# Properties of quasihyperbolic balls and geodesics on Banach spaces

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# Abstract

We deal with Banach manifolds, which are obtained by defining a conformal metric on non-trivial subdomains of a given Banach space. An example of such metric is the quasihyperbolic metric on a domain of a Banach space. It is obtained from the norm-induced metric by adding a weight, which depends on ly on distance to the boundary of the domain. We present results on convexity a nd starlikeness of quasihyperbolic and distance ratio metric balls on Banach spa ces. In particular, problems related to these metrics on a punctured Banach spac e are considered [RT12]. We also discuss our recent work on existence and sm oothness of quasihyperbolic geodesics on Banach spaces [RTxx].
This presentation is based on joint work (in process) with Jarno Talponen.

# References


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# Preliminaries

Let $\mathrm{X}$ be a Banach space. Open and closed balls in $\mathrm{X}$ are

$$\mathbf{U}(x,r) := \mathbf{U}_{\|\cdot\|}(x,r) := \{y \in X : \|x-y\| < r\}, \text{ and}$$

$$\mathbf{B}(x,r) := \mathbf{B}_{\|\cdot\|}(x,r) := \{y \in X : \|x-y\| \le r\}.$$

The norm length of an arc $\gamma$ in $\mathrm{X}$ is denoted by $\ell(\gamma)$.

A set $\Omega \subset \mathrm{X}$ is called *convex* if the line segment

$$[x,y] := \{tx + (1-t)y : t \in [0,1]\} \subset \Omega \text{ for all } x, y \in \Omega,$$

and *starlike* with respect to $x_0 \in \Omega$ if

$$[x_0,y] := \{tx_y + (1-t)y : t \in [0,1]\} \subset \Omega \text{ for all } y \in \Omega,$$

Note that a set $\Omega$ is convex if and only if it is starlike with respect to every point $x_0 \in \Omega$.

# Uniform convexity and smoothness

The next two essential moduli are related to the geometry of Banach spaces. The *modulus of convexity* $\delta_{\mathrm{X}}(\epsilon),\ 0 < \epsilon \leq 2$, is defined by

$$\delta_{\mathrm{X}}(\epsilon) = \inf \left\{ 1 - \|x + y\|/2 :\ x, y \in \mathrm{X},\ \|x\| = \|y\| = 1,\ \|x - y\| = \epsilon \right\}.$$

The *modulus of smoothness* $\rho_{\mathrm{X}}(\tau),\ t > 0$ is defined by

$$\rho_{\mathrm{X}}(\tau) = \sup \left\{ (\|x+y\| + \|x-y\|)/2 - 1,\ x, y \in \mathrm{X},\ \|x\| = 1,\ \|y\| = \tau \right\}.$$

The Banach space $\mathrm{X}$ is called *uniformly convex* if $\delta_{\mathrm{X}}(\epsilon) > 0$ for $\epsilon > 0$ and *uniformly smooth* if

$$\lim_{\tau \to 0^+} \frac{\rho_{\mathrm{X}}(\tau)}{\tau} = 0.$$

A space $\mathrm{X}$ is uniformly convex (resp. uniformly smooth) of power type $p \in [1, \infty)$ if $\delta_{\mathrm{X}}(\epsilon) \geq K\epsilon^p$ (resp. $K\tau^p \leq \rho_{\mathrm{X}}(\tau)$) for some $K > 0$.

# Quasihyperbolic metric

Let $\mathrm{X}$ be a Banach space and $\Omega \neq X$ a domain. For $x \in \Omega$, let $d(x)$ denote the distance $d(x, \partial\Omega)$. We define the *quasihyperbolic length* of $\gamma$ by

$$\ell_k := \int_\gamma \frac{\|dx\|}{d(x)}$$

Then the *quasihyperbolic distance* of points $x, y \in \Omega$ is the number

$$k_\Omega(x, y) := \inf_\gamma \ell_k(\gamma)$$

where the infimum is taken over all rectifiable arcs $\gamma$ joining $x$ and $y$ in $\Omega$.

Quasihyperbolic balls are

$$\begin{aligned} \mathbf{U}_k(x, r) &:= \{y \in \Omega : k_\Omega(x, y) < r\}, \\ \mathbf{B}_k(x, r) &:= \{y \in \Omega : k_\Omega(x, y) \leq r\}. \end{aligned}$$

# Distance-ratio metric

Let $\mathrm{X}$ be a Banach space, and let $\Omega \neq \mathrm{X}$ be a domain. Write

$$a \vee b := \max\{a, b\}, \qquad a \wedge b := \min\{a, b\}.$$

The *distance-ratio metric*, or *j-metric*, on $\Omega$ is defined by

$$j_\Omega(x, y) := \log\left(1 + \frac{\|x - y\|}{d(x) \wedge d(y)}\right), \qquad x, y \in \Omega.$$

Again, the balls with respect to the $j$-metric are

$$\begin{aligned} \mathbf{U}_j(x, r) &:= \{y \in \Omega : j_\Omega(x, y) < r\}, \\ \mathbf{B}_j(x, r) &:= \{y \in \Omega : j_\Omega(x, y) \leq r\}. \end{aligned}$$

It is well known that the norm metric, the quasihyperbolic metric and the distance-ratio metric define the same topology on $\Omega$.

# Local convexity

If $\mathrm{X}$ is a topological vector space, then we say that a subset $C \subset \mathrm{X}$ is *locally convex* if each point $x \in C$ has an open neighborhood $U$ such that $U \cap C$ is convex.

**Note.** It is not required that $U \subset C$, or that $U \cap C$ is open.

## Lemma 1

Let $C$ be a closed set of a topological vector space $\mathrm{X}$. Then the following conditions are equivalent:

1. $C$ is a convex subset of $\mathrm{X}$.
2. $C$ is locally convex and connected.

# Remarks

The local convexity of a *subset* must not be confused with the local convexity of a *space*.

It is easy to check that if $A$ and $B$ are mutually disjoint, closed, locally convex subsets of a locally convex space $X$, then $A \cup B$ is locally convex. Clearly $A \cup B$ is not convex.

The local convexity of the *space* is to ensure that there exist small convex neighborhoods for each point of $x \in A$ (resp. $y \in B$) so that the intersection with $A$ (resp. with $B$) will be convex.

It is tempting to ask whether local convexity could be replaced by 'local starlikeness' in the above result.

This is not the case: the 'bow-tie' subset of $\mathbb{R}^2$

$$\{(x, y) \in \mathbb{R}^2 : \ |y| \leq |x| \leq 1\}$$

is closed, connected, locally convex away from the origin, starlike with respect to the the origin, but not convex.

# Main results: quasihyperbolic balls are starlike for $r \leq \log 2$

The following result is a generalization the second part of [K08, Theorem 1.1].

**Theorem A**

Let $\mathrm{X}$ be a Banach space, $\Omega \subset \mathrm{X}$ a domain with $\partial\Omega \neq \emptyset$ and suppose that $j$ is the $j$-metric defined on $\Omega$. Then each $j$-ball $\mathbf{B}_j(x_0, r)$, $x_0 \in \Omega$, is starlike for radii $r \leq \log 2$.

# Main results: convexity of quasihyperbolic and $j$-balls

The following result improves [MV06, Theorem 2.13] by removing the requirement of uniform convexity of the domain, and generalizes the first part of [K08, Theorem 1.1].

**Theorem B**

Let $\mathrm{X}$ be a Banach space and $\Omega \subset \mathrm{X}$ a convex domain with $\partial\Omega \neq \emptyset$. Then all quasihyperbolic balls and $j$-balls on $\Omega$ are convex.

The following can be verified, e.g., by differentiating with respect to $t$.

**Lemma 2**

Let $a, b, c, d > 0$ be constants such that $a/c = b/d$. Then

$$\frac{ta + (1-t)b}{tc + (1-t)d} = \frac{a}{c} \quad \text{for } t \in [0, 1].$$

# Further results: uniformly convex and smooth spaces

### Theorem C

Let $\mathrm{X}$ be a Banach space, which is uniformly smooth and uniformly convex, both of power type 2. Consider $\Omega = \mathrm{X} \setminus \{0\}$ endowed with the $j$-metric. Then there exists a constant $R > 0$ such that all $j$-balls of radius $r \leq R$ are convex.

# Further results: quasihyperbolic convexity

In [MV06, 2.14] Martio and Väisälä asked whether the quasihyperbolic balls of convex domains of uniformly convex Banach spaces are quasihyperbolically convex.

More precisely, given two points $a$ and $b$ of the quasihyperbolic ball $B \subset \Omega$, does there exist a geodesic $\gamma$ joining $a$ and $b$, which is contained in the ball $B$.

Here the domain $\Omega$ was assumed to be convex and the length of the geodesic is measured with respect to the quasihyperbolic metric.

It turns out the the answer is negative, as the following counterexample shows.

# Example 1/2

Let $\Omega = \{(x, y) \in \mathbb{R}^2 : \; y < 0\}$ and we will first consider $\Omega$ as a subset of $\ell^\infty(2) = (\mathbb{R}^2, \| \cdot \|_\infty)$. Let $x = (0, 1)$, and

$$r = \ln(2) = \int_1^2 t^{-1} \, dt.$$

We study the ball $\mathbf{B}_k(x, r)$.
Put $a = (-1, -2)$, $b = (1, -2)$ and observe that $\{ta + (1 - t)b : \; t \in [0, 1]\}$ is included in $\mathbf{S}_k(x, r)$. An intuition, which helps in computing the quasihyperbolic lengths of paths, is that one can move to the directions $(-1, -1)$, $(0, -1)$ and $(1, -1)$ at the same cost because of the choice of the norm. Note that $z_2 \geq -2$ for any $(z_1, z_2) \in \mathbf{B}_k(x, r)$.
It is easy to see that any path $\gamma \subset \mathbf{B}_k(x, r)$, joining $a$ and $b$ must have quasihyperbolic length at least

$$\int_{-1}^1 \frac{1}{2} \, dt = 1.$$

# Example 2/2

However, the straight line $\gamma_0$ through $(0, -3)$ has length

$$2\int_0^1 \frac{1}{3-t}\,dt = \ln(\frac{9}{4}) < 1.$$

The existence of geodesics is clear in this choice of space. Thus $\mathbf{B}_k(x, r)$ is not quasiconvex.

This example does not change considerably if one considers the domain $\Omega = (-6, 6) \times (0, 6)$ instead.

Observe that the space $\ell^\infty(2)$ is certainly not uniformly convex.

However, since the quasihyperbolic metric depends continuously on the selection of the norm, we could apply the space $\ell^p(2)$ for large $p < \infty$ in place of $\ell^\infty(2)$ to produce similar examples, in which case we are dealing with uniformly convex spaces.

# Quasihyperbolic geodesics

Next we consider quasihyperbolic geodesics in Banach spaces. Our first result is the following:

**Theorem D**

Let $(f_n) \in \ell^1 \setminus c_{00}$, $\Omega = \{(x_n) \in c_0 : \sum f_n x_n > 0\}$ and we consider $\Omega$ in the quasihyperbolic metric. Given any pair of distinct points $x, y \in \Omega$ there is no geodesic between them.

# Proof 1/2

Without loss of generality we may assume by rotating the coordinates that $f = (f_n) \in \ell^1$ is coordinatewise non-negative. Observe that $\langle f, x\rangle = \mathrm{dist}_{\|\cdot\|}(x, \partial\Omega)$ for $x \in \Omega$. We denote by $M \subset \mathbb{N}$ the infinite subset of indices $m$ such that $f_m > 0$.

Fix $x = (x_n), y = (y_n) \in \Omega,\ x \neq y$. Assume to the contrary to the statement of the theorem that $\gamma\colon [0,1] \to \Omega$ is a quasihyperbolic geodesic joining $x$ and $y$. Let $n \in \mathbb{N}$ such that $x_n - y_n = \delta \neq 0$. Without loss of generality $\delta > 0$. We may assume that $\gamma$ is parametrized by its norm length. Clearly $\ell_{\|\cdot\|}(\gamma) \geq \delta$.

Define a sequence $(\gamma_k)$ of modifications of $\gamma$. Let $e_n^*\colon c_0 \to \mathbb{R}$ be the functional given by $e_n^*(z) = z_n$ for each $z \in c_0$. Let $\alpha\colon [0,1] \to \mathbb{R}$ be a broken line such that $\alpha(0) = \alpha(1) = 0$ and $\alpha(1/2) = \delta/2$. We let $\gamma_k$ be such that $e_n^*(\gamma_k(t)) = \max(\gamma(t), \alpha(t))$ for $1 \leq n \leq k,\ t \in [0,1]$ and $e_n^*(\gamma_k(t)) = e_n^*(\gamma(t))$ for $n > k$ and $t \in [0,1]$.

# Proof 2/2

We let $\gamma_k$ be such that $e_n^*(\gamma_k(t)) = \max(\gamma(t), \alpha(t))$ for $1 \leq n \leq k$, $t \in [0,1]$ and $e_n^*(\gamma_k(t)) = e_n^*(\gamma(t))$ for $n > k$ and $t \in [0,1]$.

Observe that $\ell_{\|\cdot\|}(\gamma_k) = \ell_{\|\cdot\|}(\gamma)$ and $\ell_{\mathrm{qh}}(\gamma_k) \leq \ell_{\mathrm{qh}}(\gamma)$ for $k$.

f $t \mapsto e_m^*(\gamma_k(t))$ and $t \mapsto e_m^*(\gamma(t))$ are not same functions for an integer $m \in M$, then $\ell_{\mathrm{qh}}(\gamma_k) < \ell_{\mathrm{qh}}(\gamma)$. Therefore we obtain that $e_m^*(\gamma_k(1/2)) = e_m^*(\gamma(1/2))$ for $m \in M$ and $k \in \mathbb{N}$.

This means that $\limsup_{m\to\infty} e_m^*(\gamma(1/2)) \geq \delta/2$ and thus contradicts the fact that $\gamma(1/2) \in c_0$.

# Smoothness of geodesics

Let $\Omega \subset \mathrm{X}$ be a non-trivial path-connected domain and $\omega\colon \Omega \to (0, \infty)$ a continuous function. We define a (conformal) metric $d_\omega$ on $\Omega$ by

$$d_\omega(x, y) = \inf_\gamma \int_\gamma \omega \; dt$$

where the infimum is taken over rectifiable paths in $\Omega$ joining $x$ and $y$.

**Theorem E**

Let $\mathrm{X}$ be a uniformly convex Banach space with modulus of convexity $\delta_{\mathrm{X}}$ of power type 2. Let $w\colon \mathrm{X} \to [0, \infty]$ be a continuously Gateaux differentiable weigth function. Then every $d_\omega$-geodesic $\gamma$ is $C^1$.