# Properties of hyperbolic type balls

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Workshop on Modern Trends in
Classical Analysis and Applications

*The First Chinese-Finnish Seminar*

Turku
August 17-18, 2012

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## Hyperbolic metric

- well-known in $\mathbb{B}^n$ and $\mathbb{H}^n$
- has many applications in mathematics and physics
- in the case $n = 2$ can be generalized by the Riemann mapping theorem
- in the case $n > 2$?
  - No useful counterpart for the Riemann mapping theorem exists
  - SOLUTION: hyperbolic type metrics

## Hyperbolic type metrics

- $k$ the quasihyperbolic metric
- $j$ the distance ratio metric ($j$-metric)
- $\alpha$ the Apollonian metric
- $\delta$ the Seittenranta's metric
- etc.

# Definitions

## Notation

- A domain $G \subsetneq \mathbb{R}^n$ is starlike w.r.t. $x \in G$ if for all $y \in G$ the line segment $[x, y]$ is contained in $G$ and $G$ is strictly starlike w.r.t. $x$ if each half-line from the point $x$ meets $\partial G$ at exactly one point.
- For a distance $d$ in $G$ we define the metric ball for $x \in G$ and $r > 0$ by $B_d(x, r) = \{y \in G \colon d(x, y) < r\}$.
- The Euclidean balls, spheres: $B^n(x, r)$ and $S^{n-1}(x, r)$.
- We denote the unit ball $B^n(0, 1)$ by $\mathbb{B}^n$ and the upper half-space by $\mathbb{H}^n = \{z \in \mathbb{R}^n \colon z_n > 0\}$.
- The hyperbolic distances are denoted by $\rho_{\mathbb{B}^n}$ and $\rho_{\mathbb{H}^n}$.

# Definitions

## Distances $k$ and $j$

Let $G \subsetneq \mathbb{R}^n$ be a domain. We define

- the *quasihyperbolic distance* for $x, y \in G$ by

$$k_G(x,y) = \inf_{\alpha \in \Gamma_{xy}} \int_\alpha \frac{|dz|}{d(z)},$$

  where $d(z) = d(z, \partial G)$ and $\Gamma_{xy}$ is the collection of all rectifiable curves in $G$ joining $x$ and $y$.

- the *j-distance* for $x, y \in G$ by

$$j_G(x,y) = \log\left(1 + \frac{|x-y|}{\min\{d(x), d(y)\}}\right).$$

Note that $k_{\mathbb{H}^n} = \rho_{\mathbb{H}^n}$ and $j_G \le k_G$ for all $G$.

# Definitions

## Distance $\alpha$

The cross-ratio $|a, b, c, d|$ for $a, b, c, d \in \mathbb{R}^n$ is defined by

$$|a, b, c, d| = \frac{|a-c||b-d|}{|a-b||c-d|}.$$

Let $G$ be a proper subdomain of $\overline{\mathbb{R}^n}$. The Apollonian distance is defined for $x, y \in G$ by

$$\alpha_G(x, y) = \sup_{a,b \in \partial G} \log |a, x, y, b| = \sup_{a,b \in \partial G} \log \frac{|a-y||x-b|}{|a-x||y-b|}.$$

Note that $\alpha_G$ is a metric if and only if the complement of $G$ is not contained in a sphere in $\overline{\mathbb{R}^n}$, [Beardon '98, Theorem 1.1].

# Definitions

## Distance $\delta$

The Seittenranta's distance is defined for $x, y \in G \subset \overline{\mathbb{R}^n}$ with $\operatorname{card} G \geq 2$ by

$$\begin{aligned} \delta_G(x,y) &= \sup_{a,b\in\partial G} \log(1 + |a,x,b,y|) \\ &= \sup_{a,b\in\partial G} \log\left(1 + \frac{|a-b||x-y|}{|a-x||y-b|}\right). \end{aligned}$$

- $\delta_G$ is always a metric [Seittenranta '99, Thm 3.3].
- $\alpha_G$ and $\delta_G$ are Möbius invariant.
- $\alpha_{\mathbb{B}^n} = \delta_{\mathbb{B}^n} = \rho_{\mathbb{B}^n}$ and $\delta_G = j_G$ for $G = \mathbb{R}^n \setminus \{0\}$.

# Motivation

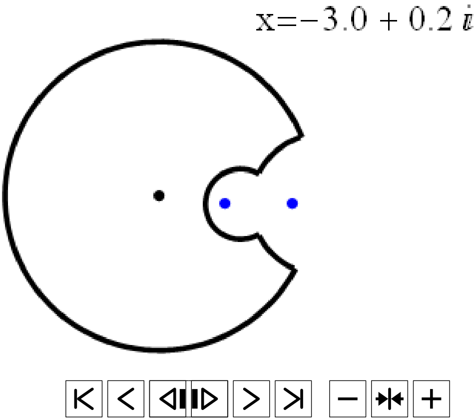
Motivation for this research comes from open problem posed by M. Vuorinen in 2007 [Vuorinen ’07, 8.1]:

**Open problem 1**

*Does the exists $r_0$ such that $B_m(x, r)$ is convex for $r \in (0, r_0)$?*

The problem has recently been studied by various authors [R.K. ’08a, R.K. ’08b, R.K. ’09, R.K. ’10, R.K.-Rasila-Talponen ’10, Martio-Väisälä ’11, Rasila-Talponen ’12, Väisälä ’07, Väisälä ’09].

# Diversity of shapes, fixed radius $r = 1.2$.

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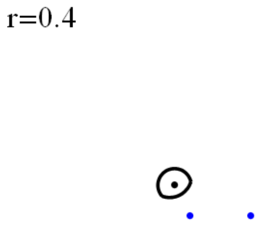
# Influence on radius, fixed center $x = -3/2 + i$.



# History

## Known results

- quasihyperbolic metric in convex domains, $n \geq 2$ [Martio-Väisälä '11]
- quasihyperbolic metric in $\mathbb{R}^n \setminus \{0\}$, $n \geq 2$ [R.K. '08a]
- $j$-metric general domain [R.K. '08b]
- quasihyperbolic metric in general domain $n = 2$ [Väisälä '09]
- quasihyperbolic and $j$-metric in banach spaces [Rasila-Talponen '12]

This talk considers the Apollonian and Seittenranta's metrics and it is based on [R.K. '12].

## Apollonian balls

For $x, y \in \mathbb{R}^n$ and $r > 0$ we define the Apollonian ball and sphere, respectively, to be

$$B^r_{x,y} = \{z \in \mathbb{R}^n : r|x - z| < |y - z|\},$$

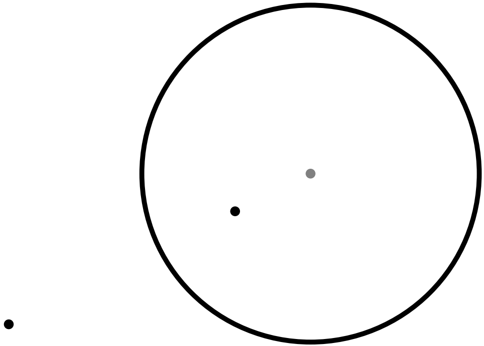
$$S^r_{x,y} = \{z \in \mathbb{R}^n : r|x - z| = |y - z|\}.$$

For $x, y \in \mathbb{R}^2$ and $c > 0$, $c \neq 1$, we have [Krzyz '71, p.5, Exercise 1.1.25]

$$S^c_{x,y} = S^{n-1}\left(\frac{y - c^2 x}{1 - c^2}, \frac{c|x - y|}{|1 - c^2|}\right). \tag{1}$$

In the case $c = 1$ the Apollonian ball is a half-space.

# Apollonian circle



## Lemma 1

*Let* $G = \mathbb{R}^n \setminus \{-e_1, e_1\}$, $x \in G$ *and* $r > 0$. *We denote*

$$B_c = B^n\left(e_1 \frac{1+c^2}{c^2-1}, \frac{2c}{|c^2-1|}\right), B_d = B^n\left(e_1 \frac{1+d^2}{1-d^2}, \frac{2d}{|1-d^2|}\right)$$

*for* $c = e^r|x+e_1|/|x-e_1|$ *and* $d = e^r|x-e_1|/|x+e_1|$. *Then*

$$B_\alpha(x,r) = \begin{cases} B_c \setminus \overline{B_d}, & \text{if } c < 1 \text{ and } d \geq 1, \\ \mathbb{R}^n \setminus (\overline{B_c} \cup \overline{B_d}), & \text{if } c > 1 \text{ and } d > 1, \\ B_d \setminus \overline{B_c}, & \text{if } c \geq 1 \text{ and } d < 1. \end{cases}$$

*Moreover, the complement of* $B_\alpha(x,r)$ *is always disconnected.*

## Remark

*Lemma 1 can be generalized for any twice punctured space:*
*Let* $y,z \in \mathbb{R}^n$ *with* $y \neq z$, $G = \mathbb{R}^n \setminus \{y,z\}$, $x \in G$ *and* $r > 0$. *We denote*

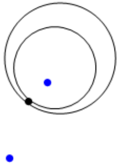
$$B_c = B^n\left(\frac{z - yc^2}{1-c^2}, \frac{|y-z|c}{|1-c^2|}\right), \quad B_d = B^n\left(\frac{y - zd^2}{1-d^2}, \frac{|y-z|d}{|1-d^2|}\right)$$

*for* $c = e^r|x-z|/|x-y|$ *and* $d = e^r|x-y|/|x-z|$. *Then*

$$B_\alpha(x,r) = \begin{cases} B_c \setminus \overline{B_d}, & \text{if } c < 1 \text{ and } d \geq 1, \\ \mathbb{R}^n \setminus (\overline{B_c} \cup \overline{B_d}), & \text{if } c > 1 \text{ and } d > 1, \\ B_d \setminus \overline{B_c}, & \text{if } c \geq 1 \text{ and } d < 1. \end{cases}$$

*Moreover, the complement of* $B_\alpha(x,r)$ *is always disconnected.*

# Apollonian metric disks in $\mathbb{R}^2 \setminus \{1+i, -i\}$



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# Ice cream cone lemma

## Lemma 2

*Let* $x, y \in \mathbb{R}^n$ *with* $x \neq y$ *and* $r \in (0,1)$. *Then*

$$\bigcup_{t \in (0,1]} B^r_{x,z} = A \cup B^r_{x,y},$$

*where* $z = x + t(y - x)$ *and*

$$A = \left\{ a \in \mathbb{R}^n : \measuredangle(a, x, y) < \arcsin r, |a| < \frac{|x-y|}{\sqrt{1-r^2}} \right\}.$$

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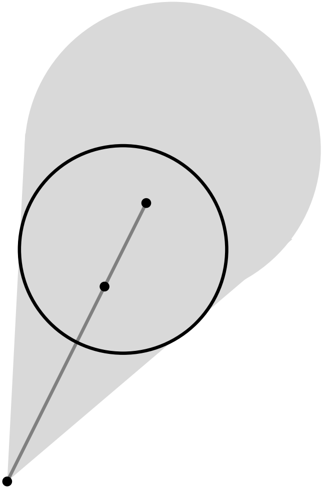
# Ice cream cone lemma

$$\bigcup_{t\in(0,1]} B^r_{x,z} = A \cup B^r_{x,y}$$



# Starlike domains

**Theorem 3**

*Let $G \subsetneq \mathbb{R}^n$ be a starlike domain with respect to $x \in G$ such that the complement of $G$ is not contained in any $(n-1)$-dimensional sphere and $r > 0$. Then $B_\alpha(x, r)$ is strictly starlike with respect to $x$.*

**Proof.**

Let us assume that $B_\alpha(x, r)$ is not starlike with respect to $x$. Then there exists $y, z \in G$ such that $y$ is contained in the line segment $(x, z)$, $\alpha_G(x, z) < r$ and $\alpha_G(x, y) = r' \geq r$. Now $B^{r'}_{x,y} \subset G$ and $S^r_{x,z}$ contains a point on $\partial G$. By Lemma 2 this is a contradiction. □

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# PART II: Seittenranta's metric

## Lemma 4

*Let* $G = \mathbb{R}^n \setminus \{-e_1, e_1\}$, $x \in G$ *and* $r > 0$. *Then for* $B_c = B^c_{-e_1,x}$ *and* $B_d = B^d_{e_1,x}$ *we have*

$$B_\delta(x,r) = \begin{cases} B_c \cap B_d, & \text{if } c \le 1 \text{ and } d \le 1, \\ B_c \setminus \overline{B_d}, & \text{if } c \le 1 \text{ and } d > 1, \\ B_d \setminus \overline{B_c}, & \text{if } c > 1 \text{ and } d \le 1, \\ \mathbb{R}^n \setminus (\overline{B_c} \cup \overline{B_d}), & \text{if } c > 1 \text{ and } d > 1, \end{cases}$$

*where* $c = |x - e_1|(e^r - 1)/2$ *and* $d = |x + e_1|(e^r - 1)/2$.

## Theorem 5

*Let* $G = \mathbb{R}^n \setminus \{-e_1, e_1\}$, $x \in G$ *and* $r_0 = \log(1 + 2/\max\{|x - e_1|, |x + e_1|\})$*. Then* $B_\delta(x, r)$ *is convex for all* $r \in (0, r_0]$ *and is not convex for* $r > r_0$*.*

## Proof.

By Lemma 4 the metric ball $B_\delta(x, r)$ is convex if and only $c \leq 1$ and $d \leq 1$, which is equivalent to

$$r \leq \min\left\{\log\left(1 + \frac{2}{|x - e_1|}\right), \log\left(1 + \frac{2}{|x + e_1|}\right)\right\}$$

and the assertion follows. □

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## Remark

*(1) Theorem 5 is true for any domain* $G = \mathbb{R}^n \setminus \{y, z\}$ *with* $y, z \in \mathbb{R}^n$ *and* $a \neq b$*, if we replace* $r_0$ *by*

$$r_1 = \log\left(1 + \frac{|y-z|}{\max\{|x-y|, |x-z|\}}\right).$$

*(2) In Theorem 5 (and the above generalization) the radius* $r_0$ *(*$r_1$*) is sharp in the sense that for* $r \in (0, r_0)$ *(*$r \in (0, r_1)$*) the metric balls* $B_\delta(x, r)$ *are strictly convex.*
*(3) Note that* $B_\delta(x, r)$ *is not starlike for* $r > r_0(r_1)$ *in Theorem 5 (in the above remark (2) ).*

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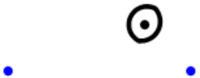
# Seittenranta's metric disks in $\mathbb{R}^2 \setminus \{1, -1\}$



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## Theorem 6

*Let* $G = \mathbb{B}^n \setminus \{0\}$*,* $x \in G$ *and* $r_0 = \log(1 + 1/(1 - |x|))$*. Then* $B_\delta(x, r)$ *is convex for all* $r \in (r, r_0]$ *and is not convex for* $r > r_0$*.*

## Lemma 7

*Let* $x \in \mathbb{B}^n \setminus \{0\}$ *and* $r > 0$*. Then the set*

$$A = \{y \in \mathbb{B}^n \setminus \{0\} \colon \log\left(1 + |x - y|/(|y|(1 - |x|))\right) < r\}$$

*is convex for* $r \in (0, \log(1 + 1/(1 - |x|))]$ *and not convex for* $r > \log(1 + 1/(1 - |x|))$*, and the set*

$$B = \{y \in \mathbb{B}^n \setminus \{0\} \colon \log\left(1|x - y|/(|x|(1 - |y|))\right) < r\}$$

*is strictly convex.*

# An example of Theorem 6

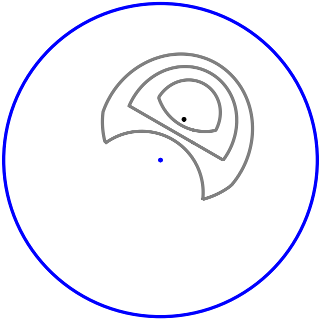

Figure: Disks $B_\delta(x, r)$ of Seittenranta's metric in the domain $\mathbb{B}^2 \setminus \{0\}$ with $r \in \{r_0 - 1/3, r_0, r_0 + 1/3\}$, where $r_0 = \log(1 + 1/(1 - |x|))$. The black dot is the origin, the black circle is the unit circle and the gray dot is the point $x$.

# Open problems

## Open problem 2

*Let* $G = \mathbb{B}^n \setminus \{0\}$ *and* $x \in G$. *Does there exists* $r_0 = r_0(|x|) > 0$ *such that* $B_\alpha(x, r)$ *is convex for all* $r \in (0, r_0]$*?*

## Open problem 3

*(1) If* $G \subsetneq \mathbb{R}^n$ *is a convex domain and* $x \in G$, *is* $B_\delta(x, r)$ *convex for all* $r > 0$*?*
*(2) If* $G \subsetneq \mathbb{R}^n$ *is starlike domain with respect to* $x \in G$, *is* $B_\delta(x, r)$ *starlike with respect to* $x$ *for all* $r > 0$*?*

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