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ON THE DISTRIBUTION OF $\mathfrak{B}$-FREE NUMBERS AND NON-VANISHING FOURIER COEFFICIENTS OF CUSP FORMS

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Abstract. We study properties of $\mathfrak{B}$-free numbers, that is numbers that are not divisible by any member of a set $\mathfrak{B}$. First we formulate the most-used procedure for finding them (in a given set of integers) as easy-to-apply propositions. Then we use the propositions to consider Diophantine properties of $\mathfrak{B}$-free numbers and their distribution on almost all short intervals. Results on $\mathfrak{B}$-free numbers have implications to non-vanishing Fourier coefficients of cusp forms, so this work also gives information about them.

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1. Introduction. Let $\mathfrak{B} = \{b_i\}_{i\in\mathbb{N}}$ be a set of integers greater than 1 such that

$$\sum_{b\in\mathfrak{B}} \frac{1}{b} < \infty \quad \text{and} \quad \gcd(b_i, b_j) = 1 \text{ whenever } i \neq j. \tag{1}$$

We say that a natural number n is $\mathfrak{B}$-free if it is not divisible by any element of the set $\mathfrak{B}$. The notion of $\mathfrak{B}$-free numbers was first introduced by Erdős **[9]** as a generalisation of square-free integers.

In this paper, we will prove that certain natural subsets of integers contain $\mathfrak{B}$-free numbers. Compared to the most-studied number theoretic sequence, the prime numbers, $\mathfrak{B}$-free numbers are much easier to handle. This is to be expected since the set of $\mathfrak{B}$-free numbers has positive density

$$\prod_{b\in\mathfrak{B}} \left(1 - \frac{1}{b}\right).$$

Besides having independent interest, results on $\mathfrak{B}$-free numbers have implications to non-vanishing Fourier coefficients of cusp forms as was first noticed by Balog and Ono **[7]**. We will discuss this application in Section 2.

We will formulate the usual procedure for finding $\mathfrak{B}$-free numbers in a given set $\mathcal{A} \subset \mathbb{N} \cap [x, 2x]$ as easy-to-apply propositions in Section 4. The propositions demonstrate that if we have enough so-called type I information, we can find $\mathfrak{B}$-free numbers. By saying that we have type I information up to D, we mean that, for any bounded complex coefficients a_d, we can give an asymptotic formula for the so-called type I sum

$$\sum_{\substack{dm \in \mathcal{A} \\ d \le D}} a_d = \sum_{d \le D} a_d |\mathcal{A}_d|,$$

where

$$\mathcal{A}_d = \{a \in \mathcal{A} \mid a \equiv 0 \,(\mathrm{mod}\, d)\} \tag{2}$$

and $|\mathcal{C}|$ denotes the cardinality of the set $\mathcal{C}$.

More precisely Proposition 7 below shows that we can find $\mathfrak{B}$-free numbers in a dense enough set $\mathcal{A}$ if we have type I information up to $D = x^{1/2+\varepsilon}$. We will actually prove a more complicated-looking generalisation, Proposition 8 below. It can often save the day when we cannot quite obtain this good type I information for arbitrary coefficients a_d.

The difference to the case of primes is that type I information is not sufficient for detecting primes in the set $\mathcal{A}$, but one needs some additional information about $\mathcal{A}$. Often this additional information is in the form of so-called type II information on certain ranges $[M_1, M_2]$, which means information about asymptotic behaviour of the so-called type II sums

$$\sum_{\substack{mn \in \mathcal{A} \\ M_1 < m < M_2}} a_m b_n,$$

where a_m and b_n are bounded complex coefficients. Vinogradov was the first one to split a sum over primes into type I and type II sums, and since then this splitting has been used in numerous applications. In our case, we will be able split a (weighted) sum over $\mathfrak{B}$-free integers into type I sums only.

As a first illustrative application, we prove in Section 5 the following result on Diophantine properties of $\mathfrak{B}$-free numbers.

THEOREM 1. *Let α be irrational and $\theta < 1/2$. Then, for any $\beta \in \mathbb{R}$, there are infinitely many $\mathfrak{B}$-free numbers n such that*

$$\|\alpha n + \beta\| < n^{-\theta}.$$

Here $\|x\|$ denotes the distance from x to the nearest integer.

This improves a result of Alkan, Harman and Zaharescu [**4**] who showed that the theorem holds for any $\theta < 1/3$. They contended themselves with the exponent $1/3$ since type II information disappears there. However, as mentioned, type I information is enough for finding $\mathfrak{B}$-free numbers. On the other hand, for primes the best known result is that, for any $\varepsilon > 0$, $\|\alpha p\| < p^{-1/3+\varepsilon}$ for infinitely many primes p (see [**14**]) exactly due to the break in type II information.

Gaps between $\mathfrak{B}$-free numbers have enjoyed some recent interest. The latest results can be found in the paper [**13**] by Kowalski, Robert and Wu. In particular, they showed that the gap between consecutive $\mathfrak{B}$-free numbers of size x is $O(x^{7/17})$. Like its predecessors, this result depends on good estimates for exponential sums over monomials.

In this paper, we do not attack individual gaps but instead study the gaps on average. For this we write

$$E(x,h) = |\{n \sim x \mid [n, n+h] \text{ does not contain any } \mathfrak{B}\text{-free numbers}\}|,$$

where $n \sim x$ means $n \in \mathbb{N} \cap [x, 2x)$. Plaksin [**16**] has proved that

$$E(x,h) \ll xh^{-1} \log x \quad \text{for } h \le x^{1/13-\varepsilon} \tag{3}$$

and

$$E(x,h) \ll xh^{-1/3} \quad \text{for } h \ge x^{\varepsilon}. \tag{4}$$

Here and later ε denotes an arbitrarily small positive constant. Choosing the parameters in the argument leading to (4) more carefully, one would actually obtain

$$E(x,h) \ll \frac{x^{1+\varepsilon}}{h} + x^{2/3+\varepsilon} h^{1/3}.$$

Here we will prove the following estimates for $E(x, h)$.

THEOREM 2. (i) *Let $\delta > 0$ and $h \le x^{1/6-\delta}$. Then, for any $\theta < 1$,*

$$E(x,h) \ll_{\delta,\theta} x/h^{\theta}.$$

(ii) *When $\mathfrak{B} \subset \mathbb{P}$, this holds for any $h \ge 1$.*

The main point of part (i) is the removal of the logarithm in (3), so that unlike Plaksin's results, this theorem gives non-trivial information even in very short intervals. Qualitative results of this sort have been obtained by Alkan in [**1**] and [**2**].

Again, the results for $\mathfrak{B}$-free numbers are much stronger than what is known for primes for which Peck [**15**] has proved that the exceptional set has size $x^{5/4+\varepsilon}/h^2$. For large h, this is better than Theorem 2, which is optimized for small h. However, it should not be difficult to gain better results for large h.

2. Applications to non-vanishing Fourier coefficients. For a positive integer N and an even positive integer k, we write $S_k^*(N, \chi)$ for the space of holomorphic primitive cusp forms of weight k, conductor N and nebentypus χ. Each $f \in S_k^*(N, \chi)$ has a Fourier expansion

$$f(z) = \sum_{n=1}^{\infty} \lambda_f(n) n^{(k-1)/2} e(nz)$$

in the upper half plane. The normalised Fourier coefficients $\lambda_f(n)$ are also eigenvalues of Hecke operators and in particular multiplicative.

In the case of the discriminant function

$$\Delta(z) = \sum_{n=1}^{\infty} \tau(n) e(nz) = e(z) \prod_{n=1}^{\infty} (1 - e(nz))^{24} \in S_{12}^*(1, \chi_0)$$

Lehmer has famously conjectured that the Fourier coefficients $\tau(n)$ (called Ramanujan's τ-function) never vanish, and this conjecture is still wide open.

In the general case Serre [17] has shown that, for any $\varepsilon > 0$ and any non-CM $f \in S_k^*(N, \chi)$,

$$|\{p \le x \mid \lambda_f(p) = 0\}| \ll_f \frac{x}{(\log x)^{3/2-\varepsilon}}, \tag{5}$$

so $\lambda_f(p)$ can vanish only rarely. Taking

$$\mathfrak{B} = \{p \in \mathbb{P} \mid \lambda_f(p) = 0\} \cup \{p^2 \mid \lambda_f(p) \ne 0\},$$

we see from (5) that any result for $\mathfrak{B}$-free numbers implies a result for the set of n for which $\lambda_f(n)$ is non-vanishing.

In particular, Theorem 1 implies that there are infinitely many integers n such that $\lambda_f(n) \ne 0$ and $\|\alpha n + \beta\| < n^{-1/2+\varepsilon}$, so that Lehmer's conjecture cannot completely fail even in such a sparse set of integers.

From the application's point of view, short intervals are probably more interesting. In the case of all short intervals, Kowalski, Robert and Wu [13] have made a comprehensive study. While considering almost all intervals here, we are able to take advantage of Theorem 2(ii) thanks to them: They refined Serre's argument to yield

$$|\{p \le x \mid \lambda_f(p^\nu) = 0 \text{ for some } \nu \ge 1\}| \ll_f \frac{x}{(\log x)^{3/2-\varepsilon}}.$$

This lets us take

$$\mathfrak{B} = \{p \in \mathbb{P} \mid \lambda_f(p^\nu) = 0 \text{ for some } \nu \ge 1\},$$

so that $\mathfrak{B} \subset \mathbb{P}$. Besides rewriting Theorem 2(ii) for non-vanishing Fourier coefficients, we also immediately get following corollary.

COROLLARY 3. *Assume that $f(z) \in S_k^*(N, \chi)$ does not have complex multiplication. Let*

$$i_f(n) = \max\{k \ge 0 \mid \lambda_f(n + j) = 0 \textit{ for } 0 < j \le k\}.$$

Then, for any $\theta < 1$,

$$\frac{1}{x} \sum_{n \sim x} i_f(n)^\theta = O_{f,\theta}(1). \tag{6}$$

This is much superior to the conditional estimates present in [3], where Alkan shows that if (5) was improved to

$$|\{p \le x \mid \lambda_f(p) = 0\}| \ll x^\rho$$

for some $\rho \in (0, 1)$, one would have (6) for any $\theta < (1 - \rho)/2$.

3. Auxiliary results. In this section, we present a couple of auxiliary results that we will need later. We start with a fundamental lemma of sieve.

LEMMA 4. *Let $z \ge 2$ and $y = z^s$ with $s \ge 2$. Write $P(z) = \prod_{p<z} p$. There exist two sequences $(\lambda_q^\pm)_{q \mid P(z)}$ of real numbers such that*

(i) $\lambda_1^{\pm} = 1$, $|\lambda_q^{\pm}| \leq 1$ *for all* q, *and* $\lambda_q^{\pm} = 0$ *for* $q \geq y$;
(ii) *For all* $Q \mid P(z)$, $Q > 1$,

$$\sum_{q|Q} \lambda_q^- \leq 0 \leq \sum_{q|Q} \lambda_q^+;$$

(iii) *One has*

$$\sum_{q|P(z)} \frac{\lambda_q^{\pm}}{q} = (1 + O(e^{-s})) \prod_{p<z} \left(1 - \frac{1}{p}\right). \tag{7}$$

Proof. See for instance [**10**, Lemma 5]. □

The following lemma shows that while looking for $\mathfrak{B}$-free numbers, we can restrict to a set $\mathfrak{B}$ with numbers with at most two prime factors. This trick was also utilized by Alkan and Zaharescu [**5**].

Lemma 5. *Let* $\mathfrak{B} \subset \mathbb{N}$ *satisfy* (1). *Then there exists a set* $\mathfrak{B}' \subset \mathbb{N}$ *such that* (1) *holds with* $\mathfrak{B}$ *replaced by* $\mathfrak{B}'$,

$$\mathfrak{B}' \subset \mathbb{P} \cup \{p^2 \mid p \in \mathbb{P}\} \cup \{p_1p_2 \mid p_1, p_2 \in \mathbb{P}\}$$

and every $\mathfrak{B}'$*-free number is also* $\mathfrak{B}$*-free.*

Proof. We define a sequence $\{b_i'\}_{i\in\mathbb{N}} = \mathfrak{B}'$ satisfying all the conditions. If $b_i \in \mathbb{P}$, we take $b_i' = b_i$. Otherwise we take b_i' to be the product of two largest (possibly not distinct) prime factors of b_i. Clearly the elements of $\mathfrak{B}'$ are pairwise co-prime and every $\mathfrak{B}'$-free number is also $\mathfrak{B}$-free. Furthermore, by co-primality

$$\sum_{b'\in\mathfrak{B}'} \frac{1}{b'} \leq \sum_{\substack{b\in\mathfrak{B} \\ b\in\mathbb{P}}} \frac{1}{b} + \sum_{p\in\mathbb{P}} \frac{1}{p^2} < \infty.$$

□

4. Method for finding $\mathfrak{B}$-free numbers. The basic framework used in most subsequent works on $\mathfrak{B}$-free numbers was already present in Erdős's first paper [**9**] on the topic. However, it seems that the argument, with some refinements, has been repeated for each problem. Although it is not difficult to apply the method for each problem separately, it is instructive to have a general formulation from which one clearly sees, what exactly needs to be known about a sequence in order to find $\mathfrak{B}$-free numbers. In this section, we will give such a formulation. In it and its proof, we will need small parameters $\varepsilon > \eta > 0$. The implied (and explicit) constants may depend on ε and any implied (and explicit) constants in the assumptions but not on η.

Throughout we assume the following.

Assumption 6. Assume that $\omega(d)$ is a multiplicative function such that, for every $p \in \mathbb{P}$ and $k \in \mathbb{N}$, one has

$$\omega(p^k) \in [0, p^k), \quad \text{and} \quad \omega(p), \omega(p^2) \ll 1.$$

Recalling the notation $\mathcal{A}_d$ from (2), we write

$$|\mathcal{A}_d| = \frac{\omega(d)}{d}H + R_d, \tag{8}$$

where one tries to choose $H \geq 1$ and $\omega(d)$ so that R_d is small at least on average.

We start with a simple statement, which shows that we can find $\mathfrak{B}$-free numbers from a dense enough sequence $\mathcal{A}$ as soon as the level of distribution is $\geq x^{1/2+\delta}$ (that is (10) below holds). By dense enough we mean here that

$$H \gg x^{1/2} \quad \text{and} \quad \sum_{x^\varepsilon < p < x^{\varepsilon+\varepsilon^2}} \frac{\omega(p)}{p} \gg_\varepsilon 1 \tag{9}$$

for any $\varepsilon > 0$ and large enough x.

PROPOSITION 7. *Let $\mathfrak{B} = \{b_i\}$ be a sequence of integers satisfying* (1) *and let $\mathcal{A} \subseteq [x, 2x] \cap \mathbb{N}$ satisfy* (8) *with some H and ω such that* (9) *holds for any $\varepsilon > 0$ and any large enough x. Assume that, for any bounded real coefficients a_d and for some $\delta > 0$,*

$$\sum_{d \leq x^{1/2+\delta}} a_d R_d = o(H). \tag{10}$$

Then

$$|\{n \in \mathcal{A} \mid n \text{ is } \mathfrak{B}\text{-free}\}| \gg H$$

for any large enough x.

This is already enough to prove Theorem 1. However, in the situation in Theorem 2 and many other applications, one can conclude something like (10) only for special coefficients a_d. Next more general proposition gives us some flexibility.

There we will need the notation

$$P(\mathcal{M}) = \max\{p \in \mathbb{P} \mid p \mid m \text{ for some } m \in \mathcal{M}\},$$

that is, $P(\mathcal{M})$ is the greatest prime factor occurring in the set $\mathcal{M}$. We also write $p(n)$ for the smallest prime factor of n.

PROPOSITION 8. *Let $\varepsilon > 0$ and let $\mathfrak{B} = \{b_i\}$ be a sequence of integers satisfying* (1) *and let $\mathcal{A} \subseteq [x, 2x] \cap \mathbb{N}$ satisfy* (8)*. Let $\alpha_1, \alpha_2 \in [0, 1)$, $M_i = x^{\alpha_i}$ and let $y \in [x^\varepsilon, x]$ be a parameter depending on x. Assume that either $\alpha_1 = 0$ or $\omega(d)$ is identically* 1*, and that*

$$\mathcal{M}_2 \subseteq \left\{n \in \left[M_2, M_2^{1+\varepsilon}\right] \mid p \mid n \implies p > M_2^\varepsilon\right\}$$

is such that $\sum_{m \in \mathcal{M}_2} \omega(m)/m \gg_\varepsilon 1$. Assume that the following three conditions hold for any small enough η.

(i) *For any constant a and any sieve coefficients λ_q as in Lemma* 4 *with $y = M_1^\varepsilon$ and $z = M_1^\eta$ (in case $\alpha_1 = 0$, take just $\lambda_1 = 1$),*

$$\sum_{\substack{q,k,m_2 \\ q \leq M_1^\varepsilon, M_1 \leq qk \leq M_1^{1+\varepsilon} \\ m_2 \in \mathcal{M}_2}} \lambda_q R_{aqkm_2} = o(H);$$

(ii) *There is a constant C such that, for any set $\mathcal{C}$ of integers in the interval* $[1, \max\{y, x^\eta M_1^{1+\varepsilon}\}]$,

$$\sum_{c\in\mathcal{C}} |\mathcal{A}_c| \le C \sum_{c\in\mathcal{C}} \frac{\omega(c)}{c} H + o(H);$$

(iii) $M_1 M_2 y > 2x$ *and*

$$\sum_{\substack{b\in\mathfrak{B}, b>y \\ p(b)<\max\{M_1^{1+\varepsilon}, P(\mathcal{M}_2)\}}} |\mathcal{A}_b| = o(H). \tag{11}$$

The estimate (11) *holds in particular if either*

(a) $\mathfrak{B} \subset \mathbb{P}$ *and* $y \ge \max\{M_1^{1+\varepsilon}, P(\mathcal{M}_2)\}$ *or*

(b) $\max\{M_1^{1+\varepsilon}, P(\mathcal{M}_2)\} \cdot \max_{\substack{b\in\mathfrak{B} \\ b>y}} |\mathcal{A}_b| = o(H)$.

Then

$$|\{n \in \mathcal{A} \mid n \text{ is } \mathfrak{B}\text{-free}\}| \gg H$$

for any large enough x.

Notice that Proposition 7 follows from Proposition 8 taking $\varepsilon = 1/\lceil 2/\delta \rceil$, $M_1 = 1$, $M_2 = y = x^{1/2+\varepsilon}$ (that is, $\alpha_1 = 0$ and $\alpha_2 = 1/2 + \varepsilon$) and

$$\mathcal{M}_2 = \left\{n \in \left[M_2, M_2^{1+\varepsilon}\right] \mid p \mid n \implies M_2^{\varepsilon} < p < M_2^{\varepsilon+\varepsilon^2}\right\}.$$

Before going into the proof of Proposition 8, we discuss the conditions (i)–(iii) a bit. The conditions (i) and (ii) amount to gathering type I information, whereas the condition (iii) tells how much type I information is needed, that is how large the parameters y and M_i must be.

In (i), there is a free variable k, which is sometimes of benefit. Other times, one might choose $M_1 = 1$ still keeping the freedom to choose the set $\mathcal{M}_2$ so that type I information is available.

In (ii), an upper bound of correct order of magnitude suffices. So if there are, for instance, prime numbers involved, one can use an upper bound sieve. This was the case in [**19**], where Wu showed that, for each sufficiently large x, the interval $[x, x + x^{3/4+\varepsilon}]$ contains primes p such that $p + 2$ is $\mathfrak{B}$-free — Wu was able to conclude a result like (ii) for $y = x^{3/4}$ using the Brun-Titchmarsh theorem.

Proof of Proposition 8. The essentials of the proof are contained in earlier papers on $\mathfrak{B}$-free numbers. However, for completeness sake, we prove the general formulation combining the arguments from the literature.

By Lemma 5, we can assume that $\mathfrak{B}$ consists of numbers with at most two prime factors, so that $\omega(b) \ll 1$ for all $b \in \mathfrak{B}$. Throughout we will assume that x is large. Let

$$\mathcal{M}_1 = \left\{n \in \left[M_1, M_1^{1+\varepsilon}\right] \mid p \mid n \implies p > M_1^{\eta}\right\},$$

where η is a very small positive constant. We will consider

$$A = \sum_{\substack{n\in\mathcal{A} \\ n\ \mathfrak{B}\text{-free}}} c(n) \qquad \text{with weights} \quad c(n) = \sum_{\substack{n=m_1 m_2 k \\ m_i\in\mathcal{M}_i}} 1.$$

Writing, for $i = 1, 2$,

$$\alpha_i' = \begin{cases} \alpha_i & \text{if } \alpha_i \neq 0, \\ 1 & \text{otherwise,} \end{cases}$$

we have, for $n \leq 2x$,

$$c(n) \leq \sum_{\substack{m_1 | n \\ m_1 \in \mathcal{M}_1}} \sum_{\substack{m_2 | n \\ m_2 \in \mathcal{M}_2}} 1 \leq \sum_{\substack{m_1 | n \\ m_1 \in \mathcal{M}_1}} 2^{1+1/(\alpha_2' \varepsilon)} \leq 2^{2+1/(\alpha_1' \eta)+1/(\alpha_2' \varepsilon)}, \tag{12}$$

so it is enough to prove that $A \gg H$.

Let $l = l(\mathfrak{B}) \in \mathbb{N}$ be such that

$$\sum_{k=l+1}^{\infty} \frac{\omega(b_k)}{b_k} < \frac{1}{2^{2+1/(\alpha_2' \varepsilon)} C} \sum_{m \in \mathcal{M}_2} \frac{\omega(m)}{m} \prod_{b \in \mathfrak{B}} \left(1 - \frac{\omega(b)}{b}\right), \tag{13}$$

where C is the constant in condition (ii). Such l exists since by assumptions on ω, $\mathfrak{B}$ and $\mathcal{M}_2$, the right-hand side is $\gg_{\varepsilon, \mathfrak{B}} 1$.

Now clearly

$$A \geq A_1 - A_2 - A_3 - A_4,$$

where

$$A_1 = \sum_{\substack{n \in \mathcal{A} \\ b_k | n \implies k > l}} c(n), \quad A_2 = \sum_{\substack{b \in \mathfrak{B} \\ b_l < b \leq x^{\alpha_1' \eta/2}}} \sum_{n \in \mathcal{A}_b} c(n)$$

$$A_3 = \sum_{\substack{b \in \mathfrak{B} \\ x^{\alpha_1' \eta/2} < b \leq y}} \sum_{n \in \mathcal{A}_b} c(n) \quad \text{and} \quad A_4 = \sum_{\substack{b \in \mathfrak{B} \\ y < b \leq 2x}} \sum_{n \in \mathcal{A}_b} c(n).$$

Our splitting is similar to that in [**20**]. In [**13**], A_2 and A_3 were treated as one sum, but in doing that there is a slight problem due to definitions of two parameters η and l depending on each other. This problem is easy to overcome by working as here, that is as in [**20**].

We estimate the sums A_i one by one. By (12),

$$A_2 \leq 2^{1+1/(\alpha_2' \varepsilon)} \sum_{\substack{b \in \mathfrak{B} \\ b_l < b \leq x^{\alpha_1' \eta/2}}} \sum_{m_1 \in \mathcal{M}_1} \sum_{\substack{n \in \mathcal{A} \\ b | n, m_1 | n}} 1.$$

By the definition of $\mathcal{M}_1$, $(b, m_1) = 1$ above. Hence

$$
\begin{aligned}
A_2 &\le 2^{1+1/(\alpha_2'\varepsilon)} \sum_{\substack{b\in\mathfrak{B} \\ b_l<b\le x^{\alpha_1'\eta/2}}} \sum_{m_1\in\mathcal{M}_1} |\mathcal{A}_{bm_1}| \\
&\le 2^{1+1/(\alpha_2'\varepsilon)} CH \sum_{\substack{b\in\mathfrak{B} \\ b_l<b\le x^{\alpha_1'\eta/2}}} \sum_{m_1\in\mathcal{M}_1} \frac{\omega(b)\omega(m_1)}{bm_1} + o(H) \\
&\le \frac{H}{2} \prod_{b\in\mathfrak{B}} \left(1 - \frac{\omega(b)}{b}\right) \sum_{m_1\in\mathcal{M}_1} \frac{\omega(m_1)}{m_1} \sum_{m_2\in\mathcal{M}_2} \frac{\omega(m_2)}{m_2} + o(H)
\end{aligned}
$$

by condition (ii) and (13).

Writing $f(x) = o_\eta(|g(x)|)$ when, for any fixed $\eta > 0, f(x)/|g(x)| \to 0$ when $x \to \infty$, we see by (12), (ii) and (1) that

$$
\begin{aligned}
A_3 &\le 2^{2+1/(\alpha_1'\eta)+1/(\alpha_2'\varepsilon)} \sum_{\substack{b\in\mathfrak{B} \\ x^{\alpha_1'\eta/2}<b\le y}} |\mathcal{A}_b| \\
&\le 2^{2+1/(\alpha_1'\eta)+1/(\alpha_2'\varepsilon)} \left(CH \sum_{\substack{b\in\mathfrak{B} \\ x^{\alpha_1'\eta/2}<b\le y}} \frac{\omega(b)}{b} + o(H) \right) = o_\eta(H).
\end{aligned}
$$

Next we treat A_4. Only those n divisible by both m_1m_2 with $m_i \in \mathcal{M}_i$ and by some $b \in \mathfrak{B}$ with $b > y$ give non-zero contribution to A_4. Since $M_1M_2y > 2x$, we must have $(m_1m_2, b) > 1$. Now

$$
p(b) \le p((b, m_1m_2)) \le \max\left\{M_1^{1+\varepsilon}, P(\mathcal{M}_2)\right\}
$$

and hence we have

$$
A_4 \le 2^{2+1/(\alpha_1'\eta)+1/(\alpha_2'\varepsilon)} \sum_{\substack{b\in\mathfrak{B}, b>y \\ p(b)<\max\{M_1^{1+\varepsilon}, P(\mathcal{M}_2)\}}} |\mathcal{A}_b| = o_\eta(H)
$$

by (12) and (11). It is clear that (iii)(a) implies (11). Also (iii)(b) implies it since $b \in \mathfrak{B}$ are pairwise co-prime, so that at most $\max\{M_1^{1+\varepsilon}, P(\mathcal{M}_2)\}$ distinct b give non-trivial contribution to the sum in (11).

Consider then A_1. Following **[20]** we introduce some notation. For $\sigma = \{k_1, \dots, k_i\} \subset \{1, \dots, l\}$, we write $|\sigma| = i$ and $d_\sigma = b_{k_1}b_{k_2}\cdots b_{k_i}$ with conventions $|\emptyset| = 0$ and $d_\emptyset = 1$.

Let

$$
R = \sum_{\sigma\subseteq\{1,\dots,l\}} (-1)^{|\sigma|} \sum_{\substack{m_1,m_2 \\ m_i\in\mathcal{M}_i}} R_{d_\sigma m_1m_2} = \sum_{\sigma\subseteq\{1,\dots,l\}} (-1)^{|\sigma|} R(\sigma),
$$

say. We have by the inclusion-exclusion principle

$$A_1 = \sum_{\sigma\subseteq\{1,\dots,l\}} (-1)^{|\sigma|} \sum_{\substack{n\in\mathcal{A}\\ d_\sigma|n, m_1m_2|n\\ m_i\in\mathcal{M}_i}} 1 = \sum_{\sigma\subseteq\{1,\dots,l\}} (-1)^{|\sigma|} \sum_{m_i\in\mathcal{M}_i} |\mathcal{A}_{d_\sigma m_1 m_2}|$$

since b_i are pairwise co-prime and $(d_\sigma, m_1m_2) = 1$ when $m_i \in \mathcal{M}_i$. We get

$$A_1 = H \sum_{\sigma\subseteq\{1,\dots,l\}} \frac{(-1)^{|\sigma|}\omega(d_\sigma)}{d_\sigma} \sum_{m_i\in\mathcal{M}_i} \frac{\omega(m_1m_2)}{m_1m_2} + R.$$

Since $\omega(d)$ is identically 1 when $M_1 > 1$, $\omega(m_1m_2) = \omega(m_1)\omega(m_2)$ here, so that

$$A_1 \geq H \prod_{b\in\mathfrak{B}} \left(1 - \frac{\omega(b)}{b}\right) \sum_{m_1\in\mathcal{M}_1} \frac{\omega(m_1)}{m_1} \sum_{m_2\in\mathcal{M}_2} \frac{\omega(m_2)}{m_2} + R.$$

Next we want to show that $R = o(H)$. In case $\alpha_1 = 0$, this follows immediately from condition (i). Hence, we can assume that ω is identically one. Next we transform the sum over $\mathcal{M}_1$ using Lemma 4. Taking $z = M_1^\eta$ and $y = M_1^\varepsilon$ we get

$$\begin{aligned} R(\sigma) &\leq \sum_{\substack{M_1\leq qk\leq M_1^{1+\varepsilon}\\ m_2\in\mathcal{M}_2}} \left(\lambda_q^+ |\mathcal{A}_{d_\sigma qkm_2}| - \lambda_q^- H \frac{1}{d_\sigma qkm_2}\right) \\ &= \sum_{\substack{M_1\leq qk\leq M_1^{1+\varepsilon}\\ m_2\in\mathcal{M}_2}} \lambda_q^+ R_{d_\sigma qkm_2} + H \sum_{\substack{M_1\leq qk\leq M_1^{1+\varepsilon}\\ m_2\in\mathcal{M}_2}} \frac{\lambda_q^+ - \lambda_q^-}{d_\sigma qkm_2}. \end{aligned}$$

The first sum is $o(H)$ by assumption (i). On the other, hand the second sum is

$$\begin{aligned} &\frac{1}{d_\sigma} \sum_{m_2\in\mathcal{M}_2} \frac{1}{m_2} \sum_{q|P(M_1^\eta)} \frac{\lambda_q^+ - \lambda_q^-}{q} \left(\varepsilon \log M_1 + O\left(M_1^{\varepsilon-1}\right)\right) \\ &\quad\ll e^{-\varepsilon/\eta} \log M_1 \prod_{p<M_1^\eta} \left(1 - \frac{1}{p}\right) + o(1) \ll e^{-\varepsilon/\eta}/\eta + o(1) \end{aligned}$$

by (7) and Mertens' formula. Hence, in this case $R = o(H) + O(H \cdot e^{-\varepsilon/\eta}/\eta)$.

Collecting everything together, we obtain that

$$\begin{aligned} A &\geq A_1 - A_2 - A_3 - A_4 \\ &\geq \frac{H}{2} \prod_{b\in\mathfrak{B}} \left(1 - \frac{\omega(b)}{b}\right) \sum_{m_1\in\mathcal{M}_1} \frac{\omega(m_1)}{m_1} \sum_{m_2\in\mathcal{M}_2} \frac{\omega(m_2)}{m_2} + O(H \cdot e^{-\varepsilon/\eta}/\eta) + o_\eta(H). \end{aligned} \tag{14}$$

Notice that the sum over m_1 above is

$$\sum_{\substack{M_1\leq m_1\leq M_1^{1+\varepsilon}\\ p|m_1 \implies p>M_1^\eta}} \frac{1}{m_1} \gg_\varepsilon 1.$$

Hence by our assumptions on $\mathfrak{B}$, ω and $\mathcal{M}_2$, the first term in (4) is $\gg H$, so it dominates the second term when η is small enough. Letting then $x \to \infty$, we see that indeed $A \gg H$. □

5. Proof of theorem 1. In this section, we derive Theorem 1 from Proposition 7 with

$$\mathcal{A} = \{n \sim x \mid \|\alpha n + \beta\| < x^{-\theta}/2\}.$$

Here $x = q^2$ with q any large denominator in the continued fraction expansion of α. We further take $H = x^{1-\theta}$ and $\omega(d) = 1$. We show that (10) holds for any $\varepsilon < (1/2 - \theta)/2\,(> 0)$.

We take advantage of the following standard finite Fourier expansion.

LEMMA 9. *Let*

$$\chi_\delta(x) = \begin{cases} 1 & \text{if } \|x\| \le \delta, \\ 0 & \text{else.} \end{cases}$$

Then there exist coefficients $c_l^\pm$ *such that*

$$2\delta - \frac{1}{L} + \sum_{0<|l|<L} c_l^- e(lx) \le \chi_\delta(x) \le 2\delta + \frac{1}{L} + \sum_{0<|l|<L} c_l^+ e(lx),$$

where

$$|c_l^\pm| \ll \min\left\{\delta + \frac{1}{L}, \frac{1}{l}\right\}.$$

Proof. See [**6**, Sections 2.2–2.3]. □

We will bound the arising exponential sums by Vinogradov's estimate for type I exponential sums; a convenient form for us is the following direct consequence of [**11**, formula (1.6.1)].

LEMMA 10. *Assume that* $|\alpha - a/q| < 1/q^2$ *with* $(a, q) = 1$*. Then, for any function* $f : \mathbb{N} \to \mathbb{R}_+$ *such that* $1 \le rf(r) \le N$ *whenever* $1 \le r \le R$*, one has*

$$\sum_{1\le r\le R} \left| \sum_{f(r)\le n<2f(r)} e(\alpha nr) \right| \ll \left(\frac{N}{q} + R + q\right) \log(NRq).$$

Returning to the proof of Theorem 1, let $L = x^{\theta+\varepsilon}$. By Lemma 9

$$|\mathcal{A}_d| = \sum_{k\sim x/d} \chi_{x^{-\theta}/2}(\alpha dk + \beta) = \frac{H}{d}(1 + O(x^{-\varepsilon})) + O\left(\sum_{0<|l|<L} c_l \sum_{k\sim x/d} e((\alpha dk + \beta)l)\right)$$

for some $c_l \ll \min\{1/l, x^{-\theta}\}$. Hence (10) holds if

$$\sum_{d \le x^{1/2+\varepsilon}} \sum_{l \sim L'} \left| \sum_{k \sim x/d} e(dkl\alpha) \right| = o(HL'/\log x)$$

for any $L' \le L$. The left-hand side is

$$\ll x^{\varepsilon/2} \sum_{1 \le r \le x^{1/2+\varepsilon} L'} \left| \sum_{f(r) \le k < 2f(r)} e(\alpha k r) \right|,$$

where $f(r) = x/d$ with $d \sim r/(2L')$ is chosen so that the corresponding exponential sum has maximal absolute value. Applying Lemma 10 with $N = 2xL'$, this is

$$\ll \left(\frac{xL'}{q} + x^{1/2+\varepsilon} L' + q \right) x^{\varepsilon} \ll HL' x^{\theta - 1/2 + 2\varepsilon} = o(HL'/\log x).$$

Hence, Proposition 7 is applicable and Theorem 1 follows since there are infinitely many choices for the denominator q.

6. Proof of theorem 2. We prove Theorem 2 by showing that the conditions of Proposition 8 hold for $\mathcal{A} = \mathcal{A}(n) = [n, n+h]$ with few exceptional n.

Let ε be very small compared to δ and $1-\theta$. For the proof of Theorem 2(i), we choose $\omega(d) = 1$, $H = h+1$, $M_1 = 1$, $M_2 = (x/h)^{3/5-\delta/2}$, $y = x^{1/2}/(\log^3 x)$ and

$$\mathcal{M}_2 = \{ m_2 = n_1 n_2 \in [M_2, M_2^{1+\varepsilon}] \mid n_i \in [M_2^{i/3}, M_2^{(1+\varepsilon)i/3}], \\ p \mid n_i \implies p \in [M_2^{\varepsilon i}, M_2^{\varepsilon(i+1)}] \}.$$

We start from condition (iii). Since $h < x^{1/6-\delta}$, $M_2 y > 2x$. Further the interval $[x, 2x+h]$ contains $\ll x/b$ numbers divisible by $b \in \mathfrak{B}$. As the number of b giving positive contribution to the sum (11) is at most $P(\mathcal{M}_2) \le M_2^{3\varepsilon}$, we get in total at most $hx^{1+3\varepsilon}/y$ intervals $[n, n+h]$ containing a number counted in (11). Hence, there are $\ll hx^{1+3\varepsilon}/y = o(x/h)$ exceptions to (iii).

The work of Plaksin gives that, for any bounded coefficients a_d,

$$\sum_{d \le D} a_d R_d = o(H) \tag{15}$$

with $\ll D^2(\log D)^4/h$ exceptions, which implies condition (ii) with $C = 1$ and $o(x/h)$ exceptions.

Establishing condition (i) is more involved. We study the second moment

$$\sum_n f(n) \left(\sum_{m \in \mathcal{M}_2} R_{am} \right)^2 = \sum_n f(n) \left(\sum_{\substack{akm \in \mathcal{A}(n) \\ m \in \mathcal{M}_2}} 1 - H \sum_{m \in \mathcal{M}_2} \frac{1}{am} \right)^2, \tag{16}$$

where $f(n)$ is a smooth weight function such that $f(n) = 1$ for $n \in [x, 2x]$, $f^{(k)}(x) \ll x^{-k}$ for any $k \ge 0$, and f is supported on $[x/2, 5x/2]$ (See for instance [**11**, Appendix 5] for construction of such function). Condition (i) follows with $o(x/h^{\theta})$ exceptions if we can show that (16) is $o(h^{2-\theta} x)$.

REMARK 11. The sum in (16) is essentially a type I sum over almost all short intervals. Such sums are often handled by applying Perron's formula to change into a question about mean values of Dirichlet polynomials (see for instance [**11**, Chapter 9]) and then applying known results on these. A mean value result where the Dirichlet polynomials are exactly of the form corresponding to our sum has been proved by Deshouillers and Iwaniec [**8**]. However, as we are working on very short intervals, we cannot afford to transform into Dirichlet polynomials and use their result. A crucial observation here is that the proof of the mean value result in [**8**] is carried out by essentially transforming the problem to estimating a type I sum in almost all short intervals. This sum is then transformed to sums of Kloosterman sums. We will avoid losing too much by not going back and forth.

We square out and see that (16) is

$$\sum_n f(n)\Bigg(\sum_{\substack{akm\in[n,n+h]\\ m\in\mathcal{M}_2}} 1\Bigg)^2 - 2\frac{h+1}{a}\Bigg(\sum_{m\in\mathcal{M}_2}\frac{1}{m}\Bigg)\sum_n f(n)\sum_{\substack{akm\in[n,n+h]\\ m\in\mathcal{M}_2}} 1$$
$$+\frac{(h+1)^2}{a^2}\Bigg(\sum_{m\in\mathcal{M}_2}\frac{1}{m}\Bigg)^2\sum_n f(n)$$
$$= S_1 - 2S_2 + S_3,$$

say.

For $n_1\in\mathcal{A}(n)$, $|f(n_1)-f(n)|\ll h/n$, so

$$S_2 = \frac{h+1}{a}\Bigg(\sum_{m\in\mathcal{M}_2}\frac{1}{m}\Bigg)\sum_n\sum_{\substack{akm\in[n,n+h]\\ m\in\mathcal{M}_2}} f(akm) + O(h^3)$$
$$= \frac{(h+1)^2}{a}\Bigg(\sum_{m\in\mathcal{M}_2}\frac{1}{m}\Bigg)\sum_{m\in\mathcal{M}_2}\sum_k f(akm) + O(h^3) \tag{17}$$

Notice that, for any $\alpha>0$,

$$\sum_k f(\alpha k) = \sum_k \frac{1}{\alpha}\int_{\alpha(k-1/2)}^{\alpha(k+1/2)} f(t) + O(|t-\alpha k|f'(t))dt = \frac{\hat{f}(0)}{\alpha} + O(1),$$

where $\hat{f}(z)=\int f(t)e(-tz)dt$ is the Fourier transform of $f(x)$. Applying this to (17) and to S_3, we are left with showing that

$$S_1 = \frac{(h+1)^2}{a^2}\Bigg(\sum_{m\in\mathcal{M}_2}\frac{1}{m}\Bigg)^2\hat{f}(0) + o(h^{2-\theta}x).$$

Squaring the definition of S_1 out,

$$S_1 = \sum_n f(n)\sum_{m,m'\in\mathcal{M}_2}\sum_{\substack{akm\in[n,n+h]\\ ak'm'\in[n,n+h]}} 1.$$

The diagonal terms ($akm = ak'm'$) contribute

$$\ll h \sum_{m,m' \in \mathcal{M}_2} \sum_{\substack{x/2 \le akm \le 5x/2+h \\ akm = ak'm'}} 1 \ll h \sum_{m,m' \in \mathcal{M}_2} \frac{x}{[m, m']}$$

$$\ll hx \sum_{\substack{d \le M_2^{1+\varepsilon} \\ p \mid d \implies p > M_2^{\varepsilon}}} \frac{1}{d} \left(\sum_{\substack{m \le M_2^{1+\varepsilon}/d \\ p \mid m \implies p > M_2^{\varepsilon}}} \frac{1}{m} \right)^2 \ll hx.$$

On the other hand, the non-diagonal terms contribute

$$2 \sum_{j=1}^{h} (h + 1 - j) \sum_{m,m' \in \mathcal{M}_2} \sum_{akm = ak'm'+j} f(akm) + O(h^3)$$

Writing g for $\gcd(m, m')$ only terms with $ag \mid j$ give non-trivial contribution. Hence, we can rearrange the main term above to

$$\begin{aligned} &2 \sum_{\substack{g,j \\ 1 \le agj \le h}} (h + 1 - agj) \sum_{\substack{bg, b'g \in \mathcal{M}_2 \\ (b,b')=1}} \sum_{kb = k'b'+j} f(abgk) \\ &= 2 \sum_{\substack{g,j \\ 1 \le agj \le h}} (h + 1 - agj) \sum_{\substack{bg, b'g \in \mathcal{M}_2 \\ (b,b')=1}} \sum_{k \equiv j\bar{b} \, (\mathrm{mod}\, b')} f(abgk) \\ &= 2 \sum_{\substack{g,j \\ 1 \le agj \le h}} (h + 1 - agj) \sum_{\substack{bg, b'g \in \mathcal{M}_2 \\ (b,b')=1}} \sum_{l} f(abb'g \cdot l + abgj\bar{b}), \end{aligned} \tag{18}$$

where $\bar{b}$ is the inverse of $b \, (\mathrm{mod}\, b')$.

Recalling that f and its derivatives are very smooth, we apply the Poisson summation formula (see [**12**, (4.24)]) to the sum over l getting that (18) is

$$2 \sum_{\substack{g,j \\ 1 \le agj \le h}} (h + 1 - agj) \sum_{\substack{bg, b'g \in \mathcal{M}_2 \\ (b,b')=1}} \frac{1}{abb'g} \sum_{k} \hat{f}\left(\frac{k}{abb'g}\right) e\left(\frac{jk\bar{b}}{b'}\right). \tag{19}$$

If $g|k| > K := M_2^2 x^{2\varepsilon - 1}$, iterated partial integration gives that $\hat{f}(k/(abb'g)) \ll x^{-100}$, so we can restrict the sum over k to the range $g|k| \le K$. The main term is obtained for $k = 0$ and is

$$\begin{aligned} &\frac{2\hat{f}(0)}{a} \sum_{1 \le g \le h/a} \sum_{\substack{bg, b'g \in \mathcal{M}_2 \\ (b,b')=1}} \frac{1}{bb'g} \sum_{j=1}^{\lfloor h/(ag) \rfloor} (h + 1 - agj) \\ &= \frac{2\hat{f}(0)}{a} \sum_{1 \le g \le h/a} \sum_{\substack{bg, b'g \in \mathcal{M}_2 \\ (b,b')=1}} \frac{1}{bb'g} \left(\frac{(h+1)^2}{2ag} + O(h) \right) \end{aligned}$$

$$= \frac{(h+1)^2}{a^2}\hat{f}(0)\left(\sum_{g\geq 1}\sum_{\substack{bg,b'g\in\mathcal{M}_2\\(b,b')=1}}\frac{1}{bg}\cdot\frac{1}{b'g} - \sum_{g>h/a}\frac{1}{g^2}\sum_{\substack{bg,b'g\in\mathcal{M}_2\\(b,b')=1}}\frac{1}{bb'}\right) + O(hx)$$

$$= \frac{(h+1)^2}{a^2}\hat{f}(0)\left(\sum_{m\in\mathcal{M}_2}\frac{1}{m}\right)^2 + O(hx).$$

After the truncation and removal of the main term $k = 0$ from (19), we are left with

$$\sum_{1\leq g\leq h/a}\sum_{\substack{j\\1\leq agj\leq h}}\sum_{\substack{bg,b'g\in\mathcal{M}_2\\(b,b')=1}}\sum_{\substack{k\\0<g|k|\leq K}} F(h,g,j,k,b,b')e\left(\frac{jk\bar{b}}{b'}\right), \tag{20}$$

where

$$F(h,g,j,k,b,b') = 2(h+1-agj)\frac{1}{abb'g}\hat{f}\left(\frac{k}{abb'g}\right) \ll \frac{hx}{bb'g} \ll \frac{ghx}{M_2^2}.$$

Essentially the same average of Kloosterman sums was also reached by Deshouillers and Iwaniec in [**8**, end of page 309], the only difference, besides differently named parameters, being that their F-function has different size ($gLT/(MN)$ in their notation). The similarity of the outcomes is not a coincidence but very natural in light of Remark 11. We can actually readily conclude from [**8**, Section 4] for $M = M_2^{2(1+\varepsilon)/3}$, $N = M_2^{(1+\varepsilon)/3}$, $L = x/(MN)$ and $T = x^{1+\varepsilon}/h$ that (20) is

$$\ll x^{\varepsilon}\frac{\frac{ghx}{M_2^2}}{\frac{gLT}{MN}}LMN(T^{1/2}M^{3/4}N + T^{1/2}MN^{1/2} + M^{7/4}N^{3/2})$$
$$\ll x^{10\varepsilon}h^2\left(\left(\frac{x}{h}\right)^{1/2}M_2^{5/6} + M_2^{5/3}\right) \ll x^{10\varepsilon}h^2\left(\frac{x}{h}\right)^{1-\delta/3} \ll hx$$

since ε is small compared to δ and $h \leq x^{1/6-\delta}$.

Now we have shown that (16) is $O(hx)$. This is essentially the required $o(h^{2-\theta}x)$ when h is large, which can be assumed by putting the implied constant in Theorem 2 large enough. From the proof of Proposition 8, one sees that this “essentially $o(h^{2-\theta}x)$” implies Theorem 2(i).

Next we consider the case $\mathfrak{B} \subset \mathbb{P}$. Now we can assume that $h \in [x^{1/7}, x^{7/16}]$ by part (i) and the result in [**13**] that the gap between consecutive $\mathfrak{B}$-free numbers is $O(x^{7/17})$. We take ε, ω, H and y as in part (i) and choose $M_1 = x^{9/20}$, $M_2 = x^{1/10}$ and

$$\mathcal{M}_2 = \{m \in [M_2, M_2^{1+\varepsilon}] \mid p \mid m \implies p \in [M_2^{\varepsilon}, M_2^{2\varepsilon}]\}.$$

Now there are no exceptions to (iii) by (iii)(a). Further (15) gives (ii) with $o(x/h)$ exceptions.

Consider then the condition (i). This time we have chosen our parameters corresponding to Dirichlet polynomial result of Watt [**18**]. Watt’s approach is similar to that of Deshouillers and Iwaniec and we could quote his Kloosterman sum result to give an estimate to an expression corresponding to (20). However, as we now have larger

h, it is possible and perhaps illustrative to use the traditional approach of Dirichlet polynomials, which lets us directly cite work on primes in almost all short intervals (recall also Remark 11).

Analogously to [**11**, Lemma 9.3] (that is transformation of the problem to one on mean values of Dirichlet polynomials by Perron's formula), (i) holds with $O(x/h^{\theta})$ exceptions, if we can show that, for $T = x^{1+\varepsilon}/h$,

$$\int_{-T}^{T} \left|K_1\left(\tfrac{1}{2}+it\right)K_2\left(\tfrac{1}{2}+it\right)M\left(\tfrac{1}{2}+it\right)\right|^2 dt \ll x/h^{\theta+\varepsilon}. \tag{21}$$

Here

$$M(s) = \sum_{m \le M_1^{\varepsilon} M_2^{1+\varepsilon}} \left(\sum_{\substack{m=qm_2 \\ q \le M_1^{\varepsilon}, m_2 \in \mathcal{M}_2}} \lambda_q \right) m^{-s} \quad \text{and} \quad K_i(s) = \sum_{k \le K_i} k^{-s}$$

with $K_1 = M_1^{1+\varepsilon}$, $K_2 = x/(M_1 M_2) = x^{9/20}$. Actually this does not yet take into account the cross-condition $M_1 \le kq \le M_1^{1+\varepsilon}$. However, this can be handled with the truncated Perron's formula (for details of such removal, see [**11**, Section 3.2]), the only differences are that we need to show that the above holds when $K_1(\frac{1}{2}+it)$ is replaced by $K_1(\frac{1}{2}+iu+it)$ for some $|u| \le T$ and coefficients of $M(s)$ are slightly modified.

By the Cauchy-Schwarz inequality and Watt's mean value theorem [**18**], (21) is

$$\ll \left(\int_{-T}^{T} \left|K_1\left(\tfrac{1}{2}+it\right)^4 M\left(\tfrac{1}{2}+it\right)^2\right| dt \right)^{1/2} \left(\int_{-T}^{T} \left|K_2\left(\tfrac{1}{2}+it\right)^4 M\left(\tfrac{1}{2}+it\right)^2\right| dt \right)^{1/2}$$
$$\ll T^{1+\varepsilon}(1 + (M_1^{\varepsilon} M_2^{1+\varepsilon})^2/T^{1/2}) \ll T^{1+\varepsilon} \ll \frac{x}{h^{\theta+\varepsilon}}$$

finishing the proof of Theorem 2.

The proof of part (i) of Theorem 2 actually shows that, for $h \le x^{1/6-\delta}$, one has

$$E(x, h) \ll xg(h)/h$$

for any function $g(h)$ tending to infinity with h. The current proof of part (ii) does not give such a result for larger h but using the method in the proof of part (i), one could get a slightly improved result also in this case.

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