Local convexity properties in triangular ratio metric

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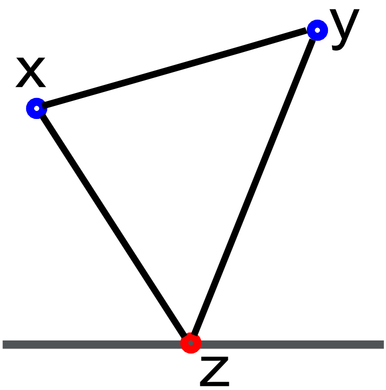
- In a domain D triangular ratio metric is defined as

$$s_D(x,y) = \sup_{z\in\partial D} \frac{|x-y|}{|x-z|+|y-z|},$$

where x and y are points inside domain D, and z is an edge point of D.
- In all domains D the inequality $0 \leq s_D(x,y) \leq 1$ holds.

- For example, in an upper half plane



$$\sup_{z \in \partial D} \frac{|x-y|}{|x-z|+|y-z|},$$

- For $x \in D$ we define a metric ball $B_s(x,r)$, where $r \in [0,1]$, as
$$B_s(x,r) = \{y \in D | s_D(x,y) < r\}$$

Let us first focus on the domain $D = \mathbb{R}^n \setminus \{z\}$, where $z \in \mathbb{R}^n$.

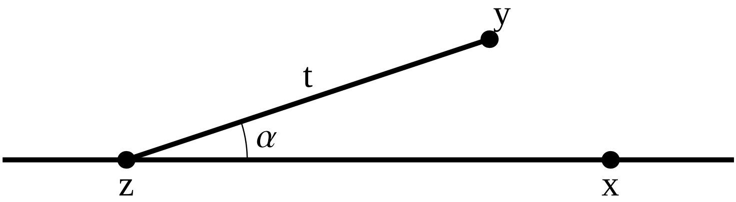
Nonconvex balls

If $x \in D$, the metric ball $B_s(x, r)$ is nonconvex for all $r > \frac{1}{2}$.

Convex balls

If $x \in D$, the metric ball $B_s(x, r)$ is convex for all $r \leq \frac{1}{2}$.

In both proofs it is sufficient to consider only the cases $n = 2$,$z = 0$ and $x = 2$.



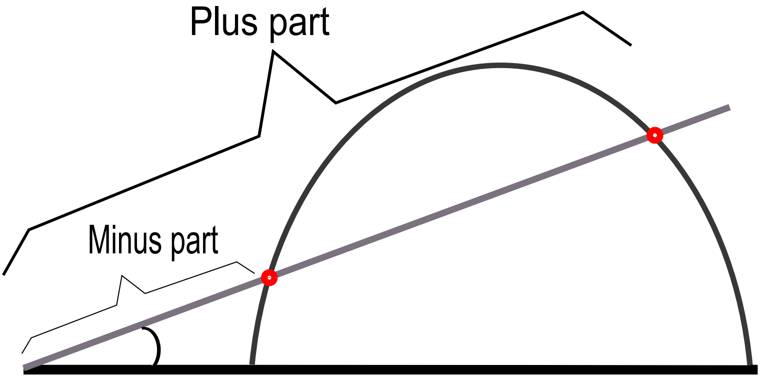
- By the definition we can write

$$\frac{|x-y|}{|x-z|+|y-z|} = \frac{|2-y|}{2+t} = r \Leftrightarrow |2-y| = r(2+t).$$

- By the law of cosines we can write

$$|2-y| = \sqrt{t^2+4+4t\cos\alpha}.$$

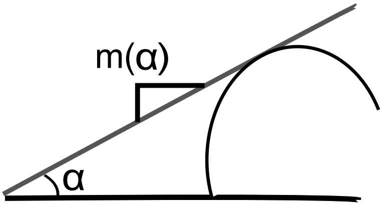
- By combining these two we can solve t with respect of α
- $t(\alpha) = \frac{2\left(r^2+\cos\alpha\pm\sqrt{(1+\cos\alpha)(\cos\alpha+2r^2-1)}\right)}{1-r^2}.$



- When proving nonconvexity for $r > \frac{1}{2}$ we are only interested in the minus part.
- It is enough to consider only the upper part of the axis, because it is mirrored to the lower part of the axis.

- We need to count the slope of the tangent of the $B_s(x,r)$ contour.
- This can be obtained by the formula

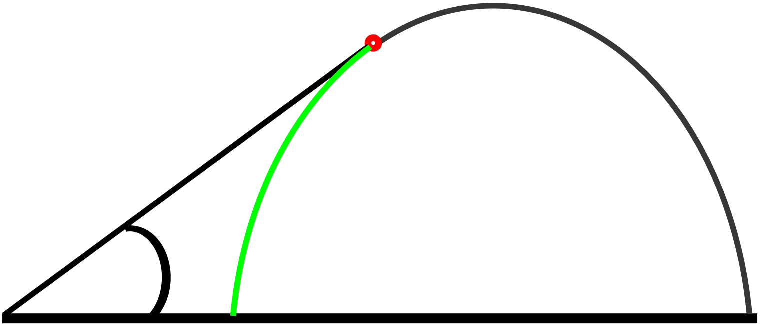
$$m(\alpha) = \frac{t(\alpha) + \tan\alpha\, t'(\alpha)}{-t(\alpha)\tan\alpha + t'(\alpha)}$$



- When $r > \frac{1}{2}$ and α is sufficiently small($0 < \alpha < \frac{\pi}{3}$), $m(\alpha) < 0$.
- Therefore $B_s(x, r)$ is nonconvex for $r > \frac{1}{2}$.

- Next we need to show that $B_s(x,r)$ is convex for $r \leq \frac{1}{2}$.
- For that recall one previously obtained formula.
- $t(\alpha) = \frac{2\left(r^2+\cos\alpha \pm \sqrt{(1+\cos\alpha)(\cos\alpha+2r^2-1)}\right)}{1-r^2}$.
- To prove this we need both plus and minus part.

- Let us first select the minus part
- $t_{\text{minus}}(\alpha) = \frac{2\left(r^2+\cos\alpha-\sqrt{(1+\cos\alpha)(\cos\alpha+2r^2-1)}\right)}{1-r^2}.$
- Following green arch is determined by this minus part

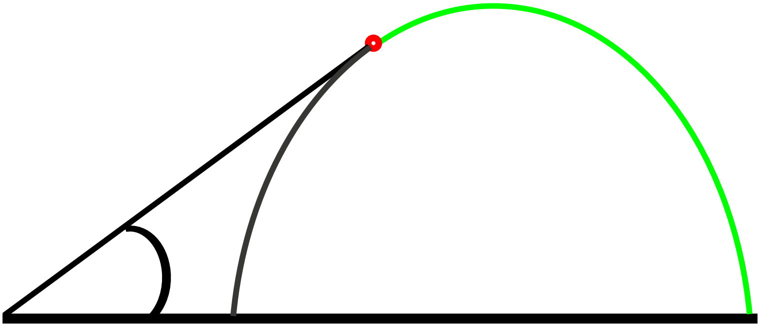


- Previously we counted the slope of the tangent by

$$m_{\text{minus}}(\alpha) = \frac{t_{\text{minus}}(\alpha) + \tan\alpha\, t'_{\text{minus}}(\alpha)}{-t_{\text{minus}}(\alpha)\tan\alpha + t'_{\text{minus}}(\alpha)}$$

- It can be shown that $m_{\text{minus}}(0) > 0$ for all $r < \frac{1}{2}$ and $m_{\text{minus}}(0) \to \infty$ when $r \to \frac{1}{2}$.
- This part of theorem is proven by showing that $m'_{\text{minus}}(\alpha) \leq 0$ for all $\alpha \in (0, \arccos(1-2r^2))$, where $r \in (0, \frac{1}{2}]$.
- If $m'_{\text{minus}}(\alpha) > 0$ for some $\alpha \in (0, \arccos(1-2r^2))$, then slope of the tangent would increase in this angle and ball $B_s(x, r)$ would not be convex.

- Let us now select the plus part
- $t_{\text{plus}}(\alpha) = \frac{2\left(r^2+\cos\alpha+\sqrt{(1+\cos\alpha)(\cos\alpha+2r^2-1)}\right)}{1-r^2}.$
- Following green arch is determined by this plus part



- We count the slope of the tangent as before

$$m_{\text{plus}}(\alpha) = \frac{t_{\text{plus}}(\alpha) + \tan\alpha\, t'_{\text{plus}}(\alpha)}{-t_{\text{plus}}(\alpha)\tan\alpha + t'_{\text{plus}}(\alpha)}$$

- It can be shown that $m_{\text{plus}}(\alpha) \to -\infty$ when $\alpha \to 0$.
- To prove the theorem it is sufficient to show that $m'_{\text{plus}}(\alpha) \geq 0$ for all $\alpha \in (0, \arccos(1-2r^2))$.
- If $m'_{\text{plus}}(\alpha) < 0$ for some $\alpha \in (0, \arccos(1-2r^2))$, slope of the tangent would decrease for that α and ball $B_s(x, r)$ would not be convex.

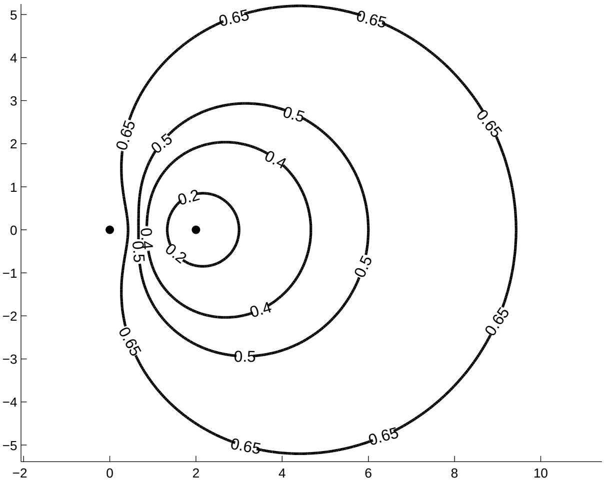


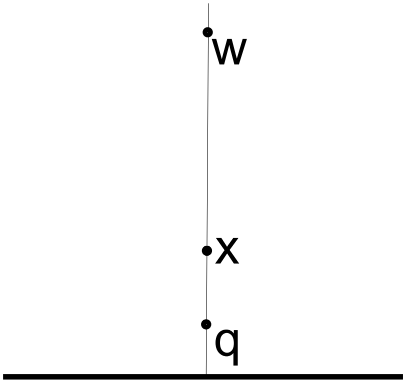
Figure: Contours of the metric ball $B_s(x, r)$ in $\mathbb{R}^2 \setminus \{0\}$ where $x = 2$ and $r = 0.2, 0.4, 0.5$ and 0.65.

- The next domain is the upper half plane.

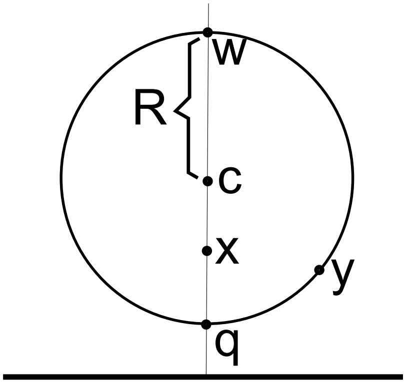
Convexity of balls in the upper half plane

Let $x \in \mathbb{H}^n$. Then balls $B_s(x, r)$ are Euclidean balls for all $r \in (0, 1)$ and therefore convex.

- It is sufficient to consider only the case $n = 2$ and $x = (0, a)$ where $a > 0$.



- First we select points q and w such that satisfy the condition $s_{\mathbb{H}^2}(x, q) = s_{\mathbb{H}^2}(x, w) = r$, where $r \in (0, 1)$.
- Real part of points q and w is zero.



- Then we count a circle that goes through points w and q.

$$c = \frac{w_2 + q_2}{2}, R = \frac{w_2 - q_2}{2},$$

where w_2 and q_2 are imaginary parts of points w and q.

- y is an arbitrary point from the circle.

- y can be expressed by using polar coordinates

$$y = (R\cos t, c + R\sin t),$$

where $t \in [0, 2\pi]$.

- It can be shown that $s_{\mathbb{H}^2}(x, y) = r$ for all $t \in [0, 2\pi]$, what proves the theorem.

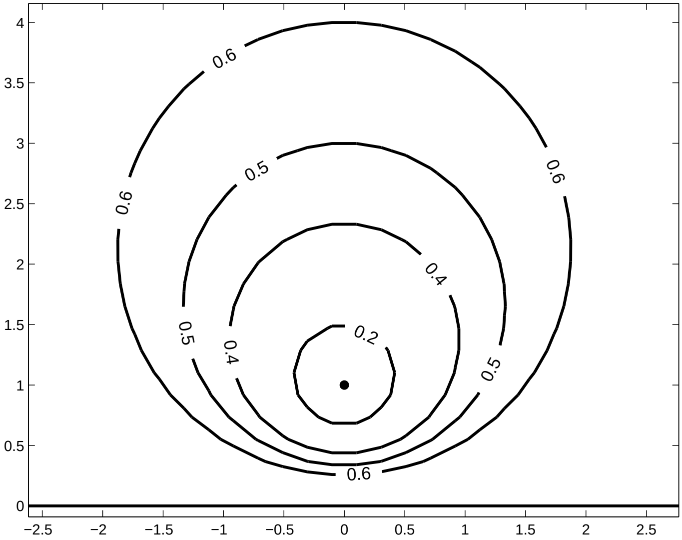


Figure: Contours of the metric ball $B_s(x, r)$ in $\mathbb{H}^2$ where $x = (0, 1)$ and $r = 0.2, 0.4, 0.5$ and 0.6.

- Previous results can be combined to obtain a following corollary

Convexity in the punctured half space

Let $x \in \mathbb{H}^n \setminus \{z\}$ where $z \in \mathbb{H}^n$. Then balls $B_s(x, r)$ are convex for all $r \in (0, \frac{1}{2})$.

- However, in a general case this is not an strict upper bound.

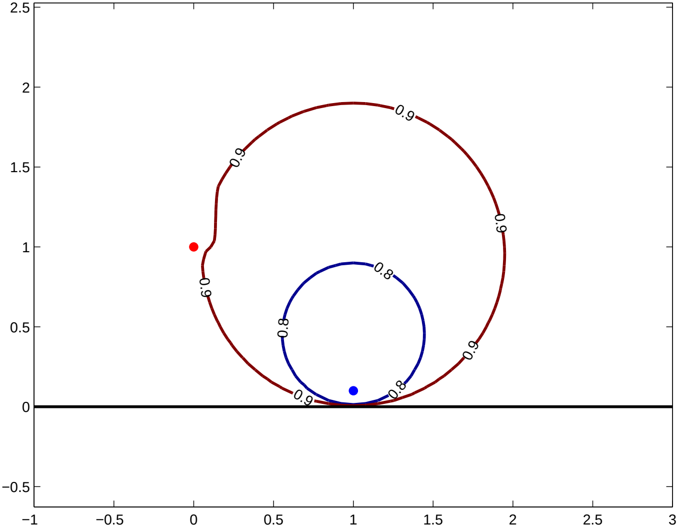


Figure: Contours of the metric ball $B_s(x, r)$ in $\mathbb{H}^2 \setminus \{z\}$ where $z = (0, 1)$, $x = (1, \frac{1}{10})$ and $r = \{0.8, 0.9\}$

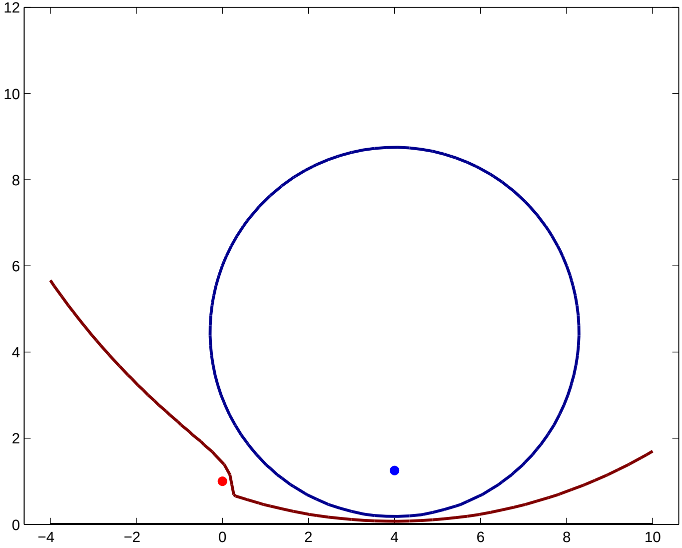


Figure: Contours of the metric ball $B_s(x, r)$ in $\mathbb{H}^2 \setminus \{z\}$ where $z = (0, 1)$, $x = (4, \frac{5}{4})$ and $r = \{0.75, 0.9\}$

- In some special cases the better upper bound can be achieved.
- Following result is proven by R.Klén.

Convexity in the punctured half space

Let $x = (x_1, x_2) \in \mathbb{H}^2 \setminus \{e_n\}$, where $x_2 < |x_1|$ and

$$r \in \left(0, \frac{\sqrt{x_1^2 + x_2^2} - \sqrt{2}x_2}{|x_1| + x_2}\right].$$

Then the ball $B_s(x, r)$ is convex.