (C\log(P+T+|t_0|))^{4k+1} \max_{\beta_0\leq\sigma\leq 1} N(\sigma, P+T+|t_0|)P^{2k(\sigma-1)} \end{aligned}$$

which finishes the proof. □

Proposition 3.1 is a consequence of the lemma but there are several small complications: the variable r, the cross-condition $r \cdot n_1 \cdots n_k \sim M$ and the larger ranges $P_1 \leq p \leq P_2$. These are not difficult to overcome but make the proof quite technical.

Proof of Proposition 3.1. Let us write

$$\sum_{\substack{\varrho=\beta+i\gamma \\ \beta\geq\beta_0 \\ Q\leq|\gamma|\leq T}} |M(\varrho)|^2 M^{2(\beta-1)} = \left(\sum_{\substack{\varrho=\beta+i\gamma \\ \beta\geq\beta_0 \\ Q\leq|\gamma|\leq R}} + \sum_{\substack{\varrho=\beta+i\gamma \\ \beta\geq\beta_0 \\ R<|\gamma|\leq T}} \right) |M(\varrho)|^2 M^{2(\beta-1)} =: Z_1 + Z_2,$$

say. (If $R \geq T$, then let $Z_2 = 0$ and the above holds with $=$ replaced by $\leq$).

Let us first consider Z_1. In this sum the saving will come from the sum of r^ϱ in $M(s)$. Note that in the sum defining $M(s)$, $r \geq M/P_2^k = R$, so that Lemma 2.1 implies that, for $|\gamma| \leq R$,

(4)

$$\begin{aligned} M(\varrho) &= \sum_{\substack{n_1,\ldots,n_k \\ P_1\leq n_j\leq P_2}} \frac{\Lambda(n_1)\cdots\Lambda(n_k)}{(n_1\cdots n_k)^\varrho} \sum_{r\sim M/(n_1\cdots n_k)} \frac{1}{r^\varrho} \\ &= \sum_{\substack{n_1,\ldots,n_k \\ P_1\leq n_j\leq P_2}} \frac{\Lambda(n_1)\cdots\Lambda(n_k)}{(n_1\cdots n_k)^\varrho} \left(\frac{M^{1-\varrho}}{(n_1\cdots n_k)^{1-\varrho}} \cdot \frac{2^{1-\varrho}-1}{1-\varrho} + O\left(\frac{M^{-\beta}}{(n_1\cdots n_k)^{-\beta}}\right)\right) \\ &\ll M(1)\frac{M^{1-\beta}}{1+|\gamma|} + P_2^k M^{-\beta} = M(1)\frac{M^{1-\beta}}{1+|\gamma|} + \frac{M^{1-\beta}}{R}. \end{aligned}$$

Hence

$$\begin{aligned} Z_1 = \sum_{\substack{\varrho=\beta+i\gamma \\ \beta\geq\beta_0 \\ Q\leq|\gamma|\leq R}} |M(\varrho)|^2 M^{2(\beta-1)} &\ll M(1)^2 \sum_{\substack{\varrho=\beta+i\gamma \\ Q\leq|\gamma|\leq R}} \frac{1}{(1+|\gamma|)^2} + R^{-2}N(\beta_0, R) \\ &\ll M(1)^2 Q^{-1/2} + R^{-1/2} = o(M(1)^2+1) \end{aligned}$$

by Lemma 2.3(ii) and since $Q, R \to \infty$.

Now we can concentrate to bounding Z_2. Here the saving will eventually come from Lemma 3.2.

Let us first reduce the variables n_j to dyadic intervals: Let us split the Dirichlet polynomial $M(s)$ dyadically into ℓ^k Dirichlet polynomials of the shape

$$M_{\mathbf{Q}}(s) = \sum_{\substack{n_1\cdots n_k r\sim M \\ n_j\sim Q_j}} \frac{\Lambda(n_1)\cdots\Lambda(n_k)}{(n_1\cdots n_k r)^s}.$$

Notice that

$$Z_2 = \sum_{\substack{\varrho=\beta+i\gamma \\ \beta\geq\beta_0 \\ R<|\gamma|\leq T}} |M(\varrho)|^2 M^{2(\beta-1)} \ll \ell^{2k} \max_{\substack{\mathbf{Q} \\ Q_j\in[P_1,P_2]}} \sum_{\substack{\varrho=\beta+i\gamma \\ \beta\geq\beta_0 \\ R<|\gamma|\leq T}} |M_{\mathbf{Q}}(\varrho)|^2 M^{2(\beta-1)}. \tag{5}$$

From now on we let $\mathbf{Q}$ be that reaching the maximum.

Our next task is to dispose of the cross-condition $(n_1\cdots n_k)\cdot r \sim M$ in the definition of $M_{\mathbf{Q}}(s)$. For this we use a variant of Perron's formula. Write $U = Q_1\cdots Q_k$,

$$U(s) := \sum_{\substack{n_1,\ldots,n_k \\ n_j\sim Q_j}} \frac{\Lambda(n_1)\cdots\Lambda(n_k)}{(n_1\cdots n_k)^s} =: \prod_{j=1}^k Q_j(s), \quad \text{and} \quad R(s) := \sum_{M/(2^k U)\leq r\leq 2M/U} \frac{1}{r^s}.$$

We may assume that $M-1/4$ is an integer. Then, by Perron's formula in form [8, Lemma 2.2], we have, for any $\varepsilon>0$,

$$\begin{aligned} M_{\mathbf{Q}}(\varrho) &= \frac{1}{\pi}\int_{-T}^{T} U(\varrho+it)R(\varrho+it)\frac{\sin(t\log(2M))-\sin(t\log M)}{t}ds + O\left(\frac{M^{1-\beta+\varepsilon}}{T}\right) \\ &=: \int_{-T}^{T} F(\varrho,t)dt + O\left(\frac{M^{1-\beta+\varepsilon}}{T}\right), \end{aligned}$$

say. Here

$$\begin{aligned} F(\varrho,t) &\ll |U(\varrho+it)||R(\varrho+it)|\min\left\{1,\frac{1}{|t|}\right\} \\ &\ll |U(\varrho+it)|k\left(\frac{M}{U}\right)^{1-\beta}\min\left\{1,\frac{1}{|t|}\right\} \ll kM^{1-\beta}\min\left\{1,\frac{1}{|t|}\right\} \end{aligned} \tag{6}$$

We want eventually, after an application of Hölder's inequality, to apply Lemma 3.2 to $\sum_\varrho |Q_j(\varrho+it)|^{2k}Q_j^{2k(\beta-1)}$ but this can be done only when $\gamma+t$ is not too small. However, the complementary region will be easy to handle (a similar argument is used for instance in [8, Proof of Lemma 7.5]). To do this, we write

$$\int_{-T}^{T} F(\varrho,t)dt = \int_{\substack{t\in[-T,T] \\ |t+\gamma|>R^{1/5}}} F(\varrho,t)dt + \int_{\substack{t\in[-T,T] \\ |t+\gamma|\leq R^{1/5}}} F(\varrho,t)dr =: J_1(\varrho)+J_2(\varrho),$$

say. We have

(7)

$$\begin{aligned} \sum_{\substack{\varrho=\beta+i\gamma \\ \beta\geq\beta_0 \\ R\leq|\gamma|\leq T}} M^{2(\beta-1)}|M_{\mathbf{Q}}(\varrho)|^2 &\ll \sum_{\substack{\varrho=\beta+i\gamma \\ \beta\geq\beta_0 \\ R\leq|\gamma|\leq T}} M^{2(\beta-1)}\left(|J_1(\varrho)|^2+|J_2(\varrho)|^2+O\left(\frac{M^{2-2\beta+\varepsilon}}{T^2}\right)\right) \\ &=: E_1+E_2+O\left(\frac{M^\varepsilon\log T}{T}\right), \end{aligned}$$

say.

Now using (6)

$$
\begin{aligned}
E_2 &\ll \sum_{\substack{\varrho=\beta+i\gamma\\ \beta\geq\beta_0\\ R\leq|\gamma|\leq T}} M^{2(\beta-1)}\left(\int_{-\gamma-R^{1/5}}^{-\gamma+R^{1/5}}|F(\varrho,t)|dt\right)^2\\
&\ll \sum_{\substack{\varrho=\beta+i\gamma\\ \beta\geq\beta_0\\ R\leq|\gamma|\leq T}} M^{2(\beta-1)}\left(R^{1/5}\cdot k\frac{M^{1-\beta}}{|\gamma|}\right)^2 \ll R^{-1/3}
\end{aligned}
$$

by Lemma 2.3(ii).

On the other hand changing the order of summation and integration and applying (6) we get

$$
\begin{aligned}
E_1 &\ll \int_{-T}^{T}\int_{-T}^{T} \sum_{\substack{\varrho=\beta+i\gamma\\ \beta\geq\beta_0\\ R\leq|\gamma|\leq T, |\gamma+t_j|>R^{1/5}}} M^{2(\beta-1)}F(\varrho,t_1)\overline{F(\varrho,t_2)}dt_1dt_2\\
&\ll (\log T)^2k^2 \max_{|t_1|,|t_2|\leq T} \sum_{\substack{\varrho=\beta+i\gamma\\ \beta\geq\beta_0\\ R\leq|\gamma|\leq T, |\gamma+t_j|>R^{1/5}}} |U(\varrho+it_1)||U(\varrho+it_2)|U^{2(\beta-1)}\\
&\ll (\log T)^2k^2 \max_{|t|\leq T} \sum_{\substack{\varrho=\beta+i\gamma\\ \beta\geq\beta_0\\ R\leq|\gamma|\leq T, |\gamma+t|>R^{1/5}}} |U(\varrho+it)|^2U^{2(\beta-1)}.
\end{aligned}
$$

By Hölder's inequality and Lemma 3.2

$$
\begin{aligned}
E_1 &\ll (\log T)^2k^2 \max_{|t|\leq T}\prod_{j=1}^{k}\left(\sum_{\substack{\varrho=\beta+i\gamma\\ \beta\geq\beta_0\\ R\leq|\gamma|\leq T, |\gamma+t|>R^{1/5}}} |Q_j(\varrho+it)|^{2k}Q_j^{2k(\beta-1)}\right)^{1/k}\\
&\ll (\log T)^2C^k\prod_{j=1}^{k}\left(\frac{\log T}{R^{1/5}}+(\log T)^{4k+1}\max_{\beta_0\leq\sigma\leq1}N(\sigma,3T)Q_j^{2k(\sigma-1)}\right)^{1/k}\\
&\ll (\log T)^2C^k\left(\frac{\log T}{R^{1/5}}+(\log T)^{4k+1}\max_{\beta_0\leq\sigma\leq1}N(\sigma,3T)P_1^{2k(\sigma-1)}\right).
\end{aligned}
$$

Hence recalling (5) and (7) and Lemma 2.3(i) and (iii)

$$
\begin{aligned}
Z_2 &\ll \frac{(\log T)^{2k+3}C^k}{\min\{R^{1/5},M^{\varepsilon}\}}+(\log T)^{6k+3}C^k\max_{\beta_0\leq\sigma\leq1}N(\sigma,3T)P_1^{2k(\sigma-1)}\\
&\ll o(1)+(C\log T)^{6k+18}\max_{\beta_0\leq\sigma\leq1-c(\log T)^{-2/3}(\log\log T)^{-1/3}}\left(\frac{P_1^{2k}}{T^{\theta}}\right)^{\sigma-1}\\
&\ll o(1)+\exp\left((6k+18)(\log\log T+\log C)-c\frac{(\log T)^{1/3}}{(\log\log T)^{1/3}}\right),
\end{aligned}
$$

and the claim follows from the upper bound for k.

□

4. Proof of Theorem 1.1

Let us first fix some notation. Let x, y, z, u and ε be as in Theorem 1.1, and define δ by

$$xe^{2\delta} = x + z, \quad \text{so that } \delta \sim \frac{z}{2x}.$$

Let η, η' be small positive constants and let $\theta = \theta(\eta)$ be as in Lemma 2.3(i). Let

$$\begin{aligned}
&L = \log x, \quad L_2 = \log\log x, \quad \lambda = \frac{3}{14} - \eta, \quad k = \left\lceil \max\left\{u, \frac{2}{2-\theta}\right\}\right\rceil, \\
&T = \frac{x^{\eta'}}{\delta} \asymp \frac{x^{1+\eta'}}{z} \quad Q = \frac{z^2}{x}, \quad M = \left(\frac{x}{Q}\right)^{1/2}, \\
&P_1 = M^{1/k-1/(2k^2)}, \quad P_2 = 2^\ell P_1 \sim M^{1/k-1/(4k^2)} \quad \text{where } \ell = \left\lfloor \frac{\log M}{4k^2 \log 2}\right\rfloor, \\
&\text{and} \quad M(s) = \sum_{\substack{n_1\cdots n_k r \sim M \\ P_1 \le n_j \le P_2}} \frac{\Lambda(n_1)\cdots\Lambda(n_k)}{(n_1\cdots n_k r)^s}.
\end{aligned} \tag{8}$$

Let

$$S = \sum_{\substack{x \le n := n_1 \cdots n_{2k} r_1 r_2 q \le xe^{2\delta} \\ P_1 \le n_j \le P_2 \\ n_1 \cdots n_k r_1 \sim M \\ n_{k+1}\cdots n_{2k} r_2 \sim M}} \Lambda(n_1)\cdots\Lambda(n_{2k})\Lambda(q) \min\left\{\log \frac{e^{2\delta}x}{n}, \log\frac{n}{x}\right\}. \tag{9}$$

Notice that $n_j \le P_2 \le (x/Q)^{1/u} \le y$, $r_j \le 2M/P_1^k = 2M^{1/2k} \le y$ and $q \le (x+z)/M^2 \le 2Q \le y$, so only y-smooth numbers are counted in S. We have chosen S so that, after an application of Perron formula and moving the integration region, we can apply Proposition 3.1 to the resulting sum over zeros of the zeta function. Somewhat similar sums have been considered in earlier works but there the sum has been chosen so that one gets good estimates for the corresponding Dirichlet polynomials in the zero-free region of the Riemann zeta function.

Notice that, writing $\tau(n)$ for the number of divisors of n and $\Omega(n)$ for the number of prime divisors of n, each number n is counted in S with weight at most

$$\Omega(n)^{2k+1}\tau(n)^2 \log e^\delta \ll (\log 2x)^{2k+1} x^{\varepsilon/2}\delta \ll \delta x^\varepsilon.$$

Hence

$$\Psi(x+z, y) - \Psi(x, y) \gg x^{-\varepsilon}\delta^{-1} S. \tag{10}$$

It would be possible to get a better lower bound by arguing more carefully, using averaging, but our method would not yield a correct order of magnitude lower bound.

The log-weights in S are chosen so that, similarly to Soundararajan's conditional work [14] (where different $M(s)$ is used), the variant

$$\frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} \frac{x^s}{s^2} = \begin{cases} \log x & \text{if } x \ge 1; \\ 0 & \text{otherwise.}\end{cases}$$

of Perron's formula gives

$$S = \frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} -\frac{\zeta'}{\zeta}(s)M(s)^2 x^s \frac{(e^{\delta s}-1)^2}{s^2}ds. \tag{11}$$

We use this instead of the unweighted form to save a logarithmic factor — there are points where we need to integrate the denominator $1/|s|^2$ and integrating $1/|s|$ would lose a logarithm (another, almost equivalent, way to save this logarithm is to slightly average over z as is done for instance in [6]).

We wish to move the integration path to the left, but not too much so that good zero density estimates are available. Furthermore we want to avoid the contour going too close to the zeros of the zeta function. To find suitable contour we argue similarly to [5, Section 3] but for completeness we give the construction here.

In the region $|\Im s| \le T$ the contour will be a union of horizontal and vertical line segments inside the rectangle $[1-\lambda, 1-\lambda+L_2^{-1}] \times [-iT, iT]$. Let $u \in [-T, T]$ and let us first show how to find an appropriate line segment $\sigma_0 + [iu, i(u+1)]$ with $\sigma_0 \in [1-\lambda, 1-\lambda+L_2^{-1}]$. Write $\mathcal{Z}$ for the set of zeros $\varrho = \beta + i\gamma$ of the zeta function in the rectangle $[0,1] \times [i(u-1), i(u+2)]$. Note that by Lemma 2.3(ii) we know $|\mathcal{Z}| \ll L$, so that, by Lemma 2.6, there is a set $E_0 \subset [1-\lambda, 1-\lambda+L_2^{-1}]$ of measure at least $L_2^{-1}/2$ such that

$$\sum_{\beta+i\gamma\in\mathcal{Z}} \frac{1}{|\sigma-\beta|} \ll LL_2^2 \tag{12}$$

for every $\sigma \in E_0$.

Since E_0 has measure at least $L_2^{-1}/2$, there exists $\sigma_0 \in E_0$ such that

$$\begin{aligned}\int_{\sigma_0+iu}^{\sigma_0+i(u+1)} \left|M(s)^2 x^s \frac{(e^{\delta s}-1)^2}{s^2}\right| |ds| &\le 2L_2 \int_{E_0}\int_{\sigma+iu}^{\sigma+i(u+1)} \left|M(s)^2 x^s \frac{(e^{\delta s}-1)^2}{s^2}\right| |ds| d\sigma \\ &\ll L_2 \int_{1-\lambda}^{1-\lambda+L_2^{-1}} \int_{\sigma+iu}^{\sigma+i(u+1)} \left|M(s)^2 x^s \frac{(e^{\delta s}-1)^2}{s^2}\right| |ds| d\sigma.\end{aligned}$$

Note that by Lemma 2.4 and (12), for every $s \in \sigma_0 + [iu, i(u+1)]$,

$$\frac{\zeta'}{\zeta}(s) \ll \sum_{\beta+i\gamma\in\mathcal{Z}} \frac{1}{|s-(\beta+i\gamma)|} + L \ll LL_2^2.$$

One can argue similarly to show that, for any given $u \in [-T, T]$, there exists $t_0 \in [u, u+L_2^{-1}]$ such that the line segment $[1-\lambda, 1-\lambda+L_2^{-1}] + it_0$ has the same properties. Combining these line segments we find that there is a countour $\mathcal{C}$ consisting of

- sublines of $[\sigma + iu, \sigma + i(u+1)]$ with $\sigma \in [1-\lambda, 1-\lambda+L_2^{-1}]$
- sublines of $[1-\lambda+iu, 1-\lambda+L_2^{-1}+iu]$ with $u \in [-T, T]$

such that, with $s = \sigma + it$,

$$\int_{\mathcal{C}} \left|M(s)^2 x^s \frac{(e^{\delta s}-1)^2}{s^2}\right| |ds| \ll L_2 \int_{1-\lambda}^{1-\lambda+L_2^{-1}} \int_{-T}^{T} |M(s)|^2 x^\sigma \min\left\{\delta^2, \frac{1}{|s|^2}\right\} dt d\sigma \tag{13}$$

and

$$\frac{\zeta'}{\zeta}(s) \ll LL_2^2 \quad \text{for every } s \in \mathcal{C} \cup [1-\lambda \pm iT, 1 \pm iT]. \tag{14}$$

(here we have changed T by $O(1)$ if necessary to avoid zeros of zeta-function too close to $[1-\lambda \pm iT, 1 \pm iT]$).

Moving the contour in (11) to $\mathcal{C} \cup [1-\lambda' \pm iT, 1 \pm iT] \cup [1 \pm iT, 1 \pm i\infty]$ and picking up the poles we find that

$$\begin{aligned} S &= x(e^\delta - 1)^2 M(1)^2 + \sum_\varrho{}' M(\varrho)^2 x^\varrho \left(\frac{e^{\delta\varrho}-1}{\varrho}\right)^2 \\ &\quad + O\left(\left(\int_{\mathcal{C}} + \int_{1-\lambda+iT}^{1+iT} + \int_{1+iT}^{1+i\infty}\right) \left|\frac{\zeta'}{\zeta}(s) M(s)^2 x^s \frac{(e^{\delta s}-1)^2}{s^2}\right| |ds|\right) \\ &=: x(e^\delta-1)^2 M(1)^2 + Z + O(I_{\mathcal{C}} + I_T + I_{>T}), \end{aligned} \tag{15}$$

say, where $\sum'$ means that the sum is over those zeros of the zeta-function that are to the right of the contour $\mathcal{C}$.

Let us first figure out the size of the main term $x(e^\delta-1)^2M(1)^2$. Since $P_2^k \leq M$,

(16)

$$\begin{aligned} M(1) &= \sum_{\substack{n_1,\dots,n_k \\ P_1 \leq n_j \leq P_2}} \frac{\Lambda(n_1)\cdots\Lambda(n_k)}{n_1\cdots n_k} \sum_{r \sim M/(n_1\cdots n_k)} \frac{1}{r} \asymp \sum_{\substack{n_1,\dots,n_k \\ P_1 \leq n_j \leq P_2}} \frac{\Lambda(n_1)\cdots\Lambda(n_k)}{n_1\cdots n_k} \\ &= \left(\log\frac{P_2}{P_1} + O(1)\right)^k = \left(\frac{\log M}{4k^2} + O(1)\right)^k = (1+o(1))\frac{(\log P_2)^k}{(4k-1)^k}, \end{aligned}$$

so that the main term in (15) is

$$x(e^\delta-1)^2M(1)^2 \gg \delta^2 x M(1)^2 \gg \delta^2 x \frac{(\log P_2)^{2k}}{(4k)^{2k}},$$

Let us now turn to error terms coming from the new integrals. The main tools for estimating these are (13)–(14) and Lemma 2.5. Using (14) we get

$$\begin{aligned} I_T &\ll \int_{1-\lambda}^1 \left|\frac{\zeta'}{\zeta}(s)M(s)^2\right| \frac{x^\sigma}{T^2} d\sigma \ll \frac{LL_2^2}{T^2}\int_{1-\lambda}^1 x^\sigma M(\sigma)^2 d\sigma \\ &\ll \frac{xM(1)^2 LL_2^2}{T^2} \int_{1-\lambda}^1 \left(\frac{x}{M^2}\right)^{\sigma-1} d\sigma \ll \delta^2 x M(1)^2 \cdot x^{-\eta'}. \end{aligned}$$

On the other hand, by (13) and (14) we have

$$I_{\mathcal{C}} \ll LL_2^3 \int_{1-\lambda}^{1-\lambda+L_2^{-1}} \int_{-T}^T |M(s)|^2 x^\sigma \min\left\{\delta^2, \frac{1}{|s|^2}\right\}.dtd\sigma. \tag{17}$$

Applying Lemma 2.5 we find that the inner integral is at most of order

$$\delta^2 x \cdot \left(\frac{x}{M^2}\right)^{\sigma-1}\left(1 + \frac{1}{\delta M}\right) M(1) \max_{m\sim M}\left(\sum_{\substack{n_1\cdots n_k r = m \\ P_1 \leq n_j \leq P_2}} \Lambda(n_1)\cdots\Lambda(n_k)\right).$$

Since $\frac{\log 2M}{\log P_1} < k+1$, every $m \sim M$ has at most k prime power factors from the interval $[P_1, P_2]$. These can be ordered to be $n_1, \dots, n_k$ in at most $k!$ ways and hence the maximum above is at most $k! \cdot (\log P_2)^k$. Recalling also that $M = (x/Q)^{1/2} = x/z \asymp 1/\delta$ we see that the inner integral in (17) is at most

$$\delta^2 x \cdot Q^{\sigma-1} M(1) k! (\log P_2)^k \ll \delta^2 x M(1)^2 Q^{\sigma-1}(4k)^{2k}$$

since by (16) $(\log P_2)^k \ll (4k)^k M(1)$. Hence

$$I_{\mathcal{C}} \ll \delta^2 x M(1)^2 \cdot L L_2^2 (4k)^{2k} Q^{-\lambda + L_2^{-1}}.$$

Recall that $Q = z^2/x = \exp((\frac{14}{3} + 2\varepsilon)(\log\log x + 4u\log u))$, so that

$$Q^{-\lambda+L_2^{-1}} \le Q^{-3/14+2\eta} \le \exp(-(1+\varepsilon)(\log\log x + 4u\log u)) = o(((4k)^{2k} L L_2^3)^{-1}).$$

when η is chosen small enough compared to ε. Hence $I_{\mathcal{C}} = o(\delta^2 M(1)^2 x)$.

By Lemmas 2.4 and 2.3(ii)–(iii) $\frac{\zeta'}{\zeta}(1+it) \ll \log^2 t$, so that

$$\begin{aligned} I_{>T} &\ll L^2 \int_{1+iT}^{1+iT^{10}} \left| \frac{M(s)^2 x^s}{s^2} \right| |ds| + \int_{1+iT^{10}}^{1+i\infty} \left| \frac{M(s)^2 x^s}{s^2} \log^2 |s| \right| |ds| \\ &\ll L^2 \int_{1-iT^{10}}^{1+iT^{10}} |M(s)|^2 x^\sigma \min\{T^{-2}, |s|^{-2}\} |ds| + M(1)^2 x \int_{1+iT^{10}}^{1+i\infty} \frac{\log^2 |s|}{|s|^2} |ds|. \end{aligned}$$

Applying Lemma 2.5 to the first integral we get as for $I_{\mathcal{C}}$

$$\begin{aligned} I_{>T} &\ll L^2 T^{-2} x \left(1 + \frac{T}{M}\right) M(1) k! (\log P_2)^k + \frac{M(1)^2 x}{T^4} \\ &\ll \delta^2 M(1)^2 x (4k)^{2k} x^{-\eta'} + \delta^4 M(1)^2 x = o(\delta^2 M(1)^2 x) \end{aligned}$$

by (16) and since $M \asymp 1/\delta$ and $T = x^{\eta'}/\delta$. Combining the estimates

$$I_{\mathcal{C}} + I_T + I_{>T} = o(\delta^2 x M(1)^2), \tag{18}$$

so we can concentrate on Z in (15). We have

(19)

$$\begin{aligned} Z = \sum_{\varrho}{}' M(\varrho)^2 x^\varrho \left(\frac{e^{\delta\varrho} - 1}{\varrho}\right)^2 &\ll \delta^2 x \left(\sum_{\substack{\varrho \\ |\gamma| \le Q}}{}' x^{\beta-1} |M(\varrho)|^2 + \sum_{\substack{\varrho \\ |\gamma| > Q}}{}' x^{\beta-1} |M(\varrho)|^2 \right) \\ &=: \delta^2 x (Z_1 + Z_2), \end{aligned}$$

say. Here, recalling that $x = QM^2$, and using the estimate (4),

$$Z_1 \ll M(1)^2 \sum_{\substack{\varrho = \beta + i\gamma \\ |\gamma| \le Q}}{}' \frac{Q^{\beta-1}}{(1+|\gamma|)^2} \ll M(1)^2 \exp(-(\log Q)^{1/4}). \tag{20}$$

by Lemma 2.3(ii)–(iii).

Notice that

$$P_1^{2k} = x^{o(1)+1-1/(2k)} \ge x^{o(1)+1-(2-\theta)/4} = x^{o(1)+\theta/2+(2-\theta)/4} \ge T^{\theta+\varepsilon'}$$

for some $\varepsilon' > 0$ since $\theta < 2$, so Proposition 3.1 gives $Z_2 = o(M(1)^2 + 1)$. Hence $Z = o(\delta^2 x M(1)^2)$. Combining with (18) and (15) we see that $S \gg \delta^2 x M(1)^2 \gg \delta z$ and the claim follows immediately from (10). $\square$

5. Proof of Theorem 1.3

Here we sketch the proof of Theorem 1.3 which is a variant of the proof of Theorem 1.1. We need to choose some parameters differently and start with fixing the notation. Let x, y, z, u and ε be as in Theorem 1.3, and define δ by

$$Xe^{\delta} = X + z, \quad \text{so that } \delta \sim \frac{z}{X}.$$

Let η, η' be small positive constants, let

$$L = \log X, \quad L_2 = \log\log X, \quad T = \frac{X^{\eta'}}{\delta} \asymp \frac{X^{1+\eta'}}{z}, \quad Q = z, \quad M = \frac{X}{Q},$$

and let $\lambda, \theta, k, P_1, P_2, \ell$ and $M(s)$ be as in (8).

Let us study

$$S(x) = \sum_{\substack{x \leq n_1 \cdots n_k r q \leq x e^{\delta} \\ P_1 \leq n_j \leq P_2 \\ n_1 \cdots n_k r \sim M}} \Lambda(n_1) \cdots \Lambda(n_k) \Lambda(q)$$

for $x \sim X$. Notice that as before $n_j, r \leq y$ and also $q \leq (2X + z)/M \leq 3Q \leq y$, so only y-smooth numbers are counted in S. Hence it is enough to show that $S(x) > 0$ for almost all $x \sim X$. By Perron's formula

$$S(x) = \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} -\frac{\zeta'}{\zeta}(s) M(s) x^s \frac{e^{\delta s} - 1}{s} ds.$$

We move the contour as in the proof of Theorem 1.1, getting

$$\begin{aligned} S(x) &= x(e^{\delta} - 1)M(1) + \sideset{}{'}\sum_{\varrho} M(\varrho) x^{\varrho} \frac{e^{\delta \varrho} - 1}{\varrho} \\ &\quad + \frac{1}{2\pi i} \left(\int_{\mathcal{C}} + \int_{1-\lambda+iT}^{1+iT} + \int_{1+iT}^{1+i\infty} \right) - \frac{\zeta'}{\zeta}(s) M(s) x^s \frac{e^{\delta s} - 1}{s} ds \\ &=: x(e^{\delta} - 1)M(1) + Z(x) + I_{\mathcal{C}}(x) + I_T(x) + I_{>T}(x), \end{aligned}$$

say, where $\sum'$ means that the sum is over those zeros of the zeta-function that are to the right of the contour $\mathcal{C}$. We want to show that, for almost all $x \sim X$, we have

$$Z(x) + I_{\mathcal{C}}(x) + I_T(x) + I_{>T}(x) = o(\delta x M(1)).$$

The contribution from the zeros $\varrho = \beta + i\gamma$ with $|\gamma| \leq Q$ to $Z(x)$ is $o(\delta x M(1))$ as in (20). Write $Z_2(x)$ for the contribution of rest of the zeros. Changing the order of summation and integration we see that

$$\begin{aligned} \int_{X/2}^{2X} \int_x^{2x} |Z_2(h)|^2 dh dx &= \int_{X/2}^{2X} \int_x^{2x} \Bigg| \sideset{}{'}\sum_{\substack{\varrho \\ |\gamma| > Q}} M(\varrho) h^{\varrho} \frac{e^{\delta \varrho} - 1}{\varrho} \Bigg|^2 dh dx \\ &= \sideset{}{'}\sum_{\substack{\varrho_1 \\ |\gamma_1| > Q}} \sideset{}{'}\sum_{\substack{\varrho_2 \\ |\gamma_2| > Q}} \left| M(\varrho_1) M(\varrho_2) \frac{e^{\delta \varrho_1} - 1}{\varrho_1} \frac{e^{\delta \varrho_2} - 1}{\varrho_2} \right| \left| \int_{X/2}^{2X} \int_x^{2x} h^{\varrho_1 + \overline{\varrho_2}} dh dx \right|. \end{aligned} \tag{21}$$

Integrating over h and x and using the inequality $|ab| \leq |a|^2 + |b|^2$ we get

$$\begin{aligned}\int_{X/2}^{2X}\int_{x}^{2x} |Z_2(h)|^2 dh dx &\ll \delta^2 \sideset{}{'}\sum_{\substack{\varrho_1 \\ |\gamma_1|>Q}} \sideset{}{'}\sum_{\substack{\varrho_2 \\ |\gamma_2|>Q}} |M(\varrho_1)M(\varrho_2)| \frac{X^{\beta_1+\beta_2+2}}{(1+|\gamma_1-\gamma_2|)^2} dx \\ &\ll \delta^2 X^2 \sideset{}{'}\sum_{\substack{\varrho_1 \\ |\gamma_1|>Q}} \sideset{}{'}\sum_{\substack{\varrho_2 \\ |\gamma_2|>Q}} \frac{|M(\varrho_1)|^2 X^{2\beta_1} + |M(\varrho_2)|^2 X^{2\beta_2}}{(1+|\gamma_1-\gamma_2|)^2} \\ &\ll \delta^2 X^4 \sideset{}{'}\sum_{\substack{\varrho \\ |\gamma|>Q}} |M(\varrho)|^2 M^{2(\beta-1)}\end{aligned} \tag{22}$$

by Lemma 2.3(ii). Recalling that the summation in $\sum'$ is only over zeros with real part $> 11/14 + \eta$, we see by Proposition 3.1 that

$$\int_{X/2}^{2X}\int_{x}^{2x} |Z_2(h)|^2 dh dx = o(X^2 \cdot (\delta M(1) X)^2).$$

Hence $Z(x) = o(\delta M(1) x)$ for almost all $x \sim X$.

As in the proof of Theorem 1.1, $I_T(x) = o(\delta x M(1))$. Furthermore recalling (14) and arguing analogously to (21)– (22)

$$\begin{aligned}\int_{X/2}^{2X}\int_{x}^{2x} |I_{\mathcal{C}}(x)|^2 dh dx &\ll L^2 L_2^2 X^2 \int_{\mathcal{C}} |M(s)|^2 X^{2\sigma} \left| \frac{e^{\delta s}-1}{s} \right|^2 ds \\ &\ll L^2 L_2^3 X^2 \int_{1-\lambda}^{1-\lambda+L_2^{-1}} \int_{-T}^{T} |M(s)|^2 X^{2\sigma} \min\left\{\delta^2, \frac{1}{|s|^2}\right\} dt d\sigma\end{aligned}$$

by (13). Comparing with (17) and the arguments following it we see that

$$\int_{X}^{2X}\int_{x}^{2x} |I_{\mathcal{C}}(x)|^2 dh dx \ll L^2 L_2^2 X^2 \delta^2 X^2 M(1)^2 (4k)^{2k} Q^{-2\lambda+2L_2^{-1}} = o(X^2 \cdot (\delta M(1) X)^2)$$

since

$$Q^{-2\lambda+2L_2^{-1}} \leq \exp(-2(3/14-2\eta)(14/3+\varepsilon)(\log\log x + 4u\log u)) = o(((4k-1)^{2k+1} L^2 L_2^3)^{-1})$$

when η is chosen small enough compared to ε. Hence $I_{\mathcal{C}}(x) = o(\delta M(1) x)$ for almost all $x \sim X$. Similarly $I_{>T}(x) = o(\delta M(1) x)$ for almost all $x \sim X$ and the claim follows. □

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