Math. Proc. Camb. Phil. Soc. (2009), **147**, 267 © 2009 *Cambridge Philosophical Society*
doi:10.1017/S030500410900245X *Printed in the United Kingdom*
First published online 15 April 2009

The distribution of αp modulo one

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(*Received* 28 *November* 2008)

Abstract

We prove that, for any irrational number α, there are infinitely many primes p such that $\|\alpha p\| < p^{-1/3+\epsilon}$. Here $\|y\|$ denotes the distance from y to the nearest integer. The proof uses Harman's sieve method with arithmetical information coming from bounds for averages of Kloosterman sums.

1. *Introduction*

Let α be an irrational number and let $\|y\|$ denote the distance from y to the nearest integer. In this paper we will investigate when the Diophantine inequality

$$\|\alpha p + \beta\| < p^{-\tau}$$

has infinitely many prime solutions p for $\beta = 0$. We prove the following theorem.

THEOREM 1. *Let $\epsilon > 0$ and $\tau = 1/3 - \epsilon$. Then there exist infinitely many primes p such that*

$$\|\alpha p\| < p^{-\tau}. \tag{1·1}$$

The first result of this form was obtained by Vinogradov [**12**] with the exponent $-1/5 + \epsilon$ and the latest published result is the exponent $-16/49 + \epsilon$ due to Heath–Brown and Jia [**7**]. Mikawa has claimed a proof of $-1/3$ using a different method (see [**9**]) but it has never been published.

We start by defining a set whose prime elements satisfy (1·1). Let a/q be a convergent to the continued fraction for α with a large enough denominator. Let

$$x = q^{2/(1+\tau)} \quad \text{and} \quad R = x^{(1-\tau)/2},$$

so that $q = x^{2/3-\epsilon/2}$ and $R = x^{1/3+\epsilon/2}$.

We write

$$\mathcal{A} = \{n \mid x < n \leqslant 2x,\ (n, q) = 1,\ an \equiv r \ (\mathrm{mod}\, q) \text{ for some } r \in [1, 3R]\}. \tag{1·2}$$

If $n \in \mathcal{A}$, then

$$\|\alpha n\| \leqslant \left\| \left(\alpha - \frac{a}{q}\right) n \right\| + \left\| \frac{a}{q} n \right\| \leqslant \frac{2x}{q^2} + \frac{3R}{q} \ll n^{-\tau},$$

so it is enough to find a non-trivial lower bound for the number of primes in $\mathcal{A}$. To this end we will use Harman's sieve method. Actually Harman introduced his sieve method while considering the current problem in [**2**], where he proved the theorem with the exponent $-3/10$. He further developed the sieve method in [**3**] leading to the exponent $-7/22$. For a comprehensive account of Harman's sieve method, see [**4**].

When we apply the sieve method in Section 5, we need information about the sum

$$\sum_{\substack{mn\in\mathcal{A}\\ m\sim M, n\sim N}} a(m)b(n), \tag{1·3}$$

where the coefficients are supported on integers that are co-prime to q. The notation $m \sim M$ means that $M < m \leqslant 2M$. We will also use the notation $m \asymp M$ to indicate that $cM < m < CM$ for some positive constants c and C.

We write $\mathcal{B} = (x, 2x] \bigcap \mathbb{N}$. Then one would expect that (1·3) is asymptotically equivalent to

$$\frac{3R}{q} \sum_{\substack{mn\in\mathcal{B}\\ m\sim M, n\sim N}} a(m)b(n). \tag{1·4}$$

We will assume that all the coefficient $a(m), b(n), \ldots$ are bounded. The sequences in Section 5 are actually only divisor-bounded, so that the coefficients are $\leqslant x^{\eta}$. However, this makes no difference since in all asymptotic formulae that we use the order of the error term is $x^{-\delta}$ times the main term and we can take $\eta < c\delta$ for a very small positive constant c.

We call sums in which one of the coefficients is the characteristic function of an interval type I sums. Here and later the characteristic function is taken to be supported only on integers that are co-prime to q. Such conditions can be handled using Möbius inversion. Sums with arbitrary coefficients are called type II sums.

A classical way to show an asymptotic formula relating (1·3) to (1·4) is to use a Fourier expansion to transfer the problem to that of considering the exponential sum

$$\sum_{h=1}^{H} \left| \sum_{mn\in\mathcal{B}} a(m)b(n)e(\alpha hmn) \right|. \tag{1·5}$$

All the results, until Heath–Brown's and Jia's work [**7**], were based on Vinogradov's estimates on these trigonometric type I and type II sums.

The advantage of methods depending only on estimates on (1·5) is that the results hold for arbitrary $\beta \in \mathbb{R}$. Vaughan's arguments in [**11**] lead to the following type I and type II information.

LEMMA 2. *There exists $C > 0$ such that for any $\delta > 0$*

$$\sum_{\substack{mn\in\mathcal{A}\\ m\sim M, n\sim N}} a(m)b(n) = \frac{3R}{q} \sum_{\substack{mn\in\mathcal{B}\\ m\sim M, n\sim N}} a(m)b(n) + O(x^{1-\tau-2\delta}) \tag{1·6}$$

for $x^{2\tau+C\delta} \leqslant M \leqslant x^{1-\tau-C\delta}$.

LEMMA 3. *Let $\mathcal{I}$ be a subinterval of $[N, 2N]$. Assume that*

$$b(n) = \begin{cases} 1 & \text{if } n \in \mathcal{I} \text{ and } (n, q) = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Then there exists $C > 0$ such that (1·6) holds for $M \leqslant x^{1-\tau-C\delta}$.

The problem with the type II information coming from Lemma 2 is that it disappears as τ approaches $1/3$. Our methods will not give much more genuine type II information but instead we have results for tri-linear sums, that is sums with

$$a(m) = \sum_{\substack{m=yz \\ y\sim Y, z\sim Z}} c(y)d(z). \tag{1·7}$$

Heath–Brown and Jia reduced showing (1·6) to estimation of sums with Kloosterman fractions and used a result from the geometry of numbers to prove the following type II result [**7**, lemma 6].

LEMMA 4. *Let*

$$N \ll q \quad \textit{and} \quad Y^5Z^5(Y+Z) \ll x^2.$$

Then (1·6) *holds.*

We will follow Heath–Brown and Jia's approach, but use estimates on averages of Kloosterman sums from [**1**] to get more type I and type II information. Our results essentially supersede other new arithmetical information obtained by Heath–Brown and Jia.

2. *Reduction of the problem*

In the introduction we simplified matters a bit. Indeed we need to add some smooth weights in order to use results on averages of Kloosterman sums.

Definition 5. We call a function $f(x)$ a *test function* on interval $[X, 2X]$ if $f(x) \in [0, 1]$ for all x,

$$f(x) = \begin{cases} 1 & \text{if } x \in [X, 2X], \\ 0 & \text{if } x \notin [X/2, 3X], \end{cases}$$

$f(x)$ is smooth and its derivatives satisfy $f^{(j)}(x) \ll X^{-j}$. The definiton is extended to multivariable functions in an obvious way.

Let $h(r)$ be a test function on $[R, 2R]$. We modify the definion of $\mathcal{A}$ in (1·2) a bit by letting each element of $\mathcal{A}$ to be counted with weight $h(r)$. Then we would expect that (1·3) is asymptotically equivalent to

$$\frac{\hat{h}(0)}{q} \sum_{\substack{mn\in\mathcal{B} \\ m\sim M, n\sim N}} a(m)b(n),$$

where

$$\hat{h}(x) = \int_{-\infty}^{\infty} h(y)e(xy)\, dy$$

is the Fourier transform of $h(r)$. Lemmas 2, 3 and 4 can be easily modified to this situation using partial summation.

In this section we reduce the problem of getting an asymptotic formula into the problem of bounding certain exponential sums. In the following proposition and later we write η

and δ for small enough positive constants satisfying $\eta < \delta < \epsilon$. The constant η is not necessarily the same at each occurrence. Further, for complex coefficients $\overline{a(m)}$ means the complex conjugate, but for summation variables and other integers $\overline{m}$ usually means the multiplicative inverse.

PROPOSITION 6. *Define*

$$\Omega_2 = \sum_{\substack{m_1', m_2' \sim M' \\ (m_1', m_2')=1 \\ |\mu(\beta m_1' m_2')|=1}} a(\rho m_1') \overline{a(\rho m_2')} \sum_{\substack{|k| \sim K \\ |l| \sim L}} g(|k|, |l|, m_1', m_2') e\left(\frac{-qkl\overline{m}_{2'}}{m_1'}\right).$$

Assume that $M, N \ll q$ and that for any positive integer $\rho = \beta d^2 \leqslant M$ we have for $M' = M/\rho$, $K \leqslant M'^{1+\eta}/R$ and $L \leqslant 10RM'/q$ the bound

$$\Omega_2 \ll M'^2 x^{\frac{1-3\tau}{2} - 5\delta} \tag{2·1}$$

whenever $g(k, l, m_1', m_2')$ is a smooth function supported on $[K, 2K] \times [L, 2L] \times [M', 2M'] \times [M', 2M']$ satisfying

$$\frac{\partial^{j_1+j_2+j_3+j_4} g}{\partial k^{j_1} \partial l^{j_2} \partial m_1'^{j_3} \partial m_2'^{j_4}} \ll x^{(j_1+j_2)\eta} K^{-j_1} L^{-j_2} M'^{-j_3-j_4} \quad \textit{for } j_i \geqslant 0.$$

Then

$$\sum_{\substack{mn \in \mathcal{A} \\ m \sim M, n \sim N}} a(m)b(n) = \frac{\hat{h}(0)}{q} \sum_{\substack{mn \in \mathcal{B} \\ m \sim M, n \sim N}} a(m)b(n) + O(x^{1-\tau-2\delta}). \tag{2·2}$$

Remark 7. The proof of the proposition can be easily modified to show that a similiar claim holds if the summation condition $|\mu(\beta m_1' m_2')| = 1$ is replaced by $|\mu(\beta' z_1' z_2')| = 1$. Some modifications to the statement are necessary, but they are obvious from the proof. For instance $a(\rho m_i')$ in Ω_2 have to be replaced by $c(\rho_{Y_i} y_i')$ and $d(\rho_{Z_i} z_i')$ by $\rho_{Y_i} \leqslant Y$ and $\rho_{Z_i} \leqslant Z$.

Proof of the proposition. We start by writing

$$\Omega_1 = \sum_{\substack{mn \in \mathcal{A} \\ m \sim M, n \sim N}} a(m)b(n) - \frac{\hat{h}(0)}{q} \sum_{\substack{mn \in \mathcal{B} \\ m \sim M, n \sim N}} a(m)b(n)$$

$$= \sum_{d \leqslant 2M^{1/2}} \sum_{\substack{m \sim M/d^2 \\ |\mu(m)|=1}} \sum_{n \sim N} a(md^2)b(n) \left(\Phi(md^2, n) - \frac{\hat{h}(0)}{q} \right),$$

where

$$\Phi(m, n) = \sum_{r \equiv amn \,(\mathrm{mod}\, q)} h(r).$$

Here we have eliminated the condition $mn \sim x$ that can be done by Perron's formula.

Following Heath–Brown and Jia [**7**, section 3] we use the dispersion method. By the Cauchy–Schwarz inequality

$$|\Omega_1|^2 \ll N \sum_{n\sim N} \sum_{d\leqslant 2M^{1/2}} d^{1+\eta} \left| \sum_{\substack{m\sim M/d^2\\ |\mu(m)|=1}} a(md^2)\left(\Phi(md^2,n)-\frac{\hat{h}(0)}{q}\right)\right|^2$$

$$\leqslant N \sum_{d\leqslant 2M^{1/2}} d^{1+\eta} \sum_{n \,(\mathrm{mod}\, q)} \left| \sum_{\substack{m\sim M/d^2\\ |\mu(m)|=1}} a(md^2)\left(\Phi(md^2,n)-\frac{\hat{h}(0)}{q}\right)\right|^2$$

$$= N \sum_{d\leqslant 2M^{1/2}} d^{1+\eta} \sum_{\substack{m_i\sim M/d^2\\ |\mu(m_i)|=1}} a(m_1d^2)\overline{a(m_2d^2)}\Bigg(\sum_{n\,(q)} \Phi(m_1d^2,n)\Phi(m_2d^2,n)$$

$$-\sum_{n\,(q)}\Phi(m_1d^2,n)\frac{\hat{h}(0)}{q} - \sum_{n\,(q)}\Phi(m_2d^2,n)\frac{\hat{h}(0)}{q} + \sum_{n\,(q)}\frac{\hat{h}(0)^2}{q^2}\Bigg).$$

Here

$$\sum_{n\,(\mathrm{mod}\, q)} \Phi(md^2,n) = \sum_r h(r) = \hat{h}(0) + O(1)$$

and

$$\sum_{n\,(\mathrm{mod}\, q)} \Phi(m_1d^2,n)\Phi(m_2d^2,n) = \sum_{\substack{r,s\\ \overline{am_1d^2}r\equiv\overline{am_2d^2}s\,(q)}} h(r)h(s) = \sum_{\substack{r,s\\ rm_2\equiv sm_1\,(q)}} h(r)h(s).$$

Thus

$$|\Omega_1|^2 \ll N \sum_{d\leqslant 2M^{1/2}} d^{1+\eta} \left| \sum_{\substack{m_i\sim M/d^2\\ |\mu(m_i)|=1}} a(m_1d^2)\overline{a(m_2d^2)}\left(\Psi(m_1,m_2)-\frac{\hat{h}(0)^2}{q}\right)\right| + \frac{M^2NR}{q},$$

where

$$\Psi(m_1,m_2) = \sum_{\substack{r,s\\ rm_2\equiv sm_1\,(q)}} h(r)h(s) = \sum_{\substack{r,s\\ rm_2=sm_1+ql}} h(r)h(s).$$

We write $\beta=(m_1,m_2)$. Since $(m_1m_2,q)=1$, we must have $\beta \mid l$. We write $m_i=\beta m_i'$ and $l=\beta l'$. Then

$$\Psi(m_1,m_2) = \sum_{\substack{r,s,l'\\ rm_2'=sm_1'+ql'}} h(r)h(s) = \sum_{l'} \sum_{r\equiv ql'\overline{m_2'}\,(m_1')} h(r)h\left(\frac{rm_2'-ql'}{m_1'}\right)$$

$$= \sum_{l'} \sum_{r\equiv ql'\overline{m_2'}\,(m_1')} f(r,m_1',m_2',l'),$$

say. Notice that the sum is non-zero only if $|l'|\leqslant L_0 = 10RM/(\beta d^2 q)$ and further

$$\frac{\partial^{j_1+j_2+j_3+j_4} f}{\partial r^{j_1}\partial m_1'^{j_2}\partial m_2'^{j_3}\partial l'^{j_4}} \ll r^{-j_1}m_1'^{-j_2}m_2'^{-j_3}l'^{-j_4} \quad \text{for } j_i\geqslant 0.$$

By the Poisson summation formula

$$\Psi(m_1, m_2) = \sum_{0<|l'|<L_0} \frac{1}{m_1'} \sum_k \hat{f}\left(\frac{k}{m_1'}, m_1', m_2', l'\right) e\left(\frac{-ql'\overline{m_2'}k}{m_1'}\right) + O\left(\frac{R\beta d^2}{M}\right),$$

where

$$\begin{aligned}\hat{f}\left(\frac{k}{m_1'}, m_1', m_2', l'\right) &= \int_{-\infty}^{\infty} f(\xi, m_1', m_2', l') e\left(\frac{k\xi}{m_1'}\right) d\xi \\ &= \int_{R/2}^{3R} f(\xi, m_1', m_2', l') e\left(\frac{k\xi}{m_1'}\right) d\xi\end{aligned} \tag{2·3}$$

is the Fourier transform.

Terms with $k = 0$ contribute to $\Psi(m_1, m_2)$ the amount

$$\begin{aligned}&\sum_{0<|l'|<L_0} \frac{1}{m_1'} \int_{-\infty}^{\infty} h(x) h\left(\frac{xm_2' - ql'}{m_1'}\right) dx \\ &\quad = \frac{1}{m_1'} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(x) h\left(\frac{xm_2' - ql'}{m_1'}\right) dx dl' + O\left(\frac{R\beta d^2}{M}\right) \\ &\quad = \frac{\hat{h}(0)^2}{q} + O\left(\frac{R\beta d^2}{M}\right)\end{aligned}$$

and the contribution from terms with $|k| \geqslant K_0 = \left(\frac{M}{d^2\beta}\right)^{1+\eta} / R$ is $O(R\beta d^2/M)$.

Hence we see that

$$\begin{aligned}|\Omega_1|^2 \ll N \sum_{d \leqslant 2M^{1/2}} d^{1+\eta} \sum_{\beta \leqslant 2M} \Bigg| &\sum_{\substack{m_i' \sim M/(d^2\beta) \\ |\mu(\beta m_1' m_2')|=1}} \frac{a(m_1'\beta d^2)\overline{a(m_2'\beta d^2)}}{m_1'} \\ &\cdot \sum_{\substack{0<|k|<K_0 \\ 0<|l'|<L_0}} \hat{f}\left(\frac{k}{m_1'}, m_1', m_2', l'\right) e\left(\frac{-ql'\overline{m}_2'k}{m_1'}\right)\Bigg| + x^\eta MNR.\end{aligned}$$

Substituting the Fourier transform (2·3) we see that

$$\begin{aligned}|\Omega_1|^2 \ll \max_{\substack{d \leqslant 2M^{1/2} \\ \beta \leqslant 2M}} \frac{RNd^{4+\eta}\beta^2}{M} \max_{\xi \in [R/2, 3R]} \Bigg| &\sum_{\substack{m_i' \sim M/(d^2\beta) \\ |\mu(\beta m_1' m_2')|=1}} a(m_1'\beta d^2)\overline{a(m_2'\beta d^2)} \\ &\cdot \sum_{\substack{0<|k|<K_0 \\ 0<|l'|<L_0}} g_\xi(k, m_1', m_2', l') e\left(\frac{-ql'\overline{m_2'}k}{m_1'}\right)\Bigg| + MNR,\end{aligned}$$

where $g_\xi(k, m_1', m_2', l') = (M'/d^2\beta m_1') f(\xi, m_1', m_2', l') e(k\xi/m_1')$ satisfies

$$\frac{\partial^{j_1+j_2+j_3+j_4} g}{\partial k^{j_1} \partial m_1'^{j_2} \partial m_2'^{j_3} \partial l'^{j_4}} \ll x^{j_1\eta + j_2\eta} |k|^{-j_1} m_1'^{-j_2} m_2'^{-j_3} |l'|^{-j_4}.$$

Writing $\rho = d^2\beta$ and $M' = M/\rho$, we see that

$$|\Omega_1|^2 \ll x^\eta \max_{\beta d^2 \leqslant 2M} \frac{RN\rho}{M'} \max_{\xi \in [R/2, 3R]} \Bigg| \sum_{\substack{m_i' \sim M' \\ |\mu(\beta m_1' m_2')|=1}} a(\rho m_1')\overline{a(\rho m_2')} \times \sum_{\substack{0<|k|<K_0 \\ 0<|l'|<L_0}} g_\xi(k, m_1', m_2', l') e\left(\frac{-ql'\overline{m_2'}k}{m_1'}\right)\Bigg| + MNR.$$

We can split the summations over k and l to dyadic segments $k \sim K$ and $l \sim L$ keeping a smooth weight function. This can be done by writing the function g as a sum of its differences (see for instance [**13**, page 208] for this kind of argument). Hence by the assumption (2·1) we have

$$|\Omega_1| \ll \left(RN\rho M' x^{(1-3\tau)/2-5\delta+\eta} + MNR\right)^{1/2}$$
$$\ll x^{(2-2\tau-5\delta+\eta)/2} + x^{(3-\tau)/4} \ll x^{1-\tau-2\delta}.$$

Remark 8. It is clear from the end of the previous proof why we cannot do better than $\tau = 1/3 - \epsilon$.

3. *Averages of Kloosterman sums*

We will need to do some modifications to known results on averages of Kloosterman sums. First we mention that any result for Kloosterman sums leads to a result for incomplete Kloosterman sums. This follows from the fact that, for a smooth function f supported on $[M, 2M]$ with derivatives satisfying $f^{(j)}(m) \ll M^{-j}$, we have

$$\sum_m f(m) e\left(\frac{a\overline{m}}{q}\right) = \frac{1}{|q|} \sum_n \hat{f}\left(\frac{n}{q}\right) S(-a, n; q), \tag{3·1}$$

which is a straight-forward consequence of the Poisson summation formula. Here restricting the summation to $|n| \leqslant q^{1+\eta}/M$ leads to a negligible error term by partial integration.

We have the following result for complete Kloosterman sums.

LEMMA 9. *Let r, s and d be positive pairwise coprime integers with r and s square-free. Let M, N and C be positive numbers and g a real-valued infinitely differentiable function supported on $[M, 2M] \times [N, 2N] \times [C, 2C]$ such that*

$$\left|\frac{\partial^{j+k+l} g}{\partial m^j \partial n^k \partial c^l}\right| \leqslant M^{-j} N^{-k} C^{-l} \quad \textit{for } 0 \leqslant j, k, l \leqslant 2.$$

Let $X_d = \sqrt{dMN}/(sC\sqrt{r})$. Then for any $\eta > 0$ and any complex sequences a_m and b_n one has

$$\sum_m a_m \sum_n b_n \sum_{\substack{c \\ (c,r)=1}} g(m, n, c) S(dm\overline{r}, \pm n, sc)$$
$$\ll C^\eta d^{\frac{7}{64}} sC\sqrt{r} \frac{(1 + X_d^{-1})^{\frac{7}{32}}}{1 + X_d} \left(1 + X_d + \sqrt{\frac{M}{rs}}\right)\left(1 + X_d + \sqrt{\frac{N}{rs}}\right) \|a_m\|_2 \|b_n\|_2,$$

where

$$\|a_m\|_2 = \left(\sum_m |a_m|^2\right)^{1/2}.$$

Proof. The case $d = 1$ is proved by Deshouillers and Iwaniec [**1**, theorem 9]. We can take $\theta_{rs} = 7/32$ there by the work of Kim and Sarnak [**10**]. For $d > 1$ the result can be proved by incorporating to the proof arguments from the beginning of the proof of [**5**, proposition 1], which generalises a result by Watt [**13**, proposition 4·1] from the case $d = 1$ to the case $d \geqslant 1$. We sketch the required modifications that are required to the proof of [**1**, theorem 9].

Theorem 9 of [**1**] is proved by applying [**1**, theorem 8] (or more precicely its corrollary, [**1**, theorem 13]) for the Hecke congruence group $\Gamma_0(q)$ with $q = rs$ and the cusps $\mathfrak{a} = \infty$ and $\mathfrak{b} = 1/s$. Further, [**1**, theorem 8] is based on the Kuznietsov formula (see [**1**, theorem 1]). The bases for the spaces of holomorphic and non-holomorphic cusp forms can be chosen to consist of eigenvectors of all Hecke operators $T(n)$ with $(n, rs) = 1$ (for definitions see [**8**, pages 370–371] for the holomorphic case and [**1**, pages 230–231] for the non-holomorphic case).

When applying the Kuznietsov formula, we will have the mth Fourier coefficient replaced by the mdth coefficient on [**1**, page 267]. However, since the involved cusp forms are eigenvectors of Hecke operators (with eigenvalues $\lambda_{jk}(n)$, say) we have, for $(d, rs) = 1$,

$$\lambda_{jk}(d)\psi_{jk}(\infty, m) = \sum_{l|(m,d)} l^{k-1}\psi_{jk}\left(\infty, \frac{dm}{l^2}\right)$$

in the holomorphic case (see [**8**, (14·47)]). By Möbius inversion this gives

$$\psi_{jk}(\infty, dm) = \sum_{l|(m,d)} \mu(l)l^{k-1}\lambda\left(\frac{d}{l}\right)\psi_{jk}\left(\infty, \frac{m}{l}\right).$$

Similarly in the non-holomorphic case

$$\rho_{j\infty}(md) = \sum_{l|(m,d)} \mu(l)\tau_j\left(\frac{d}{l}\right)\rho_{j\infty}\left(\frac{m}{l}\right),$$

where $\tau_j(n)$ are the corresponding eigenvalues.

Using these and the bounds $\tau_j(n) \ll n^{7/64}$ and $\lambda(n) \ll n^{(k-1)/2+\eta}$ (see [**8**, section 5·11]), we see that the terms coming from non-exceptional cusp forms contribute $d^{7/64+\eta}$ times the contribution in the case $d = 1$. We also replace X in [**1**] by X_d.

The Fourier coefficients of Eisenstein series can be handled easily since for square-free q the only cusps of $\Gamma_0(q)$ are of the form $1/w$, where $w \mid q$ (see [**1**, lemma 2·3]). Using the defintion of $\varphi_{\frac{1}{w}\infty md}(z)$ on [**1**, page 227] and the structure of $\sigma_{\frac{1}{w}}^{-1}\Gamma_0(q)\sigma_\infty$ (see [**1**, pages 240–241]) we see that this can be expressed as a linear combination of $\ll d^\eta$ Fourier coefficients $\varphi_{\frac{1}{wg}\infty m}(z)$ of Eisenstein series in $\Gamma_0(qg)$ with $g \mid d$. Thus the large sieve inequality [**1**, theorem 2] is still applicable.

Lemma 9 and (3·1) imply the following result for incomplete Kloosterman sums.

LEMMA 10. *Let r, s and d be positive pairwise coprime integers with r and s square-free. Let C, M and N be positive numbers and g a real-valued infinitely differentiable function* *supported in* $[M, 2M] \times [N, 2N] \times [C, 2C]$ *such that*

$$\left|\frac{\partial^{j+k+l} g}{\partial m^j \partial n^k \partial c^l}\right| \leqslant M^{-j} N^{-k} C^{-l} \quad \text{for } 0 \leqslant k, l \leqslant 2 \text{ and } j \geqslant 0.$$

Let $X_d = \sqrt{\frac{dN}{MCsr}}$. *Then for any* $\eta > 0$ *and any complex sequences* a_m *and* b_n *one has*

$$\sum_n b_n \sum_{\substack{c \\ (mr,sc)=1}} \sum_m g(m,n,c) e\left(\pm \frac{dn\overline{mr}}{sc}\right)$$

$$\ll (Cs)^\eta d^{\frac{7}{64}} \sqrt{MsrC} \frac{(1 + X_d^{-1})^{\frac{7}{32}}}{1 + X_d} \left(1 + X_d + \sqrt{\frac{C}{rM}}\right) \left(1 + X_d + \sqrt{\frac{N}{rs}}\right) \|b_n\|_2$$

$$+ \frac{MN^{1/2}}{s} \|b_n\|_2.$$

The previous lemma already furnishes us with some type I information, that is a result when $c(y)$ in (1·7) is the characteristic function of an interval. Next we derive a result for bilinear forms that will provide us type II information.

LEMMA 11. *Let* $X_d = \sqrt{\frac{dK}{MCR}}$ *and suppose that a bounded sequence* $a(k, r)$ *is supported on square-free* r *with* $(r, d) = 1$. *Then*

$$U = \sum_{\substack{c \sim C \\ (c,m)=1}} \sum_{m \sim M} \left| \sum_{k \sim K} \sum_{\substack{r \sim R \\ (r,c)=1}} a(k,r) e\left(\frac{\pm dk\overline{mr}}{c}\right) \right|$$

$$\ll (CMKR)^{1/2+\eta} \Bigg[CM + MRK^{1/2}(K + R)^{1/2}$$

$$+ d^{7/64} (MC)^{1/2} R^{3/2} (K + R)^{1/2} \frac{(1 + X_d^{-1})^{7/32}}{1 + X_d}$$

$$\times \left(1 + X_d + \sqrt{\frac{C}{R^2 M}}\right) \left(1 + X_d + \sqrt{\frac{K}{R}}\right) \Bigg]^{1/2}.$$

Proof. By the Cauchy–Schwarz inequality, introducing a test function $g(c, m)$ on $[C, 2C] \times [M, 2M]$ and squaring out, we have

$$U \ll (CM)^{1/2} \Bigg[\sum_{r_1,r_2} \sum_{k_1,k_2} a(k_1, r_1) \overline{a(k_2, r_2)}$$

$$\times \sum_{\substack{c \\ (mr_1r_2,c)=1}} \sum_m g(c,m) e\left(\pm\left(\frac{dk_1\overline{mr_1} - dk_2\overline{mr_2}}{c}\right)\right) \Bigg]^{1/2}.$$

Here

$$e\left(\pm\left(\frac{dk_1\overline{mr_1} - dk_2\overline{mr_2}}{c}\right)\right) = e\left(\pm\left(\frac{d\overline{mr_1r_2}}{c}(k_1r_2 - k_2r_1)\right)\right).$$

Diagonal terms (terms with $k_1r_2 = k_2r_1$) contribute to U the amount

$$\ll CM(KR)^{1/2+\eta}. \tag{3·2}$$

For non-diagonal terms we can use Lemma 10 with

$$m = m, \quad c = c, \quad r = r_1 r_2, \quad s = 1, \quad N \sim N \leqslant RK, \quad X_d = X'_d = \sqrt{\frac{dN}{MCr_1r_2}}$$

and

$$b_n = \sum_{\substack{k_1,k_2 \\ r_2k_1 - r_1k_2 = n}} a(k_1, r_1)\overline{a(k_2, r_2)}.$$

Hence the total contribution from these is

$$(CM)^{1/2+\eta}\Bigg[\sum_{r_1,r_2} d^{7/64}\sqrt{MC}R\frac{(1+X_d'^{-1})^{7/32}}{1+X'_d}\left(1 + X'_d + \sqrt{\frac{C}{R^2M}}\right)$$
$$\times\left(1 + X'_d + \sqrt{\frac{N}{R^2}}\right)\|b_n\|_2 + MN^{1/2}\|b_n\|_2\Bigg]^{1/2}.$$

We have

$$\begin{aligned}\left(\sum_{r_1,r_2}\|b_n\|_2\right)^2 &\leqslant R^2\sum_{r_1,r_2}\|b_n\|_2^2 \leqslant R^2\sum_{r_i\sim R}\sum_{n\sim N}\left(\sum_{\substack{k_1,k_2 \\ r_2k_1-r_1k_2=n}} 1\right)^2 \\ &= R^2\sum_{r_i\sim R}\sum_{\substack{k_1,k_2,k_3,k_4 \\ r_2k_1-r_1k_2=r_2k_3-r_1k_4\sim N}} 1 \\ &= R^2\sum_{r_i\sim R}\sum_{\substack{k_1,k_2,k_3,k_4 \\ r_2(k_1-k_3)=r_1(k_2-k_4) \\ r_2k_1-r_1k_2\sim N}} 1 \\ &\leqslant R^{3+\eta}K^{2+\eta}\left(1 + \frac{N}{K} + \frac{N}{R}\right).\end{aligned}$$

Hence the non-diagonal terms contribute to U the amount

$$\begin{aligned}(CM)^{1/2+\eta}\Bigg[&d^{7/64}\sqrt{MC}K^{1+\eta}R^{5/2+\eta}\frac{(1+X_d'^{-1})^{7/32}}{1+X'_d}\left(1+X'_d+\sqrt{\frac{C}{R^2M}}\right) \\ &\times\left(1+X'_d+\sqrt{\frac{N}{R^2}}\right)\left(1+\frac{N}{K}+\frac{N}{R}\right)^{1/2} \\ &+MN^{1/2}R^{3/2+\eta}K^{1+\eta}\left(1+\frac{N}{K}+\frac{N}{R}\right)^{1/2}\Bigg]^{1/2}.\end{aligned} \tag{3·3}$$

We see that the maximal contribution comes from maximal $N = RK$. This leads to the claim.

In our applications we will have $X_d \asymp 1$. When $X_d = o(1)$ the result of Lemma 9 and therefore of Lemmas 10 and 11 could be improved by taking advantage of averaging over r and s. Results to that end with $d = 1$ can be found in [**1**, theorems 10-12] and [**13**, proposition 4·1], and a technique to generalize the results for $d \geqslant 1$ from [**5**, section 3].

4. *Arithmetical information*

In this section we apply results from the previous section to obtain new type I and type II information. We consider the tri-linear sum

$$\Omega = \sum_{\substack{yzn\in\mathcal{A}\\ y\sim Y, z\sim Z, n\sim N}} c(y)d(z)b(n).$$

By Proposition 6, we need to consider Ω_2 with coefficients (1·7), so we face $\ll x^{\eta}$ sums of the form

$$\sum_{\substack{z_1\sim Z_1\\ z_2\sim Z_2\\ |\mu(\beta y_1 z_1 y_2 z_2)|=1}} \sum_{\substack{y_1\sim Y_1\\ y_2\sim Y_2}} c(\rho_{Y_1} y_1)\overline{c(\rho_{Y_2} y_2)} d(\rho_{Z_1} z_1)\overline{d(\rho_{Z_2} z_2)} \sum_{\substack{|k|\sim K\\ |l|\sim L}} g(|k|, |l|, y_1 z_1, y_2 z_2) e\left(\frac{-qkl\overline{y_2 z_2}}{y_1 z_1}\right), \tag{4·1}$$

where $Y_i \leqslant Y$, $Z_i \leqslant Z$, $\rho_{Y_i}\rho_{Z_i} = \rho$ and $Y_1 Z_1 = Y_2 Z_2 = M'$.

We will apply Lemmas 10 and 11. The sum (4·1) has some co-primality restrictions that do not appear in those lemmas. However, they can be handled using Möbius inversion and refining the support of coefficients $c(y)$ and $d(z)$.

In the case of type I sums $c(y)$ is the characteristic function of an interval. Recalling Remark 7 we can use Lemma 10 with

$$d = q, \quad N = KL, \quad C = Y_1, \quad M = Y_2, \quad r = z_2 \quad \text{and} \quad s = z_1.$$

Then $X_q = \sqrt{qKL/M'^2} \ll 1$. We see immediately that the maximal bound will be obtained for maximal $KL = 10M'^2/q$, whence $X_q \asymp 1$. We obtain

$$\Omega_2 \ll K^{2\eta} Z_1 Z_2 \left(M^{\eta} q^{7/64} M' \left(1 + \sqrt{\frac{1}{Z_1}}\right)\left(1 + \sqrt{\frac{M'^2}{qZ_1Z_2}}\right)\frac{M'}{q^{1/2}} + \frac{Y_2 M'^2}{qZ_1}\right).$$

We can assume that $M'^2/(qZ_1Z_2) = Y_1Y_2/q \leqslant Y^2/q \ll 1$ since otherwise we have type I information by Lemma 3. Hence

$$\Omega_2 \ll \frac{Z^2 M'^{2+\eta}}{q^{25/64}} + \frac{M'^{3+\eta}}{q},$$

so in order to establish (2·1), we need $M \ll q$ and

$$Z^2 \ll x^{\frac{1-3\tau}{2} + \frac{25(1+\tau)}{128} - 5\delta - \eta},$$

which holds for $Z \ll x^{25/192}$. Now $M \ll q$ follows from $Y \ll q^{1/2}$. Recalling the assumption $N \ll q$, we have by Proposition 6 and Lemma 2 (used when $q \ll N \ll x^{2/3}$) the following result.

LEMMA 12. *Let*

$$N \ll x^{2/3} \quad \textit{and} \quad Z \ll x^{25/192}.$$

Assume that the coefficients $a(m)$ are defined by (1·7) with $c(y)$ the characteristic function of an interval. Then (2·2) holds.

Next we turn our attention to type II information for tri-linear sums, that is we consider Ω with arbitrary bounded coefficients $c(y)$ and $d(z)$. This time we apply Lemma 11 to (4·1) with

$$d = q, \quad C = Y_1 Z_1 = M', \quad M = Y_2, \quad R = Z_2, \quad K = KL.$$

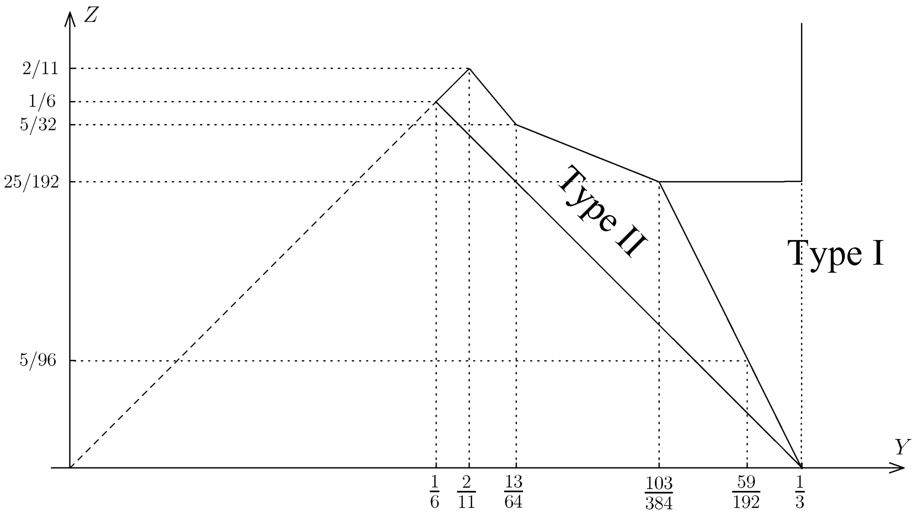

Fig. 1. Type I and type II areas with $Y \geqslant Z$ demonstrated.

Then $X_q = \sqrt{\frac{qKL}{M'^2}} \ll 1$. Again we notice that the maximal contribution comes from maximal $KL = 10M'^2/q$ in which case $X_q \asymp 1$. Hence we obtain

$$\Omega_2 \ll \frac{M'^2}{q^{1/2}}\Bigg(M'Y_2 + \frac{M'^2}{q^{1/2}}\left(\frac{M'^2}{q} + Z_2\right)^{1/2} + q^{7/64}M'Z_2\left(\frac{M'^2}{q} + Z_2\right)^{1/2}\left(1 + \sqrt{\frac{M'^2}{qZ_2}}\right)\Bigg)^{1/2}.$$

Assuming $M'Y_2 \ll x^{2/3} = qx^{\epsilon/2}$, as we essentially have to in order to get a satisfactory bound (2·1) for the first term, we get

$$\Omega_2 \ll \frac{M'^2}{q^{1/2}}\left(M'Y_2 + M'^2x^{\epsilon/4}\left(\frac{Z_2}{q}\right)^{1/2} + x^{\epsilon/2}q^{7/64}M'Z_2^{3/2}\right)^{1/2}.$$

It is easy to see that the third term dominates the second term for $M'Y_2 \ll x^{2/3}$. Thus we have (2·1) for

$$\begin{cases} MY_2 \ll x^{2/3}, \\ M^2Z_2^3 \ll x^{19/16}. \end{cases}$$

Hence Proposition 6 and Lemma 2 imply the following result.

LEMMA 13. *Assume that the coefficients* $a(m)$ *are defined by* (1·7) *and*

$$\begin{cases} N & \ll x^{2/3}, \\ Y^2Z & \ll x^{2/3}, \\ Y^2Z^5 & \ll x^{19/16}. \end{cases}$$

Then (2·2) *holds.*

Type I and type II information that we have obtained so far is demonstrated in Figure 1.

We can derive the following type II result for bilinear sums that arise from the sieve method.

LEMMA 14. *Assume that $M \in [x^{1/3}, x^{23/64}]$ and that*

$$a(m) = \sum_{m_1 \cdots m_k = m} a_1(m_1) \cdots a_k(m_k),$$

where each $a_j(m)$ is either the characteristic function of primes or integers co-prime to q in some interval $\subset [I_j, 2I_j]$ for $j = 1, \ldots, k$. Then (2·2) holds.

Proof. Using Heath–Brown's generalized Vaughan's identity [**6**] to each $a_j(m)$ with $I_j > x^{1/8}$ we can write $a(m)$ as a sum of $\ll (\log x)^C$ sums

$$\sum_{\substack{n_1 \cdots n_l = m \\ n_i \sim N_i}} b_1(n_1) \cdots b_l(n_l),$$

where $N_1 < N_2 < \cdots < N_l$, $N_1 \cdots N_l \sim M$ and b_j is the characteristic function of some interval whenever $N_i > x^{1/8}$. Now either there is some product of N_i in the interval $[x^{5/96}, x^{23/128}]$ or $N_1 \cdots N_{l-1} < x^{5/96}$ and b_l is the characteristic fuction of an interval. In the first case the sum is a type II sum and either Lemma 4 or Lemma 13 gives an asymptotic formula as can also be seen from Figure 1 (notice that $23/64 = 13/64 + 5/32 = 59/192 + 5/96$). In the second case the sum is a type I sum and Lemma 12 gives the result.

5. *Sieve asymptotic formulae*

First we introduce some standard notation. Let $\mathcal{F}$ be a finite subset of $\mathbb{N}$. Then we write $|\mathcal{F}|$ for the cardinality of $\mathcal{F}$,

$$\mathcal{F}_d = \{m \mid dm \in \mathcal{F}\}$$

and

$$S(\mathcal{F}, z) = |\{m \in \mathcal{F} \mid (m, P(z)) = 1\}|,$$

where

$$P(z) = \prod_{p<z} p.$$

The elementary *Buchstab's identity* states that

$$S(\mathcal{F}, z) = S(\mathcal{F}, w) - \sum_{w \leqslant p < z} S(\mathcal{F}_p, p),$$

where $z > w \geqslant 2$.

We are interested in the number of primes in $\mathcal{A}$, that is $S(\mathcal{A}, 2x^{1/2})$. We use Buchstab's identity to decompose this into sums that are easier to handle and decompose $S(\mathcal{B}, 2x^{1/2})$ in similar manner. We end up with decompositions

$$S(\mathcal{A}, 2x^{1/2}) = \sum_{j=1}^{k} S_j - \sum_{j=k+1}^{l} S_j \tag{5·1}$$

and

$$S(\mathcal{B}, 2x^{1/2}) = \sum_{j=1}^{k} S_j^* - \sum_{j=k+1}^{l} S_j^*,$$

where $S_j, S_j^* \geqslant 0$ and, for $j \leqslant t \leqslant k$ or $j > k$, we can find an asymptotic formula of the form

$$S_j = \frac{\hat{h}(0)}{q} S_j^*(1 + o(x^{-\delta/2})).$$

This implies that

$$S(\mathcal{A}, 2x^{1/2}) \geqslant \frac{\hat{h}(0)}{q} \left(S(\mathcal{B}, 2x^{1/2}) - \sum_{j=t+1}^{k} S_j^* \right) (1 + o(1)).$$

Here the sums S_j^* are of the form

$$\sideset{}{^\sharp}\sum_{x^\nu < p_n < \cdots < p_1 < x^\lambda} S(\mathcal{B}_{p_1 \cdots p_n}, p_n)$$

with some additional summation conditions. Here we mark the summation condition $p_1 \cdots p_{i-1} p_i^2 < x$, $i = 1, \ldots, n$ by $\sharp$. These conditions can be easily included during the decomposition phase adding an error term of a smaller order. They will be assumed when we do the decomposition and we will not write the error term to decompositions.

By changing summations to integrals using the prime number theorem and substituting $p_j = x^{\alpha_j}$, we have for example

$$\sideset{}{^\sharp}\sum_{x^\nu < p_n < \cdots < p_1 < x^\lambda} S(\mathcal{B}_{p_1 \cdots p_n}, p_n)$$
$$= \frac{x(1 + o(1))}{\log x} \int_{\alpha_1=\nu}^{\lambda} \int_{\alpha_2=\nu}^{\alpha_1^\sharp} \cdots \int_{\alpha_n=\nu}^{\alpha_{n-1}^\sharp} \omega\left(\frac{1 - \alpha_1 - \cdots - \alpha_n}{\alpha_n}\right) \frac{d\alpha_n \cdots d\alpha_1}{\alpha_1 \cdots \alpha_{n-1} \alpha_n^2},$$

where we have written $\alpha_i^\sharp = \min\{\alpha_i, (1 - \alpha_1 - \cdots - \alpha_i)/2\}$. Here $\omega(u)$ is Buchstab's function (see [**4**, sections 1·4 and A·2] or [**3**, page 244]).

Since $S(\mathcal{B}, 2x^{1/2}) = (x/\log x)(1 + o(1))$, we get the desired result if the sum of integrals corresponding to S_j^* with $t < j \leqslant k$ is strictly less than one.

Next we describe the decomposition (5·1). We start by decomposing $S(\mathcal{A}, 2x^{1/2})$ twice with Buchstab's identity getting the decomposition

$$\begin{aligned} S(\mathcal{A}, 2x^{1/2}) &= S(\mathcal{A}, z) - \sum_{z \leqslant p < 2x^{1/2}} S(\mathcal{A}_p, z) + \sum_{z \leqslant p_2 < p_1 < 2x^{1/2}} S(\mathcal{A}_{p_1 p_2}, p_2) \\ &= S_1 - S_2 + S_3, \end{aligned}$$

where $z = x^{25/528}$. We will be able to give asymptotic formulae for S_1 and S_2 and parts of S_3. We can also decompose some parts of S_3 further.

The following lemma gives asymptotic formulae for instance for S_1 and S_2.

LEMMA 15. *Assume that $M \leqslant x^{2/3}$ and $N \leqslant x^{25/192}$. Let $a(m)$ and $b(n)$ be divisor-bounded complex coefficients. Then*

$$\sum_{m \sim M} \sum_{n \sim N} a(m) b(n) S(\mathcal{A}_{mn}, z) = \frac{\hat{h}(0)}{q} \sum_{m \sim M} \sum_{n \sim N} a(m) b(n) S(\mathcal{B}_{mn}, z)(1 + o(1))$$

for $z = x^{25/528}$.

Proof. This is a fundamental sort of result in Harman's sieve method. The basic idea is to use Buchstab's identity with $w = 2$ repeatedly to split the left-hand side into sums of the type

$$\sum_{m\sim M}\sum_{n\sim N}\sum_{\substack{mnp_1\cdots p_jk\in\mathcal{A}\\ p_j<p_{j-1}<\cdots<p_1<z}} a(m)b(n).$$

When $p_1, \ldots, p_j$ can be divided into subsets $\mathcal{P}_1$ and $\mathcal{P}_2$ such that

$$M' = M\prod_{p\in\mathcal{P}_1} p \leqslant x^{2/3} \quad \text{and} \quad N' = N\prod_{p\in\mathcal{P}_2} p \leqslant x^{25/192}$$

we have a type I sum and an asymptotic formula holds by Lemma 12.

Otherwise there is $l \leqslant j$ such that $p_1, \ldots, p_{l-1}$ can be split into sets $\mathcal{P}_1$ and $\mathcal{P}_2$ such that

$$M' \leqslant x^{2/3} \quad \text{and} \quad N' \leqslant x^{25/192} \quad \text{but} \quad M'p_l > x^{2/3} \quad \text{and} \quad N'p_l > x^{25/192}.$$

Let

$$M' = x^{\alpha}, \quad N' = x^{\beta}, \quad p_l = x^{\gamma} \quad \text{and} \quad K = x/(M'N') = x^{\theta}.$$

Then $1 - (\theta + \beta) = \alpha \leqslant 2/3$. Since $4\beta \leqslant 100/192 = 19/16 - 2/3$, we have type II information by Lemma 13 with $Y = K$ and $Z = N'$ if $2\theta + \beta \leqslant 2/3$. Otherwise

$$\beta < 2(\theta+\beta) - 2/3 = 2(1-\alpha) - 2/3 < 2(1-(2/3-\gamma)) - 2/3 = 2\gamma.$$

Thus

$$2(\theta-\gamma) + (\beta+\gamma) = 2\theta + \beta - \gamma = 2 - 2\alpha - \beta - \gamma \leqslant 2 - 2(2/3-\gamma) - 25/192 < 2/3$$

and

$$\begin{aligned} 2(\theta-\gamma) + 5(\beta+\gamma) &= 2(1-\alpha-\beta) + 5\beta + 3\gamma \leqslant 2(1/3+\gamma) + 3\beta + 3\gamma \\ &< 2/3 + 11\gamma \leqslant 19/16. \end{aligned}$$

Hence we have type II information by Lemma 13 with $Y = K/p_l$ and $Z = N'p_l$. Cross-conditions can be handled using Perron's formula. For a more detailed account of this kind of argument, see for instance [**4**].

Now we turn our attention to S_3. We write $p_j = x^{\alpha_j}$ and consider the summation over H_2, where

$$\begin{aligned} H_k = \{(\alpha_1, \alpha_2, \ldots, \alpha_k) \mid {}& \alpha_1 > \alpha_2 > \cdots > \alpha_k \geqslant 25/528, \\ & \alpha_1 + \cdots + \alpha_{j-1} + 2\alpha_j < 1 \text{ for } j = 1, \ldots, k\}. \end{aligned}$$

Lemma 14 gives an asymptotic formula for sums in which some combination of α_1, α_2 and $1 - \alpha_1 - \alpha_2$ is in the interval $[1/3, 23/64]$. We also have an asymptotic formula when $p_1 = Y$ and $p_2 = Z$ satisfy conditions of one of Lemmas 4 and 13.

We write G_k for the subset of H_k on which we can immediately give an asymptotic formula using one of Lemmas 4, 13 and 14. We split the rest of H_2 into three regions

$$\begin{aligned} A &= \{(\alpha_1, \alpha_2 \in H_2) \mid \alpha_1 + 2\alpha_2 \leqslant 2/3 \quad \text{or} \quad \alpha_2 \leqslant 25/192 \\ &\qquad\qquad \text{or} \quad (\alpha_1 \leqslant 1/3 \quad \text{and} \quad \alpha_2 \leqslant 2/11)\} \setminus G_2, \\ B &= (H_2 \setminus (A \cup G_2)) \cap \{(\alpha_1, \alpha_2 \in H_2) \mid \alpha_1 \geqslant 1/3 \quad \text{and} \quad \alpha_1 + \alpha_2 \leqslant 2/3\}, \\ C &= H_2 \setminus (A \cup B \cup G_2) \end{aligned}$$

(see Figure 2).

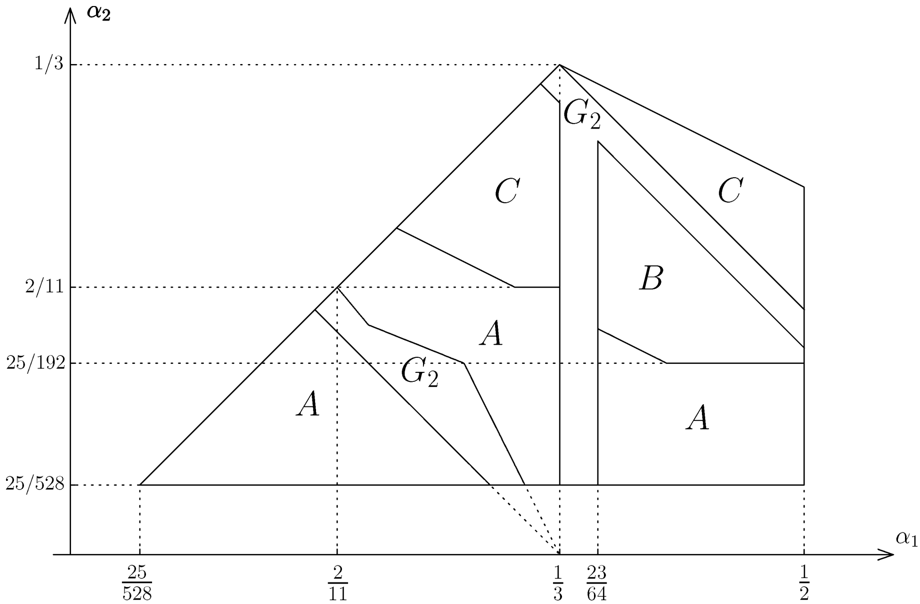

Fig. 2. Regions A, B, C and G_2 demonstrated.

In C we cannot decompose further but have to discard the whole region giving the loss

$$\iint_{(\alpha_1,\alpha_2)\in C} \omega\left(\frac{1-\alpha_1-\alpha_2}{\alpha_2}\right)\frac{d\alpha_2 d\alpha_1}{\alpha_1\alpha_2^2} < 0.53$$

by numerical integration.

In A we can use Buchstab's identity twice more giving

$$\sum_{(\alpha_1,\alpha_2)\in A} S(\mathcal{A}_{p_1p_2}, p_2) = \sum_{(\alpha_1,\alpha_2)\in A} S(\mathcal{A}_{p_1p_2}, z) - \sum_{\substack{(\alpha_1,\alpha_2,\alpha_3)\in H_3\\(\alpha_1,\alpha_2)\in A}} S(\mathcal{A}_{p_1p_2p_3}, z) + \sum_{\substack{(\alpha_1,\alpha_2,\alpha_3,\alpha_4)\in H_4\\(\alpha_1,\alpha_2)\in A}} S(\mathcal{A}_{p_1p_2p_3p_4}, p_4).$$

We can give asymptotic formulae for the first and second sums using Lemma 15 or Lemma 4. Then we can use our type II information to give asymptotic formulae for parts of the third sum and decompose in some parts further.

In B we cannot immediately decompose twice more, but we can use a role-reversals tool that was introduced in [**3**]. First we apply Buchstab's identity once to get

$$\sum_{(\alpha_1,\alpha_2)\in B} S(\mathcal{A}_{p_1p_2}, p_2) = \sum_{(\alpha_1,\alpha_2)\in B} S(\mathcal{A}_{p_1p_2}, z) - \sum_{\substack{(\alpha_1,\alpha_2,\alpha_3)\in H_3\\(\alpha_1,\alpha_2)\in B}} S(\mathcal{A}_{p_1p_2p_3}, p_3).$$

Here we have an asymptotic formula for the first term and parts of the second term. In the rest of the second term we decompose p_1 instead of $s \sim x/(p_1p_2p_3)$. We write $s = x^\beta$ and

$tp_4 = x^{\alpha_1}$ and get the decomposition

$$\sum_{\substack{(\alpha_1,\alpha_2,\alpha_3)\in H_3\setminus G_3\\(\alpha_1,\alpha_2)\in B}} S(\mathcal{A}_{p_1p_2p_3}, p_3)$$
$$= \sum_{\substack{(1-\beta-\alpha_2-\alpha_3,\alpha_2,\alpha_3)\in H_3\setminus G_3\\(1-\beta-\alpha_2-\alpha_3,\alpha_2)\in B\\(s,P(p_3))=1}} S(\mathcal{A}_{p_2p_3s}, z) - \sum_{\substack{(\alpha_1,\alpha_2,\alpha_3)\in H_3\setminus G_3\\(\alpha_1,\alpha_2)\in B\\(t,P(p_4))=1\\25/528<\alpha_4<\alpha_1/2}} S(\mathcal{A}_{tp_4p_2p_3}, p_3).$$

We can give an asymptotic formula for the first sum by Lemma 12 and some parts of the second sum are type II sums.

When we discard terms with two almost-prime variables, the resulting integral is of the form

$$\int \omega\left(\frac{\alpha_1-\alpha_4}{\alpha_4}\right)\omega\left(\frac{1-\alpha_1-\alpha_2-\alpha_3}{\alpha_3}\right)\frac{d\alpha_4 d\alpha_3 d\alpha_2 d\alpha_1}{\alpha_2\alpha_3^2\alpha_4^2}$$

(see [**4**, page 90] or [**3**, pages 250–251]).

The role reversals device can also be used in some parts of A to make more decompositions. However, it is not always benefitical to decompose more, and we have calculated the integrals numerically with a computer program which checks when it is best to decompose and when not.

The loss from A is < 0.11 and the loss from B is < 0.28. Hence the total loss is $< 0,11 + 0,28 + 0,53 = 0.92$, and so

$$|\mathcal{A}\cap\mathbb{P}| = S(\mathcal{A}, 2x^{1/2}) \gg \frac{R}{q}\frac{x}{\log x},$$

which completes the proof of Theorem 1.

Acknowledgements. The author was supported by the Finnish Cultural Foundation.

REFERENCES

[**1**] J. M. DESHOUILLERS and H. IWANIEC. Kloosterman sums and Fourier coefficients of cusp forms. *Invent. Math.* **70** (1982), 219–288.

[**2**] G. HARMAN. On the distribution of αp modulo one I. *J. London Math. Soc.* (2) **27** (1983), 9–18.

[**3**] G. HARMAN. On the distribution of αp modulo one II. *Proc. London Math. Soc.* (3) **72** (1996), 241–260.

[**4**] G. HARMAN. *Prime-detecting Sieves*, volume 33 of *London Mathematical Society Monographs (New Series)* (Princeton University Press, 2007).

[**5**] G. HARMAN, N. WATT and K. C. WONG. A new mean-value result for Dirichlet L-functions and polynomials. *Quart. J. Math.* **55** (2004), 307–324.

[**6**] D. R. HEATH-BROWN. Prime numbers in short intervals and a generalized Vaughan identity. *Can. J. Math.* **34** (1982), 1365–1377.

[**7**] D. R. HEATH-BROWN and C. JIA. The distribution of αp modulo one. *Proc. London Math. Soc.* (3) **84** (2002), 79–104.

[**8**] H. IWANIEC and E. KOWALSKI. *Analytic number theory*, volume 53 of *American Mathematical Society Colloquium Publications* (American Mathematical Society, 2004).

[**9**] C. JIA. On the distribution of αp modulo one. In *Number theoretic methods (Iizuka*, 2001), volume 8 of *Dev. Math.*, pages 151–157 (Kluwer Academic Publishers, 2002).

[**10**] H. H. KIM (with appendices by D. Ramakrishnan, H. H. Kim and P. Sarnak). Functoriality for the exterior square of GL_4 and the symmetric square of GL_2. *J. Amer. Math. Soc.* **16** (2003), 139–183.

[**11**] R. C. VAUGHAN. On the distribution of αp modulo 1. *Mathematika* **24** (1977), 135–141.

[**12**] I. M. VINOGRADOV. *The Method of Trigonometric Sums in the Theory of Numbers.* Translated from the Russian revised and annotated by K. F. Roth and A. Davenport (Wiley-Interscience, 1954).

[**13**] N. WATT. Kloosterman sums and a mean value for Dirichlet polynomials. *J. Number Theory* **53** (1995), 179–210.