This is the authors' version of the paper.
The final publication has appeared in Monatsh. Math. 164 (2011), 157–170.

Formulas for the number of gridlines

Anne-Maria Ernvall-Hytönen* Kaisa Matomäki
Pentti Haukkanen Jorma K. Merikoski

Abstract

Let $l(n)$ be the number of lines through at least two points of an $n \times n$ rectangular grid. We prove recursive and asymptotic formulas for it using respectively combinatorial and number theoretic methods. We also study the ratio $l(n)/l(n-1)$. All this originates from Mustonen's experimental results.

Keywords: Rectangular grid, Lattice points, Euler ϕ-function, Asymptotic formulas, Recursive formulas
MSC2010: 05A99, 11B37, 11N37, 11P21

1 Introduction

Let $l(n)$ be the number of lines through at least two points of an $n \times n$ rectangular grid. Sloane's database ([9], Sequence A018808) mentions an explicit formula

$$l(n) = \frac{1}{2}(f_1(n) - f_2(n)), \tag{1}$$

where

$$f_k(n) = \sum_{\substack{-n<i,j<n \\ (i,j)=k}} (n - |i|)(n - |j|) \tag{2}$$

and (i, j) denotes the greatest common divisor of i and j. There is no proof reference in the database but a proof of a generalization to $m \times n$ grids can be found in Mustonen's paper ([5], Section 3).

One would like to have a closed form expression for $l(n)$ instead of (1) which involves a double sum and does not in itself tell us much about the behaviour of $l(n)$. Besides, applying the formula (1) is computationally tedious. This motivated Mustonen [5] to investigate $l(n)$ experimentally and to state various conjectures concerning its behaviour. The aim of this paper is to widen the knowledge about $l(n)$ by proving Mustonen's conjectures.

We will first, in Section 2, prove recursive formulas for $l(n)$ using combinatorial arguments. In particular we show that the recursive formulas which Mustonen predicted in [5, Section 6] indeed hold.

*The first author is funded by the Swedish research council Vetenskapsrådet (grant 2009-721)

Next, in Section 3, we will study $l(n)$ asymptotically. Sheng [8] has shown that $l(n)$ is asymptotically equal to $9n^4/(4\pi^2)$. We will improve the error term in his asymptotic formula. We will also show that assuming the Riemann hypothesis we obtain a still better error term which corresponds to Mustonen's experimental result ([5], Section 4). Our improvements ultimately depend on known estimates on averages of averages of Euler ϕ-function.

Finally, in Section 4, we will study the asymptotic behaviour of the ratio $l(n)/l(n-1)$. We will confirm Mustonen's ([5], Section 4) prediction that the ratio is asymptotically decreasing unless all the prime factors of $n-1$ are large. The proof of this fact utilizes both recursive and asymptotic formulas for $l(n)$.

Before going further, we introduce some notation. Since we consider only integers, we write $[a, b] = \{x \in \mathbb{Z} \mid a \leq x \leq b\}$. Given integers $m, n \geq 2$, we say that a line l is a *gridline* of the rectangular grid $G = G(m, n) = [0, m-1] \times [0, n-1]$ if it goes through at least two points of G. We also say that l then *lives* in G. We write $l(m, n)$ for the number of these gridlines. In particular $l(n, n) = l(n)$.

2 Recursive formulas for $l(n)$

We first sketch, how Mustonen ([5], Section 6) experimentally found recursive formulas for $l(n)$ and $l(n-1, n)$.

Since $G(n) = G(n, n)$ can be constructed from $G(n-1)$ by adding first a new column of $n-1$ points and then a new row of n points, it is natural to look for a relation between $l(n)$, $l(n-1, n)$ and $l(n-1)$. Consider the data where $n = 3, 4, ..., 35$. A linear regression analysis suggests that

$$l(n) \approx 2l(n-1, n) - l(n-1). \tag{3}$$

The residuals

$$r(n) = l(n) - 2l(n-1, n) + l(n-1)$$

are strictly increasing except for every fourth n where the same value appears twice. This motivates to study differences

$$r(n) - r(n-1).$$

Indeed Mustonen found a simple representation for this difference (which is (6) below).

To make (4) practically applicable, a recursive formula must be found also for $l(n-1, n)$. Mustonen studied this as well and found analogously to (3) that

$$l(n-1, n) \approx 2l(n-1, n-1) - l(n-2, n-1).$$

He also found a formula for

$$s(n) = l(n-1, n) - 2l(n-1, n-1) + l(n-2, n-1)$$

(see (8) below).

We will rigorously prove the following theorem which shows that Mustonen's experimental formulas indeed hold for all $n \geq 2$.

Theorem 1. *For all $n \geq 2$,*

$$l(n) = 2l(n-1,n) - l(n-1) + r(n), \tag{4}$$

$$l(n-1,n) = 2l(n-1) - l(n-2,n-1) + s(n). \tag{5}$$

Here

$$r(n) = r(n-1) + 4(\phi(n-1) - e(n)), \tag{6}$$

$$e(n) = \begin{cases} 0 & \text{if } n \text{ is even,} \\ \phi(\frac{n-1}{2}) & \text{if } n \text{ is odd,} \end{cases}$$

or explicitly

$$r(n) = 4\sum_{i=1}^{n-1}\phi(i) - 4\sum_{i=1}^{\lfloor\frac{n-1}{2}\rfloor}\phi(i) = 4\sum_{i=\lfloor\frac{n+1}{2}\rfloor}^{n-1}\phi(i), \tag{7}$$

and

$$s(n) = \begin{cases} (n-1)\phi(n-1) & \text{if } n \text{ is even,} \\ \frac{1}{2}(n-1)\phi(n-1) & \text{if } n \equiv 1 \pmod 4, \\ 0 & \text{if } n \equiv 3 \pmod 4, \end{cases} \tag{8}$$

and $l(0) = l(0,1) = r(1) = 0$ and $l(1) = 1$.

Proof. First, we prove (4). Let us call

3-lines of $G(n)$ the lines through exactly three points of $G(n) = [0, n-1] \times [0, n-1]$, one of them in $[1, n-1] \times \{0\}$ and one in $\{0\} \times [1, n-1]$,

2-lines of $G(n)$ the lines through exactly two points of $G(n)$, located in the boundaries of $G(n)$ as above, and

1-lines of $G(n)$ the lines through the origin and exactly one other point of $G(n)$.

To find $l(n)$ recursively, we first add the numbers of lines living in $[1, n-1] \times [0, n-1]$ and, respectively, in $[0, n-1] \times [1, n-1]$. The result is $2l(n-1, n)$, but certain lines have been counted twice, namely

the $l(n-1)$ lines living in $[1, n-1] \times [1, n-1]$

and

the 3-lines in $G(n)$; let their number be $r_3(n)$.

On the other hand, the 2- and 1-lines of $G(n)$ have been ignored; let their numbers be respectively $r_2(n)$ and $r_1(n)$. In conclusion, we have

$$l(n) = 2l(n-1,n) - l(n-1) - r_3(n) + r_2(n) + r_1(n). \tag{9}$$

We still have to find recursive formulas for r_1, r_2 and r_3.

Let us study $r_3(n)$. All 3-lines in $G(n-1)$ are 3-lines also in $G(n)$. So we obtain $r_3(n)$ by adding to $r_3(n-1)$ the number of those 3-lines in $G(n)$ that go through $P = (0, n-1)$ or $Q = (n-1, 0)$.

If n is even, consider a line $l = PS$ or $l = QT$ where $S \in [0, n-1] \times \{0\}$, $T \in \{0\} \times [0, n-1]$. Then l meets $G(n)$ at even number of points, and so it is not a 3-line. Hence $r_3(n) = 0$.

Now assume that n is odd. A line $l = PS$ is a 3-line if and only if it meets $G(n)$ at a point $M_l = (m_l, \frac{n-1}{2})$ where $m_l \in \{1, ..., \frac{n-1}{2}\}$ satisfies $\gcd(m_l, \frac{n-1}{2}) = 1$. The number of such lines is the number of the m_l's, that is $\phi(\frac{n-1}{2})$. Since there are also $\phi(\frac{n-1}{2})$ 3-lines QT, we have $r_3(n) = 2\phi(\frac{n-1}{2})$.

In all,

$$r_3(n) = r_3(n-1) + \begin{cases} 0 & \text{if } n \text{ is even,} \\ 2\phi(\frac{n-1}{2}) & \text{if } n \text{ is odd.} \end{cases} \tag{10}$$

Using similar ideas, it can be shown that

$$r_2(n) = r_2(n-1) + 2\phi(n-1) \tag{11}$$

and

$$r_1(n) = r_1(n-1) + 2\phi(n-1) - \begin{cases} 0 & \text{if } n \text{ is even,} \\ 2\phi(\frac{n-1}{2}) & \text{if } n \text{ is odd.} \end{cases} \tag{12}$$

Substituting (10), (11), and (12) in (9), we obtain (4).

Second, we prove (5). Consider $G(n, n-1) = [0, n-1] \times [0, n-2]$. To find $l(n-1, n)$ recursively, we first add the numbers of lines living in $G(n-1)$ and, respectively, in $[1, n-1] \times [0, n-2]$. The result is $2l(n-1)$, but the following lines have been counted twice:

> the $l(n-2, n-1)$ lines living in $[1, n-2] \times [0, n-2]$

and

> the lines going through a point $P \in \{0\} \times [0, n-2]$, a point $Q \in \{n-1\} \times [0, n-2]$ and exactly one other point of $G(n, n-1)$; let their number be $s_2(n)$.

On the other hand, such lines PQ that do not meet $G(n, n-1)$ in any other point have been ignored; let their number be $s_1(n)$. Now

$$l(n-1, n) = 2l(n-1) - l(n-2, n-1) - s_2(n) + s_1(n). \tag{13}$$

Let us study $s_1(n)$. Choose first $P = (0, i)$. The point $Q = (n-1, j)$ applies if and only if $(i-j, n-1) = 1$. Since the number of such Q's is $\phi(n-1)$, there are $\phi(n-1)$ acceptable lines through P. Because P can be chosen in $n-1$ ways, we have

$$s_1(n) = (n-1)\phi(n-1). \tag{14}$$

A similar reasoning gives

$$s_2(n) = \begin{cases} 0 & \text{if } n \text{ is even,} \\ (n-1)\phi(\frac{n-1}{2}) & \text{if } n \text{ is odd.} \end{cases} \tag{15}$$

Now (5) follows by substituting (15) and (14) in (13).

Finally, we note that (7) can be shown simply by solving $r(n)$ from the recursive equation (6) with initial condition $r(1) = 0$. □

We now unite (4) and (5) into a single recursive formula for $l(n)$ only.

Theorem 2. *For all $n \geq 2$,*

$$l(n) = l(n-1) + 2\sum_{i=1}^{n} s(i) + 2\sum_{i=1}^{n-1} r(i) + r(n). \tag{16}$$

Proof. By adding (4) and (5), we obtain

$$l(n) = l(n-1) + l(n-1,n) - l(n-2,n-1) + r(n) + s(n),$$

and further, replacing n with $n-1, n-2, ..., 2$ gives

$$l(n) = l(n-1,n) + \sum_{i=1}^{n}(r(i) + s(i)). \tag{17}$$

On the other hand, by (4),

$$l(n-1) = 2l(n-2,n-1) - l(n-2) + r(n-1).$$

Adding this to (5) and proceeding as above yields

$$l(n-1,n) = l(n-1) + l(n-2,n-1) - l(n-2) + r(n-1) + s(n),$$

and further

$$l(n-1,n) = l(n-1) + \sum_{i=1}^{n-1} r(i) + \sum_{i=1}^{n} s(i). \tag{18}$$

Substituting (18) into (17) gives (16). □

3 An asymptotic formula for $l(n)$

Asymptotic behaviour of $l(n)$ has been studied by Sheng [8]. His more general Lemma 7 implies the following result.

Theorem 3. *For all $n \geq 2$,*

$$l(n) = \frac{3n^4}{8\zeta(2)} + O(n^3 \log n) = \frac{9n^4}{4\pi^2} + O(n^3 \log n). \tag{19}$$

Here $\zeta(s)$ denotes Riemann's zeta function.

In this paper we are able to improve the error term as follows.

Theorem 4. *Let $n \geq 2$. Then*

$$l(n) = \frac{9n^4}{4\pi^2} + O(n^3 \exp(-A(\log n)^{\frac{3}{5}}(\log\log n)^{-\frac{1}{5}}))$$

for certain constant $A > 0$. Assuming the Riemann hypothesis we have

$$l(n) = \frac{9n^4}{4\pi^2} + O(n^{\frac{5}{2}+\varepsilon}) \tag{20}$$

for any $\varepsilon > 0$.

Based on his experiments, Mustonen ([5], Section 4) has predicted earlier that (20) indeed holds.

We prove Theorem 4 using the presentation (1) for $l(n)$, so we study $f_k(n)$ defined by (2). We need the following elementary lemma (which is [1, Exercise 2.16]).

Lemma 5. *For all* $n \geq 2$,

$$\sum_{\substack{i=1\\(i,n)=1}}^{n} i = \tfrac{1}{2} n\phi(n).$$

Proof. Simply note that

$$2 \sum_{\substack{i=1\\(i,n)=1}}^{n} i = \sum_{\substack{i=1\\(i,n)=1}}^{n} i + \sum_{\substack{i=1\\(i,n)=1}}^{n} (n-i) = \sum_{\substack{i=1\\(i,n)=1}}^{n} n = n\phi(n).$$

□

Now we are able to dispose of the double summation in (2).

Lemma 6. *Let* $k \geq 1$. *For all* $n \geq 2$,

$$f_k(n+1) = 8 \sum_{i=1}^{\lfloor n/k \rfloor} (n+1-ki)(n+1-\tfrac{ki}{2})\phi(i)$$

Proof. We have

$$\begin{aligned} f_k(n+1) &= \sum_{\substack{-n\leq i,j\leq n\\(i,j)=k}} (n+1-k|i|)(n+1-k|j|) \\ &= 4 \sum_{\substack{1\leq i,j\leq \lfloor n/k \rfloor\\(i,j)=1}} (n+1-ki)(n+1-kj) + 4(n+1-k)(n+1) \\ &= 8 \sum_{i=2}^{\lfloor n/k \rfloor} (n+1-ki) \sum_{\substack{j=1\\(j,i)=1}}^{i} (n+1-kj) \\ &\quad + 4(n+1-k)^2 + 4(n+1-k)(n+1). \end{aligned}$$

By Lemma 5 we see that

$$\begin{aligned} f_k(n+1) &= 8 \sum_{i=2}^{\lfloor n/k \rfloor} (n+1-ki)(n+1-\tfrac{ki}{2})\phi(i) + 4(n+1-k)(2n+2-k) \\ &= 8 \sum_{i=1}^{\lfloor n/k \rfloor} (n+1-ki)(n+1-\tfrac{ki}{2})\phi(i). \end{aligned}$$

□

Next we apply the partial summation formula

$$\sum_{i=1}^{N} a_i b_i = \big(\sum_{i=1}^{N} a_i\big) b_N - \sum_{i=1}^{N-1} \big(\sum_{j=1}^{i} a_j\big)(b_{i+1} - b_i) \tag{21}$$

to reach sums

$$\Phi(k) = \sum_{i=1}^{k} \phi(i).$$

Lemma 7. *For all $n \geq 2$,*

$$f_1(n+1) = 4(n+2) \sum_{i=1}^{n} \Phi(i) + 8 \sum_{i=1}^{n-1} \sum_{j=1}^{i} \Phi(j). \tag{22}$$

If n is odd, then

$$f_2(n+1) = 8(n+3) \sum_{i=1}^{\frac{n-1}{2}} \Phi(i) + 32 \sum_{i=1}^{\frac{n-3}{2}} \sum_{j=1}^{i} \Phi(j). \tag{23}$$

If n is even, then

$$f_2(n+1) = 4(n+2)\Phi(\tfrac{1}{2}n) + 8(n+5) \sum_{i=1}^{\frac{n}{2}-1} \Phi(i) + 32 \sum_{i=1}^{\frac{n}{2}-2} \sum_{j=1}^{i} \Phi(j). \tag{24}$$

Proof. By Lemma 6

$$f_1(n+1) = 8 \sum_{i=1}^{n+1} (n+1-i)(n+1-\tfrac{1}{2}i)\phi(i).$$

To show (22), we start by applying partial summation (21) with

$$N = n+1, \quad a_i = \phi(i), \quad b_i = (n+1-i)(n+1-\tfrac{1}{2}i).$$

This gives

$$\begin{aligned} f_1(n+1) &= -8 \sum_{i=1}^{n} \{(n-i)[n+1-\tfrac{1}{2}(i+1)] - (n+1-i)(n+1-\tfrac{1}{2}i)\}\Phi(i) \\ &= 8 \sum_{i=1}^{n} (\tfrac{3}{2}n - i + 1)\Phi(i). \end{aligned}$$

Then we apply partial summation (21) with

$$N = n, \quad a_i = \Phi(i), \quad \text{and} \quad b_i = \tfrac{3}{2}n - i + 1$$

getting

$$f_1(n+1) = 8(\tfrac{1}{2}n+1)\sum_{i=1}^{n}\Phi(i) - 8\sum_{i=1}^{n-1}(-1)\sum_{j=1}^{i}\Phi(j)$$
$$= 4(n+2)\sum_{i=1}^{n}\Phi(i) + 8\sum_{i=1}^{n-1}\sum_{j=1}^{i}\Phi(j).$$

Similarly, for even n, we get

$$f_2(n+1) = 8\sum_{i=1}^{\frac{n}{2}}(n+1-2i)(n+1-i)\phi(i)$$
$$= 8(\tfrac{n}{2}+1)\Phi(\tfrac{n}{2}) + 8\sum_{i=1}^{\frac{n}{2}-1}(3n-4i+1)\Phi(i)$$
$$= 4(n+2)\Phi(\tfrac{n}{2}) + 8(n+5)\sum_{i=1}^{\frac{n}{2}-1}\Phi(i) + 8\cdot 4\sum_{i=1}^{\frac{n}{2}-2}\sum_{j=1}^{i}\Phi(j).$$

The case of odd n can be handled analogously. □

Remark 8. Lemma 7 would also follow from Theorem 2. However we have decided to take a more analytic path in proving the asymptotic formula.

Let us now define

$$E_\Phi(n) = \Phi(n) - \frac{3n^2}{\pi^2}. \tag{25}$$

At this point we would get Sheng's result (Theorem 3) by applying the following classical result (see e.g. [4], § I.21).

Lemma 9. *For all $n \geq 2$,*

$$E_\Phi(n) = O(n \log n).$$

This has been improved and so we could already reach a refinement of Theorem 3.

Lemma 10 (Walfisz [12], p. 144, Satz 1)**.** *For all $n \geq 2$,*

$$E_\Phi(n) = O(n(\log n)^{\frac{2}{3}}(\log\log n)^{\frac{4}{3}}).$$

Saltykov [7] presented a sharper formula but its correctness was controversial; see [2], p. 314. Finally Pétermann ([6], Section 5) falsified it.

To continue our journey toward proof of Theorem 4, we consider averages of averages of Euler ϕ-function. To this end we define

$$E_R(n) = \sum_{i=1}^{n} E_\Phi(i) - \frac{3n^2}{2\pi^2}.$$

Next we express f_1 and f_2 using E_Φ and E_R.

Lemma 11. *For all $n \geq 2$,*

$$f_1(n+1) = \frac{6}{\pi^2}(n+1)^4 + 4(n+2)E_R(n) + 8\sum_{i=1}^{n-1} E_R(i) + O(n^2).$$

If n is odd, then

$$f_2(n+1) = \frac{3}{2\pi^2}(n+1)^4 + 8(n+3)E_R(\frac{n-1}{2}) + 32\sum_{i=1}^{\frac{n-3}{2}} E_R(i) + O(n^2).$$

If n is even, then

$$\begin{aligned} f_2(n+1) = &\frac{3}{2\pi^2}(n+1)^4 + 8(n+5)E_R(\tfrac{1}{2}n-1) \\ &+ 32\sum_{i=1}^{\frac{n}{2}-2} E_R(i) + 4(n+2)E_\Phi(\tfrac{1}{2}n) + O(n^2). \end{aligned}$$

Proof. By definitions of $E_\Phi(n)$ and $E_R(n)$ we have

$$\begin{aligned} \sum_{k=1}^{K} \Phi(k) = \sum_{k=1}^{K}\left(\frac{3k^2}{\pi^2} + E_\Phi(k)\right) &= \frac{3}{\pi^2}\cdot\frac{K(K+1)(2K+1)}{6} + \frac{3K^2}{2\pi^2} + E_R(K) \\ &= \frac{K^3+3K^2}{\pi^2} + E_R(K) + O(K) \end{aligned}$$

and

$$\begin{aligned} \sum_{l=1}^{L}\sum_{k=1}^{l} \Phi(k) &= \sum_{l=1}^{L}\left(\frac{l^3+3l^2}{\pi^2} + E_R(l) + O(l)\right) \\ &= \frac{L^2(L+1)^2}{4\pi^2} + \frac{L(L+1)(2L+1)}{2\pi^2} + \sum_{l=1}^{L} E_R(l) + O(L^2) \\ &= \frac{L^4+6L^3}{4\pi^2} + \sum_{l=1}^{L} E_R(l) + O(L^2). \end{aligned}$$

The claims follow by substituting these into Lemma 7. □

Replacing n with $n-1$, we have the following

Theorem 12. *Let $n \geq 3$. Define $E_l(n)$ by*

$$l(n) = \frac{9n^4}{4\pi^2} + E_l(n). \tag{26}$$

If n is even, then

$$\begin{aligned} E_l(n) = &2(n+1)E_R(n-1) - 4(n+2)E_R(\tfrac{1}{2}n-1) \\ &- 12\sum_{i=1}^{\frac{n}{2}-2} E_R(i) + 4\sum_{i=\frac{n}{2}-1}^{n-2} E_R(i) + O(n^2). \end{aligned} \tag{27}$$

If n is odd, then

$$\begin{aligned} E_l(n) &= 2(n+1)E_R(n-1) - 4(n+4)E_R(\frac{n-3}{2}) \\ &\quad - 12\sum_{i=1}^{\frac{n-5}{2}} E_R(i) + 4\sum_{i=\frac{n-3}{2}}^{n-2} E_R(i) - 2(n+1)E_\Phi(\frac{n-1}{2}) + O(n^2). \end{aligned} \tag{28}$$

In a sense, (26), (27) and (28) together give the best possible asymptotic formula for $l(n)$. But to apply it in practice requires knowledge about E_Φ and E_R. We already mentioned results concerning E_Φ. Next lemma gives us the necessary information about E_R.

Lemma 13. *Let $n \geq 2$. Then*

$$E_R(n) = O(n^2 \exp[-A(\log n)^{\frac{3}{5}}(\log\log n)^{-\frac{1}{5}}])$$

for certain constant $A > 0$. Furthermore,

$$E_R(n) = O(n^{\frac{3}{2}+\varepsilon})$$

for all $\varepsilon > 0$ if and only if the Riemann hypothesis is true.

Proof. The first result is due to Suryanarayana and Sitaramachandra Rao [11, eq. (1.11)]. The if-part of the second claim was proved by Suryanarayana [10, eq. (3.41)] (there should be $2\pi^2$ instead of π^2 there). The converse was shown to hold by Codecà [3]. □

Now we can turn to E_l. By Theorem 12,

$$E_l(n) = O\left(\max_{m\leq n}(n|E_R(m)| + n|E_\Phi(m)|) + n^2\right).$$

Theorem 4 follows now from Lemmas 9 and 13. □

Conversely, does (20) imply the Riemann hypothesis? This would give an interesting geometric characterization of the Riemann hypothesis. Techniques used in [3] are likely to work also in this case.

4 The ratio $l(n)/l(n-1)$

Mustonen ([5], Section 4) showed experimentally that the function $\frac{l(n)}{l(n-1)}$ is asymptotically decreasing unless all the prime factors of $n-1$ are large. We prove this conjecture:

Theorem 14. *If*

$$\frac{l(n)}{l(n-1)} > \frac{l(n-1)}{l(n-2)}, \tag{29}$$

then n is even and

$$\frac{\phi(n-1)}{n} > \frac{9}{\pi^2} + o(1). \tag{30}$$

Let $p_1, ..., p_k$ be the distinct prime factors of $n-1$. Then,

$$\frac{\phi(n-1)}{n} \approx \frac{\phi(n-1)}{n-1} = \left(1 - \frac{1}{p_1}\right) \cdots \left(1 - \frac{1}{p_k}\right),$$

and so (30) is asymptotically equivalent with

$$\left(1 - \frac{1}{p_1}\right) \cdots \left(1 - \frac{1}{p_k}\right) > \frac{9}{\pi^2}.$$

of Theorem 14. Consider the difference

$$\frac{l(n)}{l(n-1)} - \frac{l(n-1)}{l(n-2)} = \frac{l(n)l(n-2) - l(n-1)^2}{l(n-2)l(n-1)}.$$

As the denominator is positive, it is enough to study the numerator. By Theorem 2,

$$l(n) - l(n-1) = l(n-1) - l(n-2) + 2s(n) + r(n-1) + r(n).$$

Hence

$$\begin{aligned} l(n)l(n-2) - l(n-1)^2 &= l(n)(l(n-2) - l(n-1)) + l(n-1)(l(n) - l(n-1)) \\ &= l(n)(2s(n) + r(n-1) + r(n)) - (l(n) - l(n-1))^2. \end{aligned}$$

To continue, notice that (7) and (25) imply $r(n) = \frac{9n^2}{\pi^2} + O(E_\Phi(n))$, and so

$$r(n-1) + r(n) = \frac{18n^2}{\pi^2} + O(|E_\Phi(n)| + |E_\Phi(n-1)|).$$

Also, we have

$$l(n) - l(n-1) = \frac{9n^3}{\pi^2} + O(|E_l(n)| + |E_l(n-1)|).$$

Consequently,

$$\begin{aligned} l(n)(2s(n) + r(n-1) + r(n)) - (l(n) - l(n-1))^2 &= -\tfrac{81}{2\pi^4}n^6 + \tfrac{9}{2\pi^2}n^4 s(n) \\ + O(n^4 (|E_\Phi(n)| + |E_\Phi(n-1)|)) &+ O(n^3(|E_l(n)| + |E_l(n-1)|)) \\ + O((|E_\Phi(n)| + |E_\Phi(n-1)|)\, E_l(n)) &+ O(E_l(n)^2 + E_l(n-1)^2). \end{aligned}$$

We know from Section 3 that

$$E_\Phi(n)n^4,\ E_l(n)n^3,\ E_\Phi(n)E_l(n),\ E_l(n)^2 = o(n^6),$$

so (29) can hold only if the main term is $> -o(n^6)$. If n is even, substituting $s(n) = (n-1)\phi(n-1) = n\phi(n-1) + O(n)$ we get the asymptotic condition:

$$\frac{\phi(n-1)}{n} > \frac{9}{\pi^2}.$$

On the other hand, if n is odd, then $s(n) = \frac{1}{2}(n-1)\phi(n-1)$ or $= 0$ depending on the residue of n modulo 4. Asymptotically, the condition never holds as

$$\frac{9}{\pi^2} > \frac{\phi(n-1)}{2n}.$$

Experiments also falsify (29) for small odd values of n. □

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A-M. Ernvall-Hytönen
Institutionen för matematik
Kungliga Tekniska Högskolan
100 44 Stockholm, Sweden.
hytonen@kth.se

K. Matomäki
Department of Mathematics
20014 University of Turku
Finland.
ksmato@utu.fi

P. Haukkanen and J. K. Merikoski

Department of Mathematics and Statistics
33014 University of Tampere
Finland
pentti.haukkanen@uta.fi and jorma.merikoski@uta.fi