Selected topics on quantum information

Mika Hirvensalo

Department of Mathematics and Statistics
University of Turku
mikhirve@utu.fi

Thessaloniki, May 2016

Quantum Information

Quantum Information

- What is quantum information?

Quantum Information

- What is quantum information?
- Reply: Quantum information is information represented in quantum systems.

Quantum Information

- What is quantum information?
- Reply: Quantum information is information represented in quantum systems.
- What is information?

Quantum Information

- What is quantum information?
- Reply: Quantum information is information represented in quantum systems.
- What is information?
- What is a quantum system?

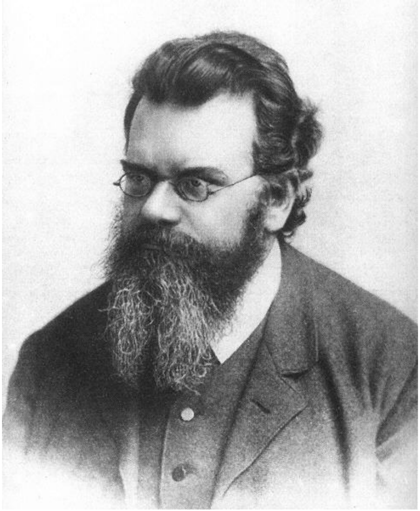
Information

- Information is difference of entropies

Information

- Information is difference of entropies
- What is entropy?

Entropy



Ludwig Boltzmann (1844-1906)

Entropy


S = k. log W
LVDWIG
BOLTZMANN
1844 - 1906
DR PHIL PAULA
BOLTZMANN
GEB. CHIARI
1891 - 1977
ARTHUR
BOLTZMANN
DIPL.ING. DR PHIL. HOFRAT
1881 - 1952
LVDWIG
BOLTZMANN
1923 - 1943
GEFALLEN BEI SMOLENSK
HENRIETTE
BOLTZMANN
1854 - 1938

Entropy

$$S = k \cdot \log W,$$

where k is a constant, W is the number of microstates corresponding to a macroscopic state

Entropy

"Elementary" entropy

$H(n)$ = number of elementary units (bits, trits, etc.) to encode n (uniform) conditions.

Entropy

"Elementary" entropy

$H(n)$ = number of elementary units (bits, trits, etc.) to encode n (uniform) conditions.

- $\{1,2\} \mapsto \{0,1\}$,

Entropy

"Elementary" entropy

$H(n)$ = number of elementary units (bits, trits, etc.) to encode n (uniform) conditions.

- $\{1, 2\} \mapsto \{0, 1\}$,
- $\{1, 2, 3\} \mapsto \{0, 11, 10\}$,

Entropy

"Elementary" entropy

$H(n)$ = number of elementary units (bits, trits, etc.) to encode n (uniform) conditions.

- $\{1, 2\} \mapsto \{0, 1\}$,
- $\{1, 2, 3\} \mapsto \{0, 11, 10\}$,
- $\{1, 2, 3, 4\} \mapsto \{00, 01, 10, 11\}$, etc.

Entropy

"Elementary" entropy

$H(n)$ = number of elementary units (bits, trits, etc.) to encode n (uniform) conditions.

- $\{1, 2\} \mapsto \{0, 1\}$,
- $\{1, 2, 3\} \mapsto \{0, 11, 10\}$,
- $\{1, 2, 3, 4\} \mapsto \{00, 01, 10, 11\}$, etc.

Binary (elementary) entropy

$H_2(n) = \log_2 n = \frac{1}{\log 2} \log n$

Entropy

"Elementary" entropy

$H(n)$ = number of elementary units (bits, trits, etc.) to encode n (uniform) conditions.

- $\{1, 2\} \mapsto \{0, 1\}$,
- $\{1, 2, 3\} \mapsto \{0, 11, 10\}$,
- $\{1, 2, 3, 4\} \mapsto \{00, 01, 10, 11\}$, etc.

Binary (elementary) entropy

$H_2(n) = \log_2 n = \frac{1}{\log 2} \log n$

Entropy measures uncertainty

Entropy

Ternary (elementary) entropy

$H_3(n) = \frac{1}{\log 3} \log n$

Entropy

Ternary (elementary) entropy

$H_3(n) = \frac{1}{\log 3} \log n$

q-ary (elementary) entropy

$H_q(n) = \frac{1}{\log q} \log n$

Entropy

Boltzmann:

Identical particles with same internal condition are indistinguishable.

Entropy

Boltzmann:

Identical particles with same internal condition are indistinguishable.

- Let l be the number of particles, each having n potential conditions $\Sigma = \{1, 2, \ldots, n\}$, $l \gg n$.

Entropy

Boltzmann:

Identical particles with same internal condition are indistinguishable.

- Let l be the number of particles, each having n potential conditions $\Sigma = \{1, 2, \ldots, n\}$, $l \gg n$.
- List the conditions of all particles: $c_1 c_2 \ldots c_l \in \{1, 2, \ldots, n\}^l$

Entropy

Boltzmann:

Identical particles with same internal condition are indistinguishable.

- Let l be the number of particles, each having n potential conditions $\Sigma = \{1, 2, \ldots, n\}$, $l \gg n$.
- List the conditions of all particles: $c_1 c_2 \ldots c_l \in \{1, 2, \ldots, n\}^l$
- Assume condition (letter) i occurs k_i times, so $k_1 + \ldots + k_n = l$ and $p_i = \frac{k_i}{l}$ is the probability (frequency) of condition i

Combinatorics:

There are

$$\frac{l!}{k_1! \dots k_n!}$$

such lists (strings of conditions)

Entropy

Combinatorics:

There are

$$\frac{l!}{k_1! \ldots k_n!}$$

such lists (strings of conditions)

Elementary entropy:

$$K \log \frac{l!}{k_1! \ldots k_n!}$$

Combinatorics:

There are

$$\frac{l!}{k_1! \dots k_n!}$$

such lists (strings of conditions)

Elementary entropy:

$$K \log \frac{l!}{k_1! \dots k_n!}$$

Per particle:

$$\frac{K}{l} \log \frac{l!}{k_1! \dots k_n!}$$

Entropy

Stirling: $\log k! = k \log k - k + O(\log k)$, so

Stirling: $\log k! = k \log k - k + O(\log k)$, so

$$\begin{aligned} & \frac{K}{l} \log \frac{l!}{k_1! \ldots k_n!} \\ = \; & \frac{K}{l}(l \log l - l + O(\log l) \\ & - \sum_{i=1}^{n} (k_i \log k_i - k_i + O(\log k_i))) \\ = \; & -K \sum_{i=1}^{n} p_i \log p_i + O(\frac{\log l}{l}). \end{aligned}$$

Entropy

For a probability distribution $(p_1, \ldots, p_n)$ of events $\{e_1, \ldots, e_n\}$, define

$$H(p_1, \ldots, p_n) = -K \sum_{i=1}^{n} p_i \log p_i$$

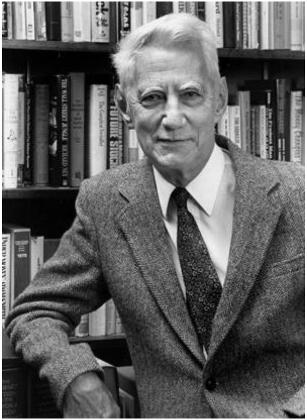
Entropy

For a probability distribution $(p_1, \ldots, p_n)$ of events $\{e_1, \ldots, e_n\}$, define

$$H(p_1, \ldots, p_n) = -K \sum_{i=1}^{n} p_i \log p_i$$

For $p_1 = \ldots = p_n = \frac{1}{n}$

$$H(\frac{1}{n}, \ldots, \frac{1}{n}) = -K \cdot n \cdot \frac{1}{n} \log \frac{1}{n} = K \log n$$



Claude Shannon (1916-2001)

- $H(p_1, \ldots, p_n)$ symmetric, continuous
- $H(\frac{1}{n}, \ldots, \frac{1}{n})$ non-negative, strictly increasing in n
- $H(p_1, \ldots, p_n) + p_n H(q_1, \ldots, q_m)$
  $= H(p_1, \ldots, p_{n-1}, p_n q_1, \ldots, p_n q_m)$

$$\Rightarrow H(p_1, \ldots, p_m) = -K \sum_{i=1}^{n} p_i \log p_i$$

Joint Entropy

Joint Entropy

Let

$$X = \{x_1, \ldots, x_n\}$$

be a random variable with distribution $p(x_1)$, ..., $p(x_n)$.

Joint Entropy

Let

$$X = \{x_1, \ldots, x_n\}$$

be a random variable with distribution $p(x_1)$, ..., $p(x_n)$.

Then

$$H(X) = -\sum_{i=1}^{n} p(x_i) \log p(x_i).$$

Joint Entropy

Let

$$X = \{x_1, \ldots, x_n\}$$

be a random variable with distribution $p(x_1)$, …, $p(x_n)$.

Then

$$H(X) = -\sum_{i=1}^{n} p(x_i) \log p(x_i).$$

If also $Y = \{y_1, \ldots, y_m\}$ is a random variable,

the *joint entropy* is

$$H(X, Y) = -\sum_{i=1}^{n} \sum_{j=1}^{m} p(x_i, y_j) \log p(x_i, y_j)$$

Conditional Entropy

Joint entropy

$$H(X, Y) = -\sum_{i=1}^{n}\sum_{j=1}^{m} p(x_i, y_j) \log p(x_i, y_j)$$

Conditional Entropy

Joint entropy

$$H(X, Y) = -\sum_{i=1}^{n} \sum_{j=1}^{m} p(x_i, y_j) \log p(x_i, y_j)$$

If we know $Y = y_j$, then

$$H(X \mid y_j) = -\sum_{i=1}^{n} p(x_i \mid y_j) \log p(x_i \mid y_j)$$

Conditional Entropy

Joint entropy

$$H(X, Y) = -\sum_{i=1}^{n}\sum_{j=1}^{m} p(x_i, y_j)\log p(x_i, y_j)$$

If we know $Y = y_j$, then

$$H(X \mid y_j) = -\sum_{i=1}^{n} p(x_i \mid y_j)\log p(x_i \mid y_j)$$

and define

$$H(X \mid Y) = \sum_{j=1}^{m} p(y_j) H(X \mid y_j).$$

Conditional Entropy

Joint entropy

$$H(X, Y) = -\sum_{i=1}^{n} \sum_{j=1}^{m} p(x_i, y_j) \log p(x_i, y_j)$$

If we know $Y = y_j$, then

$$H(X \mid y_j) = -\sum_{i=1}^{n} p(x_i \mid y_j) \log p(x_i \mid y_j)$$

and define

$$H(X \mid Y) = \sum_{j=1}^{m} p(y_j) H(X \mid y_j).$$

"Uncertainty of X when Y is known"

Conditional Entropy

Lemma

$H(X \mid Y) \leq H(X)$

Conditional Entropy

Lemma

$H(X \mid Y) \leq H(X)$

Lemma

$H(X, Y) \leq H(X) + H(Y)$

Conditional Entropy

Lemma

$H(X \mid Y) \leq H(X)$

Lemma

$H(X, Y) \leq H(X) + H(Y)$

Lemma

$H(X \mid Y) = H(X, Y) - H(Y)$

Lemma

$H(X \mid Y) \leq H(X)$

Lemma

$H(X, Y) \leq H(X) + H(Y)$

Lemma

$H(X \mid Y) = H(X, Y) - H(Y)$

(Uncertainty of X when Y is known)

Conditional Entropy

Example

Team A wins with probability $\frac{1}{2}$, $X = \{\text{win}, \text{loss}\}$. Then $H(X) = -(\frac{1}{2}\log_2\frac{1}{2} + \frac{1}{2}\log_2\frac{1}{2}) = 1$.

Conditional Entropy

Example

Team A wins with probability $\frac{1}{2}$, $X = \{\text{win}, \text{loss}\}$. Then $H(X) = -(\frac{1}{2}\log_2\frac{1}{2} + \frac{1}{2}\log_2\frac{1}{2}) = 1$.

Example

As a *home team*, A wins with $\frac{3}{4}$ probability, but as *visitor*, A wins only with $\frac{1}{3}$ probability.

$$
\begin{aligned}
H(X \mid h) &= -(\frac{3}{4}\log_2\frac{3}{4} + \frac{1}{4}\log_2\frac{1}{4}) = 0.811278\ldots, \\
H(X \mid v) &= -(\frac{1}{3}\log_2\frac{1}{3} + \frac{2}{3}\log_2\frac{2}{3}) = 0.918296\ldots
\end{aligned}
$$

Conditional Entropy

Example (Continued)

$$H(X \mid h) = -(\frac{3}{4}\log_2\frac{3}{4} + \frac{1}{4}\log_2\frac{1}{4}) = 0.811278\ldots,$$
$$H(X \mid v) = -(\frac{1}{3}\log_2\frac{1}{3} + \frac{2}{3}\log_2\frac{2}{3}) = 0.918296\ldots$$

Conditional Entropy

Example (Continued)

$$H(X \mid h) = -(\frac{3}{4}\log_2\frac{3}{4} + \frac{1}{4}\log_2\frac{1}{4}) = 0.811278\ldots,$$
$$H(X \mid v) = -(\frac{1}{3}\log_2\frac{1}{3} + \frac{2}{3}\log_2\frac{2}{3}) = 0.918296\ldots$$

Let $Y = \{0, 1\}$ be a fair coin toss for deciding if team A plays home. Then

$$H(X \mid Y) = \frac{1}{2}H(X \mid h) + \frac{1}{2}H(X \mid v) = 0.864787\ldots$$

Definition (Mutual information of X and Y)

$$I(X : Y) = H(X) - H(X \mid Y)$$

Definition (Mutual information of X and Y)

$$I(X:Y) = H(X) - H(X \mid Y)$$

$$\begin{aligned} I(X:Y) &= H(X) - H(X \mid Y) \\ &= H(X) - (H(X,Y) - H(Y)) \\ &= H(X) + H(Y) - H(X,Y) \\ &= I(Y:X) \end{aligned}$$

Definition (Mutual information of X and Y)

$$I(X:Y) = H(X) - H(X \mid Y)$$

$$\begin{aligned}
I(X:Y) &= H(X) - H(X \mid Y) \\
&= H(X) - (H(X,Y) - H(Y)) \\
&= H(X) + H(Y) - H(X,Y) \\
&= I(Y:X)
\end{aligned}$$

$I(X:Y)$

"Uncertainty of X minus uncertainty of X when Y known"

Information

Example

X is the team A result, Y is the coin toss outcome. Then

$$I(X : Y) = 1 - 0.864787\ldots = 0.135213\ldots$$

Information

Example

X is the team A result, Y is the coin toss outcome. Then

$$I(X:Y) = 1 - 0.864787\ldots = 0.135213\ldots$$

Example

Team B wins with 1/2 probability, but with 99 probability as home team and only with 5 probability as visitor. Then

$$\begin{aligned} H(X \mid h) &= -(\frac{99}{100}\log_2\frac{99}{100} + \frac{1}{100}\log_2\frac{1}{100}) = 0.0807931\ldots, \\ H(X \mid v) &= -(\frac{5}{100}\log_2\frac{5}{100} + \frac{95}{100}\log_2\frac{95}{100}) = 0.286397\ldots \end{aligned}$$

Example (Continued)

$$H(X \mid Y) = \frac{1}{2}H(X \mid h) + \frac{1}{2}H(X \mid v) = 0.274894\ldots$$

and

$$I(X : Y) = 1 - 0.274894\ldots = 0.725106\ldots$$

Quantum Entropy

John von Neumann (1903–1957)

- Quantum entropy by Gedanken Experiment (1927)
- Coincides with Shannon (and Boltzmann) entropy on classical systems

Quantum formalism revisited

n-level system $\leftrightarrow$ n perfectly distinguishable values

Formalism based on $H_n \simeq \mathbb{C}^n$ (n-dimensional Hilbert space)

- Hermitian inner product $\langle \boldsymbol{x} \mid \boldsymbol{y} \rangle = x_1^* y_1 + \ldots + x_n^* y_n$
- Norm $||\boldsymbol{x}|| = \sqrt{\langle \boldsymbol{x} \mid \boldsymbol{x} \rangle}$
- Ket-vector $|\boldsymbol{x}\rangle = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$
- Bra-vector $\langle \boldsymbol{x}| = (|\boldsymbol{x}\rangle)^* = (x_1^*, \ldots, x_n^*)$
- Adjoint matrix: $(A^*)_{ij} = A_{ji}^*$ for $m \times n$ matrix A

More formalism

- Trace: $\mathrm{Tr}(A) = \sum_{i=1}^{n} A_{ii}$
- For orthonormal basis $\{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n\}$, $\mathrm{Tr}(A) = \sum_{i=1}^{n} \langle \boldsymbol{x}_i \mid A\boldsymbol{x}_i \rangle$
- Positivity: $A \geq 0$ iff $(\forall \boldsymbol{x})\ \langle \boldsymbol{x} \mid A\boldsymbol{x} \rangle \geq 0$
- Self-adjointness: $A^* = A$
- Unitarity: $UU^* = U^*U = I$
- Normality: $A^*A = AA^*$

Kronecker product (tensor product):

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1s} \\ a_{21} & a_{22} & \dots & a_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ a_{r1} & a_{r2} & \dots & a_{rs} \end{pmatrix}, B = \begin{pmatrix} b_{11} & b_{12} & \dots & b_{1u} \\ b_{21} & b_{22} & \dots & b_{2u} \\ \vdots & \vdots & \ddots & \vdots \\ b_{t1} & b_{t2} & \dots & b_{tu} \end{pmatrix}$$

$$A \otimes B = \begin{pmatrix} a_{11}B & a_{12}B & \dots & a_{1s}B \\ a_{21}B & a_{22}B & \dots & a_{2s}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{r1}B & a_{r2}B & \dots & a_{rs}B \end{pmatrix}$$

More formalism

- $|\,\mathbf{x}\rangle\langle\mathbf{y}\,| = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \otimes (y_1^*, \ldots, y_n^*) =$
$$\begin{pmatrix} x_1 y_1^* & x_1 y_2^* & \ldots & x_1 y_n^* \\ \vdots & \vdots & \ddots & \vdots \\ x_n y_1^* & x_n y_2^* & \ldots & x_n y_n^* \end{pmatrix}$$
- $|\,\mathbf{x}\rangle\langle\mathbf{y}\,|\,|\mathbf{z}\rangle = \langle\mathbf{y} \mid \mathbf{z}\rangle\,|\mathbf{x}\rangle$
- If especially $||\mathbf{x}|| = 1$, $|\,\mathbf{x}\rangle\langle\mathbf{x}\,|$ is a projection onto a subspace generated by $\mathbf{x}$.

More formalism

Theorem (Spectral representation)

Each normal A has spectral representation

$$A = \lambda_1 \, |\mathbf{x}_1\rangle\langle\mathbf{x}_1| + \ldots + \lambda_n \, |\mathbf{x}_n\rangle\langle\mathbf{x}_n|,$$

where $\{\mathbf{x}_1, \ldots, \mathbf{x}_n\}$ is an orthonormal basis of H_n and $\lambda_1, \ldots, \lambda_n$ the eigenvalues of A.

More formalism

Theorem (Spectral representation)

Each normal A has spectral representation

$$A = \lambda_1 \, |\mathbf{x}_1\rangle\langle\mathbf{x}_1| + \ldots + \lambda_n \, |\mathbf{x}_n\rangle\langle\mathbf{x}_n|,$$

where $\{\mathbf{x}_1, \ldots, \mathbf{x}_n\}$ *is an orthonormal basis of* H_n *and* $\lambda_1, \ldots, \lambda_n$ *the eigenvalues of A.*

- If A is self-adjoint, each $\lambda_i \in \mathbb{R}$
- If A is unitary, each λ_i has $|\lambda_i| = 1$
- If A is positive, each $\lambda_i \geq 0$.
- $\text{Tr}(A) = \lambda_1 + \ldots + \lambda_n$.

State of a physical system:

Unit-trace, positive operator T:

$$T = \lambda_1 \, |\mathbf{x}_1\rangle\langle\mathbf{x}_1| + \ldots + \lambda_n \, |\mathbf{x}_n\rangle\langle\mathbf{x}_n|,$$

where $\lambda_i \geq 0$, $\lambda_1 + \ldots + \lambda_n = 1$ (density matrix).

Structure of Quantum Mechanics

State of a physical system:

Unit-trace, positive operator T:

$$T = \lambda_1 \,|\mathbf{x}_1\rangle\langle\mathbf{x}_1| + \ldots + \lambda_n \,|\mathbf{x}_n\rangle\langle\mathbf{x}_n|,$$

where $\lambda_i \geq 0$, $\lambda_1 + \ldots + \lambda_n = 1$ (density matrix).

Observables:

Self-adjoint operator A:

$$A = \mu_1 \,|\mathbf{y}_1\rangle\langle\mathbf{y}_1| + \ldots + \mu_n \,|\mathbf{y}_n\rangle\langle\mathbf{y}_n|,$$

where $\mu_i \in \mathbb{R}$ are the potential values of A

Minimal interpretation:

$$\mathbb{P}(\mu_i) = \mathrm{Tr}(T\,|\boldsymbol{y}_i\rangle\langle\boldsymbol{y}_i|)$$

is the probability of seeing value μ_i if A is observed when the system is in state T.

Example

Let $n = 2$ (quantum bit), $|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $|1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$$T = \frac{1}{2}\,|0\rangle\langle 0| + \frac{1}{2}\,|1\rangle\langle 1| = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$$

and

$$A = \sigma_z = 1\cdot\,|0\rangle\langle 0| - 1\cdot\,|1\rangle\langle 1|\,.$$

Then

$$\begin{aligned} \mathbb{P}(1) &= \mathrm{Tr}(T\,|0\rangle\langle 0|) = \mathrm{Tr}(\frac{1}{2}\,|0\rangle\langle 0|) = \frac{1}{2}, \text{ and} \\ \mathbb{P}(-1) &= \mathrm{Tr}(T\,|1\rangle\langle 1|) = \mathrm{Tr}(\frac{1}{2}\,|1\rangle\langle 1|) = \frac{1}{2}. \end{aligned}$$

Example

$$T = 1\cdot\,|0\rangle\langle 0|\,+0\cdot\,|1\rangle\langle 1| = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

and

$$A = \sigma_z = 1\cdot\,|0\rangle\langle 0|\,-1\cdot\,|1\rangle\langle 1|\,.$$

Then

$$\begin{aligned} \mathbb{P}(1) &= \mathrm{Tr}(T\,|0\rangle\langle 0|) = \mathrm{Tr}(|0\rangle\langle 0|) = 1, \text{ and} \\ \mathbb{P}(-1) &= \mathrm{Tr}(T\,|1\rangle\langle 1|) = \mathrm{Tr}(0) = 0. \end{aligned}$$

Example

$$T = \frac{1}{2}\,|0\rangle\langle 0| + \frac{1}{2}\,|1\rangle\langle 1| = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$$

and

$$A = \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = 1\cdot\,|\mathbf{y}_1\rangle\langle\mathbf{y}_1| - 1\cdot\,|\mathbf{y}_2\rangle\langle\mathbf{y}_2|,$$

where $\mathbf{y}_1 = \frac{1}{\sqrt{2}}(1,1)$ and $\mathbf{y}_2 = \frac{1}{\sqrt{2}}(-1,1)$. Then

$$\begin{aligned} \mathbb{P}(1) &= \mathrm{Tr}(T\,|\mathbf{y}_1\rangle\langle\mathbf{y}_1|) = \frac{1}{2}, \text{ and} \\ \mathbb{P}(-1) &= \mathrm{Tr}(T\,|\mathbf{y}_2\rangle\langle\mathbf{y}_2|) = \frac{1}{2}. \end{aligned}$$

Remark

The *expected value* of observable A in state T is

$$\begin{aligned}
\mathbb{E}_T(A) &= \sum_{i=1}^{n} \mu_i \mathbb{P}(\mu_i) \\
&= \sum_{i=1}^{n} \mu_i \mathrm{Tr}(T \,|\boldsymbol{y}_i\rangle\langle\boldsymbol{y}_i|) \\
&= \mathrm{Tr}(TA).
\end{aligned}$$

The State Set Structure

The State Set Structure

- If T_1 and T_2 are states, and $\lambda \in (0, 1)$, then also $\lambda T_1 + (1 - \lambda) T_2$ is. (convexity)

The State Set Structure

- If T_1 and T_2 are states, and $\lambda \in (0,1)$, then also $\lambda T_1 + (1-\lambda)T_2$ is. (convexity)
- T is *extremal* if $T = \lambda T_1 + (1-\lambda)T_2$ with $\lambda \in (0,1)$ implies $T_1 = T_2$.

The State Set Structure

- If T_1 and T_2 are states, and $\lambda \in (0, 1)$, then also $\lambda T_1 + (1 - \lambda) T_2$ is. (convexity)
- T is *extremal* if $T = \lambda T_1 + (1 - \lambda) T_2$ with $\lambda \in (0, 1)$ implies $T_1 = T_2$.

Extremals are called *pure or vector states*

The State Set Structure

- If T_1 and T_2 are states, and $\lambda \in (0,1)$, then also $\lambda T_1 + (1-\lambda) T_2$ is. (convexity)
- T is *extremal* if $T = \lambda T_1 + (1-\lambda) T_2$ with $\lambda \in (0,1)$ implies $T_1 = T_2$.

Extremals are called *pure or vector states*

Lemma

T is pure if and only if $T = |\mathbf{x}\rangle\langle\mathbf{x}|$ for some unit-length $\mathbf{x}$.

- If T_1 and T_2 are states, and $\lambda \in (0,1)$, then also $\lambda T_1 + (1-\lambda) T_2$ is. (convexity)
- T is *extremal* if $T = \lambda T_1 + (1-\lambda) T_2$ with $\lambda \in (0,1)$ implies $T_1 = T_2$.

Extremals are called *pure or vector states*

Lemma

T is pure if and only if $T = |\mathbf{x}\rangle\langle\mathbf{x}|$ for some unit-length $\mathbf{x}$.

- For a pure state $T = |\mathbf{x}\rangle\langle\mathbf{x}|$ and observable $A = \sum_{i=1}^{n} \mu_i \, |\mathbf{y}_i\rangle\langle\mathbf{y}_i|$

$$\mathbb{P}(\mu_i) = \mathrm{Tr}(T \, |\mathbf{y}_i\rangle\langle\mathbf{y}_i|) = \langle\mathbf{y}_i \,||\mathbf{x}\rangle\langle\mathbf{x}||\mathbf{y}_i\rangle\langle\mathbf{y}_i \,|\, \mathbf{y}_i\rangle = |\langle\mathbf{x} \,|\, \mathbf{y}_i\rangle|^2.$$

Pure states

Let $T = |\,\boldsymbol{x}\rangle\langle\boldsymbol{x}\,|$ be a pure state and

$$A = \mu_1 \,|\,\boldsymbol{y}_1\rangle\langle\boldsymbol{y}_1\,| + \ldots + \mu_n \,|\,\boldsymbol{y}_n\rangle\langle\boldsymbol{y}_n\,|$$

an observable. In representation

$$\boldsymbol{x} = \alpha_1 \boldsymbol{y}_1 + \ldots + \alpha_n \boldsymbol{y}_n$$

$\alpha_i = \langle\boldsymbol{y}_i \mid \boldsymbol{x}\rangle$ (amplitude of $\boldsymbol{y}_i$), so

$$\mathbb{P}(\mu_i) = |\alpha_i|^2 .$$

Pure states

Let $T = |\boldsymbol{x}\rangle\langle\boldsymbol{x}|$ be a pure state and

$$A = \mu_1 \,|\boldsymbol{y}_1\rangle\langle\boldsymbol{y}_1| + \ldots + \mu_n \,|\boldsymbol{y}_n\rangle\langle\boldsymbol{y}_n|$$

an observable. In representation

$$\boldsymbol{x} = \alpha_1\boldsymbol{y}_1 + \ldots + \alpha_n\boldsymbol{y}_n$$

$\alpha_i = \langle\boldsymbol{y}_i \mid \boldsymbol{x}\rangle$ (amplitude of $\boldsymbol{y}_i$), so

$$\mathbb{P}(\mu_i) = |\alpha_i|^2 .$$

Corollary

For each pure state T there is a nontrivial observable A such that $\mathbb{P}(\mu_1) = 1$ for a potential value μ_1 of A.

Pure states

Remark

For a pure state $T = |\mathbf{x}\rangle\langle\mathbf{x}|$ the expected value of observable A is

$$\mathbb{E}_T(A) = \mathrm{Tr}(TA) = \mathrm{Tr}(|\mathbf{x}\rangle\langle\mathbf{x}|\, A) = \langle\mathbf{x} \,||\, \mathbf{x}\rangle\langle\mathbf{x}|\, A\mathbf{x}\rangle = \langle\mathbf{x} \mid A\mathbf{x}\rangle.$$

Pure states

Remark

For a pure state $T = |\mathbf{x}\rangle\langle\mathbf{x}|$ the expected value of observable A is

$$\mathbb{E}_T(A) = \mathrm{Tr}(TA) = \mathrm{Tr}(|\mathbf{x}\rangle\langle\mathbf{x}|\,A) = \langle\mathbf{x}\,||\mathbf{x}\rangle\langle\mathbf{x}|\,A\mathbf{x}\rangle = \langle\mathbf{x}\mid A\mathbf{x}\rangle.$$

Remark

For any $\theta \in \mathbb{R}$,

$$|\,e^{i\theta}\mathbf{x}\rangle\langle e^{i\theta}\mathbf{x}\,| = |\,\mathbf{x}\rangle\langle\mathbf{x}\,|,$$

so pure state presentation as a unit-length vector is not unique.

Example

Let $|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $|1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

vector $\frac{1}{\sqrt{2}}\,|0\rangle + \frac{1}{\sqrt{2}}\,|1\rangle$ corresponds to a state

$$\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \otimes \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix},$$

and

$$\mathbb{P}(1) = \mathrm{Tr}\left(\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right) = \mathrm{Tr}\left(\begin{pmatrix} \frac{1}{2} & 0 \\ \frac{1}{2} & 0 \end{pmatrix}\right) = \frac{1}{2}.$$

This could be read directly from the vector presentation.

Example

Let $\boldsymbol{y}_1 = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$, $\boldsymbol{y}_2 = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$, and

$$A = 1\,|\boldsymbol{y}_1\rangle\langle\boldsymbol{y}_1| - 1\cdot|\boldsymbol{y}_2\rangle\langle\boldsymbol{y}_2| = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} - \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Then for vector $\boldsymbol{x} = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$

$$\mathbb{P}(-1) = \mathrm{Tr}\left(\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}\begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix}\right) = \mathrm{Tr}\left(\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\right) = 0.$$

This could be directly read from

$$\boldsymbol{x} = 1\cdot\boldsymbol{y}_1 + 0\cdot\boldsymbol{y}_2.$$

Compound Systems

Down $\rightarrow$ Up:

Tensor product construction: $T = T_1 \otimes T_2$, $A = A_1 \otimes A_2$

Down $\to$ Up:

Tensor product construction: $T = T_1 \otimes T_2$, $A = A_1 \otimes A_2$

Example

Pure state

$$\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) = \frac{1}{2}(|00\rangle + |01\rangle + |10\rangle + |11\rangle)$$

Or:

$$\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \otimes \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{pmatrix}$$

Compound Systems

Up $\to$ Down:

By partial trace: $T_1 = \mathrm{Tr}_1(T)$

Partial trace

T_1 is chosen so that $\mathrm{Tr}(T(A_1 \otimes I)) = \mathrm{Tr}(T_1 A_1)$ for each observable A_1

Up $\to$ Down:

By partial trace: $T_1 = \mathrm{Tr}_1(T)$

Partial trace

T_1 is chosen so that $\mathrm{Tr}(T(A_1 \otimes I)) = \mathrm{Tr}(T_1 A_1)$ for each observable A_1

Remark

$\mathrm{Tr}(T(A_1 \otimes I))$ is the average value of observable $A_1 \otimes I$ in state T. As well, $\mathrm{Tr}(T_1 A_1)$ is the average value of observable A_1 in state T_1. T_1 is unique and an explicit formula for $\mathrm{Tr}_1(T)$ exists.

Example

A vector

$$\frac{1}{\sqrt{2}}\left|00\right\rangle + \frac{1}{\sqrt{2}}\left|11\right\rangle$$

correspondes to a pure state

$$T = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix} \otimes \left(\frac{1}{\sqrt{2}}, 0, 0, \frac{1}{\sqrt{2}}\right) = \begin{pmatrix} \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \end{pmatrix}$$

Subsystem states:

$$T_1 = T_2 = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$$

Quantum Entropy

Quantum Entropy

von Neumann Entropy

$S = -K\mathrm{Tr}(T \log T)$

Quantum Entropy

von Neumann Entropy

$S = -K\mathrm{Tr}(T \log T)$

where $f(T)$ for

$$T = p_1 |\boldsymbol{x}_1\rangle\langle\boldsymbol{x}_1| + \ldots + p_n |\boldsymbol{x}_n\rangle\langle\boldsymbol{x}_n|$$

is defined as

$$\begin{aligned} f(T) &= f(\lambda_1 |\boldsymbol{x}_1\rangle\langle\boldsymbol{x}_1| + \ldots + \lambda_n |\boldsymbol{x}_n\rangle\langle\boldsymbol{x}_n|) \\ &= f(\lambda_1) |\boldsymbol{x}_1\rangle\langle\boldsymbol{x}_1| + \ldots + f(\lambda_n) |\boldsymbol{x}_n\rangle\langle\boldsymbol{x}_n| . \end{aligned}$$

von Neumann Entropy

$S = -K\mathrm{Tr}(T \log T)$

where $f(T)$ for

$$T = p_1 \, |\boldsymbol{x}_1\rangle\langle\boldsymbol{x}_1| + \ldots + p_n \, |\boldsymbol{x}_n\rangle\langle\boldsymbol{x}_n|$$

is defined as

$$\begin{aligned} f(T) &= f(\lambda_1 \, |\boldsymbol{x}_1\rangle\langle\boldsymbol{x}_1| + \ldots + \lambda_n \, |\boldsymbol{x}_n\rangle\langle\boldsymbol{x}_n|) \\ &= f(\lambda_1) \, |\boldsymbol{x}_1\rangle\langle\boldsymbol{x}_1| + \ldots + f(\lambda_n) \, |\boldsymbol{x}_n\rangle\langle\boldsymbol{x}_n| \, . \end{aligned}$$

Hence

$T \log T = p_1 \log p_1 \, |\boldsymbol{x}_1\rangle\langle\boldsymbol{x}_1| + \ldots + p_n \log p_n \, |\boldsymbol{x}_n\rangle\langle\boldsymbol{x}_n|$

von Neumann Entropy

$S = -K\mathrm{Tr}(T \log T)$

where $f(T)$ for

$$T = p_1 \,|\boldsymbol{x}_1\rangle\langle\boldsymbol{x}_1| + \ldots + p_n \,|\boldsymbol{x}_n\rangle\langle\boldsymbol{x}_n|$$

is defined as

$$\begin{aligned} f(T) &= f(\lambda_1 \,|\boldsymbol{x}_1\rangle\langle\boldsymbol{x}_1| + \ldots + \lambda_n \,|\boldsymbol{x}_n\rangle\langle\boldsymbol{x}_n|) \\ &= f(\lambda_1)\,|\boldsymbol{x}_1\rangle\langle\boldsymbol{x}_1| + \ldots + f(\lambda_n)\,|\boldsymbol{x}_n\rangle\langle\boldsymbol{x}_n| \,. \end{aligned}$$

Hence

$$T \log T = p_1 \log p_1 \,|\boldsymbol{x}_1\rangle\langle\boldsymbol{x}_1| + \ldots + p_n \log p_n \,|\boldsymbol{x}_n\rangle\langle\boldsymbol{x}_n|$$

and

$$S(T) = -K\mathrm{Tr}(T \log T) = -K(p_1 \log p_1 + \ldots + p_n \log p_n)$$

Quantum Entropy

Quantum Entropy

For a pure state $T = |\mathbf{x}\rangle\langle\mathbf{x}|$

$S(T) = -1 \cdot \log 1 = 0$

For a pure state $T = |\boldsymbol{x}\rangle\langle\boldsymbol{x}|$

$S(T) = -1 \cdot \log 1 = 0$

Example

Let A and B be qubits with joint state $T = |\boldsymbol{x}\rangle\langle\boldsymbol{x}|$, where $\boldsymbol{x} = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$.

For a pure state $T = |\mathbf{x}\rangle\langle\mathbf{x}|$

$S(T) = -1 \cdot \log 1 = 0$

Example

Let A and B be qubits with joint state $T = |\mathbf{x}\rangle\langle\mathbf{x}|$, where $\mathbf{x} = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$.

- $S(T) = 0$, but for subsystem states $S(T_1) = S(T_2) = 1$.

Quantum Entropy

For a pure state $T = |\mathbf{x}\rangle\langle\mathbf{x}|$

$S(T) = -1 \cdot \log 1 = 0$

Example

Let A and B be qubits with joint state $T = |\mathbf{x}\rangle\langle\mathbf{x}|$, where $\mathbf{x} = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$.

- $S(T) = 0$, but for subsystem states $S(T_1) = S(T_2) = 1$.
- Conditional entropy
  $S(T_1 \mid T_2) = S(T_1, T_2) - S(T_2) = 0 - 1 = -1$

For a pure state $T = |x\rangle\langle x|$

$S(T) = -1 \cdot \log 1 = 0$

Example

Let A and B be qubits with joint state $T = |\mathbf{x}\rangle\langle\mathbf{x}|$, where $\mathbf{x} = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$.

- $S(T) = 0$, but for subsystem states $S(T_1) = S(T_2) = 1$.
- Conditional entropy $S(T_1 \mid T_2) = S(T_1, T_2) - S(T_2) = 0 - 1 = -1$
- Mutual information:

$$I(T_1 : T_2) = S(T_1) - S(T_1 \mid T_2) = 1 - (-1) = 2$$

Theorem (Holevo Bound)

Let $\rho_1, \ldots, \rho_n$ be states of n-level quantum system, produced with probabilities $p_1, \ldots, p_n$. Let also X be a random variable with value i if ρ_i is produced, and Y any observable on H_n. Then

$$I(X:Y) \leq S(\sum_{i=1}^{n} p_i \rho_i) - \sum_{i=1}^{n} p_i S(\rho_i)$$