Seminar on automata 1996.

S. Eilenberg: Automata, Languages and Machines. Vol A,
Academic Press, New York 1974.

1. Preliminaries

Let us first recall some terminology: A *semiring* K is equipped with two binary operations which are referred as addition and multiplication. It is required that K forms an additive commutative monoid and a monoid with respect to multiplication. Neutral elements are denoted by 0 and 1 respectively. Furthermore, it is required that the additive and multiplitive structures obey natural distribution laws.

Let X be any set, and assume that K is a semiring. K is supposed to be commutative unless stated otherwise. *A K-subset* A of X is then defined to be a function $A : X \to K$. This is an obvious generalisation of the consept of an ordinary subset, which can be defined to be mappings from X to a binary semiring. If $x \in X$, then the image of x, xA in K is called the *multiplicity with which x belongs to A*. We denote the set of all K-subsets of X by K^X. The K-subset A of X is said to be *unambigious*, is xA can take only values 0 and 1 in K.

Examples of unambigious K-subsets can be given, like

$$\begin{aligned} X : X \to& K, \quad aX = 1 \text{ for all } a \in X, \\ \emptyset : X \to& K, \quad a\emptyset = 0 \text{ for all } a \in X, \\ x : X \to& K, \quad ax = \begin{cases} 1, & \text{if } a = x \\ 0, & \text{if } a \neq x \end{cases}. \end{aligned}$$

The sets in the last example are called *singletons*.

We can now define operations as the *union* of K-subsets. Let $\{A_i, i \in I\}$ be an indexed family of K-subsets of X. Then we define their union (or *sum*) to be a K-subset by

$$x\left(\bigcup_{i\in I} A_i\right) = \sum_{i\in I} xA_i.$$

This definition requires some comments, if K is not a *complete semiring*, i.e a semiring for which the sum above always exists and is well-defind. Then we must require the family A_i to be *locally finite*, i.e for each $x \in X$ $xA_i = 0$ holds for all but a finite number of i:s.

The *intersection* $A \cap B$ of two K-subsets A and B is defined to be

$$x(A \cap B) = (xA)(xB),$$

and by *multiplication of a K-subset* A by an element of $k \in K$ we mean

$$x(kA) = k(xA).$$

It is now obvious that for each K-subset A we have the following *expansion in terms of singletons*:

$$A = \sum_{x \in X} (xA)x.$$

Family $(xA)x$ is locally finite because

$$y((xA)x) = \begin{cases} xA, & \text{if } y = x \\ 0, & \text{if } y \neq x \end{cases}$$

By a *product* of two K-subsets A and B we understand

$$z(AB) = \sum_{xy=z} (xA)(yB),$$

where sum runs over all decompositions of z. If there exists only a finite number of them, then the right hand side is well-defined

Let Σ be a finite alphabet and K a commutative semiring. A $K - \Sigma$-*automaton* $\mathcal{A} = (Q, I, T)$ is given by finite set Q of *states* with K-subsets I of *initial states* and T of *terminal states*, and by a K-subset E of the cartesian product $Q \times \Sigma \times Q$, whose elements are called *transitions*. E can be extended to be a $Q \times \Sigma^* \times Q$-subset by setting $(p, s, q)E = 0$, if $s \notin \Sigma$. If the image $k = (p, \sigma, q)E$ is not the zero element, we say that the *edge*

$$p \xrightarrow{k\sigma} q$$

is in $\mathcal{A}$. Mapping $k\sigma$ is called the *label* of the edge. A *path* in $\mathcal{A}$ is a sequence of consequtive edges, for instance, let a path c be given by

$$p \xrightarrow{k_1\sigma_1} q_1 \xrightarrow{k_2\sigma_2} \quad \dots \quad \xrightarrow{k_{n-1}\sigma_{n-1}} q_{n-1} \xrightarrow{k_n\sigma_n} q.$$

If we denote $k = k_1 \dots k_n$ and $s = \sigma_1 \dots \sigma_n$, we say that the *label* of the path is $|c| = ks$ and the *length* of c is $||c|| = n = |s|$.

The *behaviour* of $\mathcal{A}$ is the K-subset of Σ^* defined to be

$$|\mathcal{A}| = \sum_{p,q \in Q} \sum_{c} (pI)\,|c|\,(qT),$$

where c runs over all paths from p to q. For each $s \in \Sigma^*$ there exists only a finite number of paths from p to q labelled as ks, $k \in K$, and the summation is therefore finite. Now

$$s\,|\mathcal{A}| = \sum_{p,q \in Q} \sum_{\substack{c:p \to q \\ |c| = ks}} (pI)k(qT).$$

Consider now the K-subset E. It is a mapping $Q \times \Sigma \times Q \to K$, and therefore E_{pq} defined by

$$\sigma E_{pq} = (p, \sigma, q)E$$

is a mapping from Σ to K, i.e a K-subset of Σ (or a K-subset of Σ^* as well, when E is extended). Hence E can be understood as a matrix

$$E : Q \times Q \to K^{\Sigma},$$

and it is called *transition matrix.* As well we can understand E as a matrix with entries

$$E_{pq} : \Sigma^* \to K,$$

where $sE_{pq} = 0$ if $s \notin \Sigma$. K^{Σ^*} forms now a semiring where addition and product are defined by

$$s(D+E)_{pq} = sD_{pq} + sE_{pq} \quad \text{and} \quad s(DE)_{pq} = s\sum_{r\in Q}(D_{pr})(E_{rq}).$$

The zero element in this semiring is given by $s\mathbf{0}_{pq} = 0$ always and unit element is given by $\mathbf{1}_{pq} = 0$ if $p \neq q$ and

$$s\mathbf{1}_{pp} = \begin{cases} 0, & \text{if } s \neq 1 \\ 1, & \text{if } s = 1 \end{cases}.$$

We can also define $E^0 = \mathbf{1}$, $E^1 = E$, and $E^{n+1} = E^n E$. We can compute that

$$sE^n_{pq} = \sum_{\substack{c:p\to q \\ |c|=ks}} k$$

and see that

$$E^n_{pq} = \sum_{\substack{c:p\to q \\ ||c||=n}} |c|\,.$$

Further, we can define

$$E^*_{pq} = \sum_{n=0}^{\infty} E^n_{pq}.$$

Then we have $E^*_{pq} = \sum_{c:p\to q} |c|$, and the behaviour of an automaton can be represented as a matrix product

$$|\mathcal{A}| = \sum_{p,q\in Q} (pI)E^*_{pq}(qT) = IE^*T,$$

Where I is understood as a row vector with entries $I_{1p} = pI \in K$ and T as a column vector with $T_{p1} = pT \in K$. In general, each K-subset of Q can be regarded as a row vector of elementes of K. Furthermore, for each $s \in \Sigma^*$ the matrix sE^* is a $Q \times Q$-matrix of elementes of K. Let us denote

$$Xs = X(sE^*).$$

Then Xs is a row vector with $(Xs)_{1q} = \sum_{p\in Q} X_p(sE^*)_{pq}$. On the other hand, if we regard X to be a column vector, we may denote

$$sX = (sE^*)X,$$

and observe that $(sX)_{p1} = \sum_{q\in Q}(sE^*)_{pq}X_q$. It is straightforward to verify that the assosiation laws as $X(st) = (Xs)t$, $(kX)s = k(Xs)$ hold. Espcesially, for each $s \in \Sigma^*$ we have

$$s\,|\mathcal{A}| = s(IE^*T) = I(sE^*)T = (Is)T = I(sT).$$

Let Σ be a finite alphabet. A K-subset A of Σ^* is said to be *recognizable*, if there exists a K-Σ-automaton $\mathcal{A}$ *recognizing* A, i.e. an automaton such that $|\mathcal{A}| = A$.

Proposition 1.1. *The class of recognizable K-subsets of Σ^* is closed under finite union, intersection, and revesal.*

By a reversal A^ρ of a K-subset A of Σ^* we understand the composite mapping

$$\Sigma^* \xrightarrow{\rho} \Sigma^* \xrightarrow{A} K.$$

Proposition 1.2. *If $f : \Gamma^* \to \Sigma^*$ is a fine morphism and A is a recognizable K-subset of Σ^*, then there exists a K-Γ-automaton recognizing Af^{-1}.*

Proposition 1.3. *Let $f : \Gamma^* \to \Sigma^*$ be a morphism satisfying $1 = 1f^{-1}$, and A a recognizable K-subset of Γ^*. Then Af if a recognizable K-subset of Σ^*.*

All the proof of propositions 1.1-1.3 are analogues of propositions referring to Σ-automata.

Proposition 1.4. *Let A be a recognizable K-subset of Σ^*. Then kA is a recognizable K-subset of Σ^*.*

Proof. Let $\mathcal{A} = (Q, I, T)$ be a K-Σ-automaton recognizing A. Then kA is recognized by

$$k\mathcal{A} = (Q, kI, T)$$

with transition matrix unchanged, since

$$|k\mathcal{A}| = (kI)E^*T = k(IE^*T) = k\,|\mathcal{A}| = kA.$$

A K-Σ-automaton is said to be *normalized*, if $I = i$ and $T = t$ are distinct singletons and if there are no edges of forms

$$q \xrightarrow{k\sigma} i, \qquad t \xrightarrow{k\sigma} q$$

for non-zero k. For normalized automaton $\mathcal{A}$ obviously holds $|\mathcal{A}| \subset \Sigma^+$.

Proposition 1.5. *Any K-Σ-automaton $\mathcal{A}$ can be converted into a normalized automaton $\mathcal{A}'$ which satisfies*

$$|\mathcal{A}'| = |\mathcal{A}| \cap \Sigma^+.$$

Proof. Let $\mathcal{A} = (Q, I, T)$ be a K-Σ-automaton. Define Q' to be $Q' = Q \cup i \cup t$, where $i \neq t$ are new states. Furthermore, define a transition matrix E' by

$$\begin{aligned}
E'_{pq} &= E_{pq}\\
E'_{iq} &= (IE)_{1q} = \sum_{p\in Q} I_p E_{pq}\\
E'_{pt} &= (ET)_{p1} = \sum_{q\in Q} E_{pq} T_q\\
E'_{it} &= (IET)_{11} = \sum_{p,q\in Q} I_p E_{pq} T_q\\
E'_{tt} &= E'_{ii} = E'_{pi} = E_{ti} = E_{tq} = 0
\end{aligned}$$

It can now be computed that ${E'}^*_{it} = IE^+T$, where $E^+ = EE^*$. Now $\mathcal{A}'$ is normalized, and

$$|\mathcal{A}'| = i{E'}^*t = {E'}^*_{it} = IE^+T = IE^*T \cap \Sigma^+ = |\mathcal{A}| \cap \Sigma^+.$$

□

Theorem 1.6 (Schüzenberger). *A K-subset A of Σ^+ is recognizable if and only if there is an integer $n > 1$ and an $n \times n$-matrix E of K-subsets of Σ such that $A = E^+_{1n}$.*

Proof. Assume first that A is recognizable. By proposition 1.5 there exists a normalized K-Σ-automaton $\mathcal{A} = (Q, i, t)$ with transition matrix E recognizing A. By renaming the states we can assume that $\mathcal{A} = (\{1, \dots, n\}, 1, n)$. Because $\mathcal{A}$ is normalized, we have $n > 1$, and finally $A = |\mathcal{A}| = 1E^*n = E^*_{1n} = E^+_{1n}$.

Assume conversely that $A = E^+_{1n}$ where E in an $n \times n$-matrix of K-subsets of Σ and $n > 1$. Let $\mathcal{A} = (\{1, \dots, n\}, 1, n)$ be a K-Σ automaton with transition matrix E. Then $|\mathcal{A}| = E^+_{1n} = A$. □

Corollary 1.7. *Let E be an $n \times n$-matrix of K-subsets of Σ. Then for any indicies $i, j \in \{1, \dots n\}$ the K-subsets E^+_{ij} and E^*_{ij} of Σ^+ and Σ^* are recognizable.*

2. The equality theorem

Now we assume that K is a subsemiring of a field F which is assumed to be commutative.

Lemma 2.1. *Let $\mathcal{A} = (Q, I, T)$ be a K-Σ-automaton. If $s\,|\mathcal{A}| = 0$ for all $s \in \Sigma^*$ satisfying $|s| < \operatorname{Card} Q \neq 0$, then $s\,|\mathcal{A}| = 0$ for all $s \in \Sigma^*$.*

Proof. By assumption K is a subsemiring of a field F. Therefore we can regard $\mathcal{A}$ as an F-Σ-automaton as well; if $|\mathcal{A}|$ is a zero mapping as a F-subset, then it is a zero mapping as a K-subset. Therefore we can assume that K is a field. Now K^Q, all the K-subsets of Q can be given a structure of a vector space over K.

The addition in the K-vector space is given by the sum of K-subsets and the scalar multiplication is the usual multiplication of a K-subset by an element of K. Let us, for example, show how the distribution law is verified. Let X_1 and X_2 be K-subsets of Q and k an element in K. Then for any $q \in Q$ we have

$$\begin{aligned} q(k(X_1 + X_2)) &= k(q(X_1 + X_1)) = k(qX_1 + qX_2) = k(qX_1) + k(qX_2) \\ &= q(kX_1) + q(kX_2) = q(kX_1 + kX_2). \end{aligned}$$

Therefore $k(X_1 + X_2) = kX_1 + kX_2$. It is obvious that K^Q is generated by the singleton K-subsets of Q. Furthermore, the singleton mappings are linearly independent, since if we have an expression

$$k_1q_1 + k_2q_2 + \ldots + k_nq_n = 0,$$

taking the image of q_i we get

$$\begin{aligned} 0 &= q_i(k_1q_1 + \ldots + k_nq_n) = q_i(k_1q_1) + \ldots + q_i(k_mq_n) \\ &= k_1(q_iq_1) + \ldots + k_n(q_iq_n) = k_i(q_iq_1) = k_i. \end{aligned}$$

Now we obtain that the dimension of K^Q is $n = \operatorname{Card} Q$. The claim becomes now: If $(Is)T = 0$, for $|s|$ satisfying $|s| < n = \operatorname{Card} Q$, then $(Is)T = 0$ for all $s \in \Sigma^*$.

We define W to be the set of those row vectors X_p which are orthogonal to T, namely

$$W = \{X \mid X \in K^Q, XT = 0\}.$$

If $\dim W = n$ then $W = K^Q$ and there is nothing left to prove. Therefore we can assume that $\dim W \leq n-1$, and furthermore, we can assume that $I, T \neq 0$. Define also V_k to be a subspace of K^Q generated by vectors Is with $|s| \leq k$

$$V_k = \langle\{Is \mid |s| \leq k\}$$

Then $V_0 = \langle I \rangle$ and obviously we have

$$V_0 \subset \ldots \subset V_{n-1} \subset W.$$

Counting the dimensions in both sides we obtain that $V_k = V_{k+1}$ for some $0 \leq k < n-1$. The subspace V_{k+2} is generated by all the vectors Is, where $|s| \leq k+2$, that is, by all vectors X and $X\sigma$, where X is in $V_{k+1} = V_k$. Therefore $V_{k+1} = V_{k+2}$ and by induction, $V_k = V_{k+p}$ for all positive p. Thus, for any $s \in \Sigma^*$ we have $Is \in W$, so $(Is)T = 0$ $\square$

Lemma 2.2. *Let* $\mathcal{A} = (Q, I_i, T)$ $i \in \{1, 2\}$ *be* K-$\Sigma-$*automata that differ only in their initial subsets. Then* $|\mathcal{A}_1| = |\mathcal{A}_2|$ *if and only if*

$$s\,|\mathcal{A}_1| = s\,|\mathcal{A}_2|$$

For all those $s \in \Sigma^*$ *satisfying* $|s| < \operatorname{Card} Q$.

Proof. One direction is trivial. Therefore, assume that $s \in \Sigma^*$ with $|s| < \operatorname{Card} Q$ satisfy $s\,|\mathcal{A}_1| = s\,|\mathcal{A}_2|$. Consider the K-Σ-automaton $\mathcal{A} = (Q, I, T)$ with $I = I_1 - I_2$. Because K can be assumed to be a field, the map $I_1 - I_2$ can always be defined in an obvious way. It is now straightforward to compute

$$s\,|\mathcal{A}| = (Is)T = ((I_1 - I_2)s)T = (I_1 s)T - (I_2 s)T = s\,|\mathcal{A}_1| - s\,|\mathcal{A}_2|\,.$$

The claim follows now directly for the previous lemma. □

Theorem 2.3 (The equality theorem). *Let* A_1 *and* A_2 *be recognizable* K-*subsets of* Σ^* *and* $\mathcal{A}_i = (Q_i, I_i, T_i)$ $i \in \{1, 2\}$ *two* K-*automata recognizing them respectively. Suppose that* $sA_1 = sA_2$ *for all* $s \in \Sigma^*$ *satisfying* $|s| < \operatorname{Card} Q_1 + \operatorname{Card} Q_2$. *Then* $A_1 = A_2$.

Proof. Without loss of generality, the sets Q_1 and Q_2 can be assumed to be disjoint. Define a K-Σ-automaton

$$\mathcal{A}_1 \cup \mathcal{A}_2 = (Q_1 \cup Q_2, I_1 \cup I_2, T_1 \cup T_2)$$

with transition matrix

$$E = \begin{pmatrix} E_1 & 0 \\ 0 & E_2 \end{pmatrix}$$

Then modify this automaton to obtain automata

$$\mathcal{B}_i = (Q_1 \cup Q_2, I_i, T_1 \cup T_2), \quad i \in \{1, 2\}$$

with transition matrix unchanged. We see that $|\mathcal{B}_i| = |\mathcal{A}_i|$ for $i \in \{1, 2\}$, for instance,

$$|\mathcal{B}_1| = \begin{pmatrix} I_1 & 0 \end{pmatrix} \begin{pmatrix} E_1^* & 0 \\ 0 & E_2^* \end{pmatrix} \begin{pmatrix} T_1 \\ T_2 \end{pmatrix} = I_1 E_1^* T_1 = |\mathcal{A}_1|\,.$$

The claim follows now from lemma 2.2. □

Now we can state a decidability result:

Theorem 2.4. *Given any two* K-Σ-*automata* $\mathcal{A}_1$ *and* $\mathcal{A}_2$, *it is decidable whether* $|\mathcal{A}_1| = |\mathcal{A}_2|$.

It is here silently assumed that the semiring K is known well enough to carry out all computations for $|s| < \operatorname{Card} Q_1 + \operatorname{Card} Q_2$.

3. The undecidability of inclusion

Since now we assume that our semiring is $\mathbb{N}_0$, the set of all nonnegative integers equipped with the natural multiplication and addition. By notation $\mathbf{k}$ we understand the set $\{0, \ldots, k-1\}$. The source of the result to be stated is well-known

Post correspondence problem. *A finite alphabet Σ and two morphisms g, $h : \Sigma^* \to \mathbf{2}^*$ are given. Decide whether there exist $s \in \Sigma^+$ such that $sg = sh$.*

We denote $\Sigma = \{\sigma_1, \ldots, \sigma_n\}$, and $x_i = \sigma_i g$, $y_i = \sigma_i h$, where x_1, $y_i \in \mathbf{2}^*$. If $s \in \Sigma^+$ is given by $s = \sigma_{i_1} \ldots \sigma_{i_k}$, then $sg = sh$ if and only if

$$x_{i_1} \ldots x_{i_k} = y_{i_1} \ldots y_{i_k}.$$

The problem can then be introduced as follows: given a finite set $\{X_1, \ldots X_n\}$ of 2×1 column vectors of binary sequencies,

$$X_i = \begin{pmatrix} x_i \\ y_i \end{pmatrix}.$$

Decide wheter there exists a sequence of indicies i_1, ..., i_k such that the upper entry equals to lower one, when the matricies are catenated componentwise.

Proposition 3.1. *Post correspondence problem is undecidable.*

Let A and B be $\mathbb{N}_0$-subsets. If $sB \leq sA$ for all $s \in \Sigma^*$, we write $B \leq A$. If $B \leq A$, we define the *difference* $A - B$ to be a $\mathbb{N}_0$-subset to be

$$s(A - B) = sA - sB.$$

More generally, we define $A \dot{-} B$ to be a $\mathbb{N}_0$-subset by

$$s(A \dot{-} B) = \begin{cases} sA - sB, & \text{if } sA \leq sB \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 3.2. *Let λ be a mapping from $\mathbf{k}^*$ to $\mathbb{N}_0^{2\times 2}$ defined by*

$$s\lambda = \begin{pmatrix} k^{|s_1|} & 0 \\ \langle s_1 \rangle & 1 \end{pmatrix},$$

where $\langle s \rangle = \sigma_0 k^n + \sigma_1 k^{n-1} + \ldots \sigma_n$ is the k-adic representation of $s = s_0 s_1 \ldots s_n$. Then λ is an injective morphism.

Proof. It is obvious that λ is injective, since the k-adic representation is unique. To prove that λ is a morphism, we observe that $\langle s_1 s_2 \rangle = k^{|s_2|}\langle s_1 \rangle + \langle s_2 \rangle$. Then it is straightforward to compute

$$\begin{pmatrix} k^{|s_1|} & 0 \\ \langle s_1 \rangle & 1 \end{pmatrix} \begin{pmatrix} k^{|s_2|} & 0 \\ \langle s_2 \rangle & 1 \end{pmatrix} = \begin{pmatrix} k^{|s_1|+|s_2|} & k^{|s_1|} \\ k^{|s_2|}\langle s_1 \rangle + \langle s_2 \rangle & 1 \end{pmatrix} = \begin{pmatrix} k^{|s_1 s_2|} & k^{|s_1|} \\ \langle s_1 s_2 \rangle & 1 \end{pmatrix}.$$

□

Theorem 3.3. *Assume that B and C are recognizable $\mathbb{N}_0$-subsets such that $B \leq C$. It is undecedable whether or not there exists $s \in \Sigma^*$ satisfying $sB = sC$.*

Proof. We will show that a decision procedure for the existence of such an element $s \in \Sigma^*$ mentioned above leads to a decision procedure for the Post correspondence problem, which is known to be undecidable. Then we can conclude that the problem 3.3. is undecidable.

First we will define a morphism $\gamma : \mathbf{2}^* \to \mathbb{N}_0^{2\times 2}$ by

$$0\gamma = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}, \quad 1\gamma = \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}.$$

From lemma 3.2. it follows that γ is injective. In fact, the choise of γ is unessential, all we need to know is that γ is injective. Suppose now that we are given two morphisms g, $h : \Sigma^* \to \mathbf{2}^*$. Consider now the compositions $g\gamma, h\gamma : \Sigma^* \to \mathbb{N}_0^{2\times 2}$, and denote

$$g\gamma = \begin{pmatrix} G_{00} & G_{01} \\ G_{10} & G_{11} \end{pmatrix}, \quad h\gamma = \begin{pmatrix} H_{00} & H_{01} \\ H_{10} & H_{11} \end{pmatrix}$$

Here G_{ij} and H_{ij} are functions $\Sigma^* \to \mathbb{N}_0$, which are entirely determined when g, h, and γ are given. But a function $\Sigma^* \to \mathbb{N}_0$ is a $\mathbb{N}_0$-subset of Σ^*. These subsets are recognizable by corollary 1.7. Define then $\mathbb{N}_0$-subsets A, B, and C to be

$$\begin{aligned} A =& \sum_{i,j=1,2} (G_{ij} - H_{ij})^2 \\ B =& 2 \sum_{i,j=1,2} G_{ij} H_{ij} \\ C =& \sum_{i,j=1,2} G_{ij}^2 + H_{ij}^2. \end{aligned}$$

Here sum and multiplication is understood in the natural way, to be union and intersection. Then we have $A + B = C$ and $B \leq C$. Furthermore, B and C are recognizable, since they are obtained from recognizable $\mathbb{N}_0$-subsets by union and intersection.

Choose now $s \in \Sigma^*$. We have

$$\begin{aligned} & sB = sC \\ \iff & sA = 0 \\ \iff & s(G_{ij} - H_{ij})^2 = 0 \\ \iff & sG_{ij} = sH_{ij} \\ \iff & sg\gamma = sh\gamma \\ \iff & sg = sh, \end{aligned}$$

since γ is injective. Now we see that a decision procedure finding such an s leads to a decision procedure for the Post correspondence problem. □

Theorem 3.4. *Let B and C be recognizable $\mathbb{N}_0$-subsets of Σ^*. It is undecidable whether or not $B \leq C$.*

Proof. We take an arbitrary instance of problem in theorem 3.3, and show that if there is a decision procedure for 3.4, then there is a decision procedure for 3.3, too.

Take any two recognizable $\mathbb{N}_0$-susbets B' and C' such that $B' \leq C'$. Define $B = B' + \Sigma^+$ and $C' = C$. Then B and C are recognizable $\mathbb{N}_0$-subsets. Now the inequality $B \leq C$ holds if and only if $sB' < sC'$ for all $s \in \Sigma^+$, since $sB' = sB + s\Sigma^+$ and $sC' = sC$. If there is a decision procedure to decide whether $B \leq C$, it would lead to a procedure to decide whether $sB' < sC'$ for all $s \in \Sigma^+$. Because $B' \leq C'$, we can conclude from this whether there exists $s \in \Sigma^+$ such that $sB' = sC'$. This is however undecidable by theorem 3.3. □