Two properties of John domains in real Banach spaces

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Notations and preliminaries

Notations

E denotes real Banach space with dimension at least 2. $D \subset E$ is a domain. The distance from z to the boundary ∂D of D is denoted by $d_D(z)$.

Comparison with the case $E = R^n$,See [1]

(1) The space E is not locally compact, and its one-point extension $\dot{E} = E \cup \{\infty\}$ is not compact. Normal family arguments are not valid in E, and many extremal problems have no solution.

Reference

[1] J. VÄISÄLÄ, The free quasiworld: freely quasiconformal and related maps in Banach spaces. *Quasiconformal geometry and dynamics (Lublin 1996), Banach Center Publications,* Vol. 48, Polish Academy of Science, Warsaw. 1999, 55-118.

(2) Several topological properties of R^n are not valid in E. For example, a ball B in the Hilbert space l_2 is homeomorphic to the domain A between two concentric spheres. But there is no freely quasiconformal map of B onto A.

(3) There is no Lebesgue measure in E. Balls have no volume. The method of moduli of path family is useless.

(4) There is no Whitney decomposition in E and packing arguments fail in E.

Notations and preliminaries

John domains,See [2], [3].

A domain D in E is called a c-John domain provided there exists a constant c such that for each pair of points z_1, z_2 in D can be joined by a rectifiable arc α in D satisfying for all $z \in \alpha$,

$$(\star) \qquad \min\{\ell(\alpha[z_1, z]),\ \ell(\alpha[z_2, z])\} \leq c\, d_D(z).$$

References

[2] O. MARTIO AND J. SARVAS, Injectivity theorems in plane and space, *Ann. Acad. Sci. Fenn. Ser. A I Math.*, (1978), 383–401.
[3] R. NÄKKI AND J. VÄISÄLÄ, John disks, *Expo. Math.* 1991, 3–43.

Notations and preliminaries

Comparison with the case $E = R^n$

If we replace the arclength $\ell(\alpha[\cdot])$ in $(\star)$ by diameter $d(\alpha[\cdot])$ or distance $|\cdot|$, we get concepts which in the case $E = R^n$ is n-quantitatively equivalent to c-John domains[2, 3]. But in Banach spaces, this lead to different properties. For example, the broken tube considered by J. Väisälä [4] can join points by arcs satisfying the diameter condition (distance condition), but it is not a John domain.

Reference

[4] J. VÄISÄLÄ, *Broken tubes in Hilbert spaces.* Preprint 390. 2004.

Notations and preliminaries

quasihyperbolic length and quasihyperbolic distance

The *quasihyperbolic length* of a rectifiable arc or a path α in the norm metric in D is the number:

$$\ell_k(\alpha) = \int_\alpha \frac{|dz|}{d_D(z)}.$$

For each pair of points z_1, z_2 in D, the *quasihyperbolic distance* $k_D(z_1, z_2)$ between z_1 and z_2 is defined in the usual way:

$$k_D(z_1, z_2) = \inf \ell_k(\alpha),$$

where the infimum is taken over all rectifiable arcs α joining z_1 to z_2 in D.

Main results

Theorem. [5]

A domain $D \subset \mathbb{R}^n$ is a John domain if and only if $G = D \setminus P$ is also a John domain, where $P = \{p_1, p_2, \cdots, p_m\} \subset D$.

In general, when P is an infinite set in D, $D \setminus P$ may not be a John domain.

Reference

[5] M. HUANG, S. PONNUSAMY AND X. WANG, FrDecomposition and removability properties of John domains, *Proc. Indian Acad. Sci. (Math. Sci.),* **118**(2008),357¨C370.

For a domain D in E, let

$$P_D = \{x_i \in D : k_D(x_i, x_j) \geq \frac{1}{2} \;\; for \;\; i \neq j\}.$$

Obviously, P_D contains at least two points.

Result 1

A domain $D \subset E$ is a c-John domain if and only if $G = D \setminus P_D$ is a c_1-John domain, where c and c_1 depend only on each other.

proof

The sufficient part is easy to prove. The main idea of the proof of the necessary part is to construct an arc in $D \setminus P_D$ satisfying the condition $(\star)$.

key point one

For all $w \in D$, there exists at most one point x_i of P_D such that $x_i \in B(w, \frac{1}{6}d_D(w))$.

key point two

Every pair of points x, y in $B(w, \frac{1}{16}r) \setminus \{x_i\}$ can be joined by an arc in $D \setminus P_D$ satisfying $(\star)$.

inner uniform domain

A domain $D \subset E$ is said to be an inner c-uniform domain if for every $x, y \in D$, there exist a rectifiable arc γ joining x and y satisfying the condition $(\star)$ and $\ell(\gamma) \leq c \inf_{\beta[x,y]\in D} \ell(\beta)$.

Remark

If replace $\inf_{\beta[x,y]\in D} \ell(\beta)$ by $|x - y|$, then we get the concept for c-uniform domain.

Application of Result 1

A domain $D \subset E$ is an inner c-uniform domain if and only if $G = D \setminus P_D$ is an inner c_1-uniform domain, where c and c_1 depend only on each other.

Theorem (see [6])

A c-John domain $D \subset R^n$ can be written as the union of domains $D_1, D_2, \ldots$ such that for each j,
(1) $\overline{D}_j$ is compact in D_{j+1}, and
(2) D_j is a c_1-John domain with $c_1 = c_1(c, n)$.

Remark

Note that here the constant c_1 depends on the dimension n.

Reference

[6],J. VÄISÄLÄ, Exhaustions of John domains. *Ann. Acad. Sci. Fenn. Ser. A I Math.,* **19**1994, 47-57.

Result 2

A c-John domain $D \subset E$ can be written as the union of domains $D_1, D_2, \ldots$ such that for each j,
(1) $\overline{D}_j$ is contained in D_{j+1}, and
(2) D_j is a c_1-John domain with $c_1 = c_1(c)$.

THANK YOU