Geometric characterizations for Patterson-Sullivan measures of geometrically finite Kleinian groups

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The basic setting:

- Our base space is $\bar{\mathbb{R}}^{n+1} = \mathbb{R}^{n+1} \cup \{\infty\}$ endowed with the chordal metric q.
- The space of Möbius transformations of $\bar{\mathbb{R}}^{n+1}$ endowed with the supremum norm of q is denoted by Möb$(n+1)$.
- The upper half-space $\mathbb{H}^{n+1} = \{(x_1, x_2, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} : x_{n+1} > 0\}$ is regarded as the $(n+1)$-dimensional hyperbolic space endowed with the hyperbolic metric d obtained from the differential $|dx|/x_{n+1}$.
- We will be considering balls of the form $\bar{B}^n(x, t)$, where $x \in \mathbb{R}^n$ and $t > 0$, i.e. closed balls of $\mathbb{R}^n$.

Let G be a Kleinian group.

- G is a discrete subset of $\text{Möb}(n+1)$.
- G is a group with respect to the combination of mappings.
- $g\mathbb{H}^{n+1} = \mathbb{H}^{n+1}$ for every $g \in G$.
- Note that $d(g(x), g(y)) = d(x, y)$ for every $g \in G$ and $x, y \in \mathbb{H}^{n+1}$, so G is essentially a discrete group of hyperbolic isometries of $\mathbb{H}^{n+1}$.

Let G be non-elementary.

- The limit set of $G = L(G) = \overline{Ge_{n+1}} \cap \bar{\mathbb{R}}^n$, where $e_{n+1} = (0, 0, \ldots, 0, 1) \in \mathbb{H}^{n+1}$.
- $L(G)$ is either uncountable or finite with at most 2 points.
- We assume that G is non-elementary, i.e. that $L(G)$ is uncountable.
- We assume that $L(G)$ is not an l-sphere or l-plane of $\bar{\mathbb{R}}^n$ for any $l \in \{1, 2, \ldots, n\}$.
- For the sake of simplicity, we assume that $\infty \notin L(G)$.

Let G be geometrically finite.

- $L(G) = L_c(G) \cup P(G)$ (disjoint union).
- $L_c(G) = \{$conical limit points of $G\}$.
- $P(G) = Gp_1 \cup Gp_2 \cup \ldots \cup Gp_m$ (pairwise disjoint union).
- Each p_i is a bounded parabolic fixed point of G.
- The case $P(G) = \emptyset$ is allowed.

Let μ be a Patterson-Sullivan measure of G.

- $\{\mu\text{-measurable sets}\} = \{\text{Borel sets of } \bar{\mathbb{R}}^{n+1}\} = \text{Bor}(n+1)$.
- $\mu(L(G)) \in]0, \infty[$ and $\mu(\bar{\mathbb{R}}^{n+1} \setminus L(G)) = 0$.
- $g \in G, A \in \text{Bor}(n+1) \implies$

  $$\mu(gA) = \int_A |g'|^\delta d\mu, \tag{0.1}$$

  where $|g'|$ is the operator norm of g' and δ is the Hausdorff dimension of $L(G)$.
- μ is essentially unique: if ν satisfies the above conditions, then $\nu = c_\nu \mu$ for some constant $c_\nu > 0$.

The geometric characterization of μ:

- According to Patterson's construction, μ is obtained as a weak limit of a sequence of measures supported by Ge_{n+1}.
- We would like to construct a measure ν using the properties of $L(G)$ as a subset of $\mathbb{R}^n$ such that $\nu = c_\nu \mu$ for some constant $c_\nu > 0$.
- According to results of Dennis Sullivan, the measure ν is sometimes obtained from the standard covering construction and sometimes from the standard packing construction; sometimes neither of the standard constructions can be used to obtain ν.
- Our main result states that if the standard constructions are modified in a suitable way, either one of them can be used to construct a suitable measure ν.

The standard covering construction:

- Let $A \subset \mathbb{R}^n$.
- Given $\varepsilon > 0$, a countable collection $\mathcal{T}$ of balls $\bar{B}^n(x,t)$ is an ε-covering of A if $t \in]0,\varepsilon]$ and $A \subset \bigcup \mathcal{T}$.
- $m_\varepsilon(A) = \inf_{\mathcal{T}} \sum_{\bar{B}^n(x,t) \in \mathcal{T}} t^\delta$.
- $\varepsilon' \in]0,\varepsilon[\implies m_{\varepsilon'}(A) \geq m_\varepsilon(A)$.
- $m(A) = \sup_{\varepsilon > 0} m_\varepsilon(A)$.

The modified gauge function:

- The main modification is to replace the standard gauge function $t \mapsto t^\delta$ by the function ψ defined by

$$\psi(x,t) = t^\delta \prod_{l=1}^{n} \gamma_l(x,t)^{\delta-l} \tag{0.2}$$

  for every $x \in \mathbb{R}^n$ and $t > 0$ such that $\bar{B}^n(x,t) \cap L(G) \neq \emptyset$, where the functions γ_l are defined as follows.
- $\rho(A,B) = \sup\{d_{\text{euc}}(a,B), d_{\text{euc}}(b,A) : a \in A, b \in B\}$.
- $\mathcal{F}_l(x,t) =$ $\{l\text{-spheres and } l\text{-planes } V \text{ of } \mathbb{R}^n \text{ such that } \bar{B}^n(x,t) \cap V \neq \emptyset\}$.
- $\gamma_l(x,t) = \frac{1}{t} \inf_{V \in \mathcal{F}_l(x,t)} \rho(\bar{B}^n(x,t) \cap L(G), \bar{B}^n(x,t) \cap V)$.
- $\gamma_l(x,t)$ measures how much $L(G)$ resembles an l-sphere or l-plane in $\bar{B}^n(x,t)$.

The modified covering construction:

- Let $A \subset L(G)$.
- Given $\varepsilon > 0$ and $v \in]0,1[$, a countable collection $\mathcal{T}$ of balls $\bar{B}^n(x,t)$ is an (ε, v)-covering of A if $t \in]0,\varepsilon]$, $A \subset \bigcup \mathcal{T}$, and $\bar{B}^n(x,vt) \cap L(G) \neq \emptyset$.
- $\bar{\nu}_\varepsilon^v(A) = \inf_{\mathcal{T}} \sum_{\bar{B}^n(x,t) \in \mathcal{T}} \psi(x,t)$.
- $\varepsilon' \in]0,\varepsilon[$ and $v' \in]0,v[\implies \bar{\nu}_{\varepsilon'}^{v'}(A) \geq \bar{\nu}_\varepsilon^v(A)$.
- $\bar{\nu}(A) = \sup_{\varepsilon > 0, v \in]0,1[} \bar{\nu}_\varepsilon^v(A)$.
- $\nu(B) = \bar{\nu}(B \cap L(G))$ for every $B \in \mathrm{Bor}(n+1)$.

ν characterizes μ geometrically:

- To show that $\nu = c_\nu \mu$ for some constant $c_\nu > 0$, we need to show that $\nu(L(G)) \in]0, \infty[$ and that ν satisfies (0.1).
- The construction of ν implies that ν satisfies (0.1).
- The result $\nu(L(G)) \in]0, \infty[$ is implied by the fact that there is a constant $c_G > 0$ such that

$$c_G^{-1}\psi(x,t) \leq \mu(\bar{B}^n(x,t)) \leq c_G \psi(x,t) \tag{0.3}$$

for every $x \in \mathbb{R}^n$ and $t \in]0, t_G[$ such that $\bar{B}^n(x, v_G t) \cap L(G) \neq \emptyset$, where $t_G > 0$ and $v_G \in]0, 1[$ are freely chosen fixed numbers.

The proof of formula (0.3), part I:

- Given $p \in P(G)$, there is an $(n+1)$-dimensional open ball $H_p \subset \mathbb{H}^{n+1}$ tangential to $\mathbb{R}^n$ at p such that the collection $\{H_p : p \in P(G)\}$ is pairwise disjoint.
- Let $x \in \mathbb{R}^n$ and $t \in]0, t_G[$ be such that $\bar{B}^n(x, v_G t) \cap L(G) \neq \emptyset$.
- $(x, t) \notin H_p$ for every $p \in P(G) \implies$

$$c_0^{-1} t^\delta \leq \mu(\bar{B}^n(x,t)) \leq c_0 t^\delta \tag{0.4}$$

  for some constant $c_0 > 0$.
- $(x, t) \in H_p$ for some $p \in P(G)$ of rank $r(p) \in \{1, 2, \ldots, n\} \implies$

$$c_1^{-1} t^\delta e^{d_p(x,t)(r(p)-\delta)} \leq \mu(\bar{B}^n(x,t)) \leq c_1 t^\delta e^{d_p(x,t)(r(p)-\delta)} \tag{0.5}$$

  for some constant $c_1 > 0$, where $d_p(x,t) = d((x,t), \partial H_p)$.
- (0.4) and (0.5) were essentially known already to Sullivan.

The proof of formula (0.3), part II:

- $(x,t) \notin H_p$ for every $p \in P(G) \implies$

  (0.6) $$c_2^{-1} \leq \gamma_l(x,t) \leq c_2$$

  for every $l \in \{1, 2, \ldots, n\}$, where $c_2 > 0$ is a constant.
- $(x,t) \in H_p$ for some $p \in P(G)$ of rank $r(p) \in \{1, 2, \ldots, n\} \implies$

  (0.7) $$c_3^{-1} \leq \gamma_l(x,t) \leq c_3$$

  for every $l \in \{1, 2, \ldots, n\} \setminus \{r(p)\}$ and

  (0.8) $$c_4^{-1} e^{-d_p(x,t)} \leq \gamma_{r(p)}(x,t) \leq c_4 e^{-d_p(x,t)},$$

  where $c_3 > 0$ and $c_4 > 0$ are constants.
- (0.4), (0.5), (0.6), (0.7), (0.8) $\implies$ (0.3).

Sources for more information:

- My PhD thesis:
  V. Ala-Mattila: *Geometric characterizations for Patterson-Sullivan measures of geometrically finite Kleinian groups*, Ann. Acad. Sci. Fenn. Math. Diss. 157 (2011), 120 pp. **PDF**
- A shorter, hopefully more accessible text:
  V. Ala-Mattila: *Patterson-Sullivan measures and geometry of limit sets of geometrically finite Kleinian groups* **PDF**

Thanks!