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GEOMETRY OF METRICS

Workshop on Modern Trends in Classical Analysis and Applications
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Abstract

We discuss a bunch of ideas of modern geometric function theory. The notion of a metric space has a central role. In particular, we study hyperbolic type metrics, such as the quasihyperbolic metric, in subdomains of the Euclidean n-dimensional space. "Metric-preserving maps" such as bilipschitz and quasiconformal maps are natural objects of study in this context. Our aim is to keep this talk as self-contained and elementary as possible.

1 Introduction

1.1 Topological space (X,τ)

$\tau \subset \mathcal{P}(X)$ **is a topology if**

(a) $A_j \in \tau \implies \cup A_j \in \tau$

(b) $A_1, \ldots, A_n \in \tau \implies \cap_{j=1}^n A_j \in \tau$

(c) $\emptyset, X \in \tau$

The sets in τ are called open sets. Closed sets are of the form $X \setminus A, A \in \tau$. The closure of a set A is denoted by $\overline{A}$ or $clos(A)$ and it is defined as $\overline{A} = \cap_{A \subset C, C \text{ closed}} C$. The interior $intA$ of A is the complement of the closure of A . The boundary ∂A of A is $clos(A) \setminus intA$.

1.2 Continuous mapping

Let (X, τ_1) and (Y, τ_2) be topological spaces. A map $f : (X, \tau_1) \to (Y, \tau_2)$ is continuous if $A \in \tau_2$ implies $f^{-1}A \in \tau_1$.

1.3 Topological mapping (=homeomorphism)

Let $f : (X, \tau_1) \to (Y, \tau_2)$ be a bijection such that both f and f^{-1} are continuous. Then f is a homeomorphism.

1.4 Remark

Topological maps f preserve "topological properties":

1. **connectedness;**
2. **number of components;**
3. $fA \in \tau_2$ **if and only if** $A \in \tau_1$
4. **a disk** $\{z \in \mathbb{C} : |z| < 1\}$ **cannot be mapped onto an annulus** $\{z \in \mathbb{C} : 1 < |z| < c\}$ **but can be mapped onto** $\mathbb{C}$

A metric space is usually equipped with the topology defined by the metric.

1.5 Metric space (X, d)

Let $d : X \times X \to [0, \infty)$ **satisfy**

(a) $d(x, y) = d(y, x), \forall x, y$

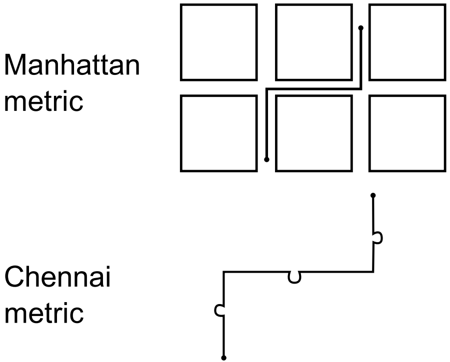
(b) $d(x, y) \le d(x, z) + d(z, y), \forall x, y, z$

(c) $d(x, y) \ge 0$ **and** $d(x, y) = 0 \iff x = y$

1.6 Examples

1. $(\mathbb{R}^n, |\cdot|)$ **is a metric space**
2. **The Möbius space** $(\overline{\mathbb{R}}^n, q), \mathbb{R}^n = \overline{\mathbb{R}}^n \cup \{\infty\}$ **is a metric space. The chordal metric** q **is defined with the help of the stereographic projection (cf. below).**
3. **Taxi cab metric also called Manhattan metric.**
4. **Chennai metric: Three wheeler taxi must avoid the obstacles (gaps, stones, rough surface of the road)**

The metrics (3) and (4) are determined by their geodesics, i.e. shortest curves, that will be defined soon. The picture is self-explaining



1.7 Remarks

1. **For a fixed** $A \subset X$ **the function** $f : X \to [0, \infty), f(x) = d(x, A) = \sup\{d(x, z) : z \in A\}$ **satisfies** $|f(x) - f(y)| \le d(x, y)$ **i.e.** f **is Lip-continuous with constant** 1 **(see below).**
2. **If** (X, d) **is a metric space, then also** (X, d^a) **is a metric space for all** $a \in (0, 1]$.
3. **More generally, if** $h : [0, \infty) \to [0, \infty)$ **is an increasing homeomorphism with** $h(0) = 0$ **such that** $h(t)/t$ **is decreasing, then** $(X, h \circ d)$ **is a metric space.**

1.8 Uniform continuity

Let $(X_j, d_j), j = 1, 2,$ **be metric spaces and** $f : (X_1, d_1) \to (X_2, d_2)$ **be a cont. map. Then** f **is uniformly continuous (u.c.) if there exists a cont. injection** $\omega : [0, t_0) \to [0, \infty)$ **such that** $\omega(0) = 0$ **and**

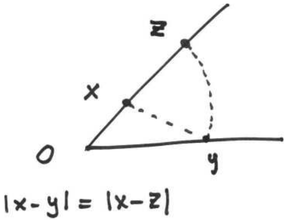
$$d_2(f(x), f(y)) \le \omega(d_1(x, y)) \,, \forall x, y \in X_1 \;\text{ with } d_1(x, y) < t_0 .$$

1.9 Remarks

1. **This definition is equivalent to the usual** (ε, δ)**-definition.**
2. **If** $\omega(t) = Lt$ **for** $t \in (0, t_0)$, **then** f **is** L**-Lipschitz (abbr.** L**-Lip)**
3. **If** $\omega(t) = Lt^a$ **for some** $a > 0$ **and all** $t \in (0, t_0)$, **then** f **is Hölder.**
4. **If** $f : X_1 \to X_2$ **is a bijection and** f **and** f^{-1} **are** L**-Lip then** f **is** L**-bilip.**
5. **A 1-bilip map is an isometry.**
6. **The map** $f : (X, |\cdot|) \to (X, |\cdot|), f(x) = 1/x, X = (0, \infty)$, **is not uniformly continuous. With** $\sigma(x, y) = |\log(x/y)|$ f **is u.c. as a map** $f : (X, \sigma) \to (X, \sigma)$.
7. **A Lip map** $h : [a, b] \to \mathbb{R}$ **has a derivative a.e.**

1.10 Exercise*

Fix $0 < a \le 1 \le b$. **Let** $g : \mathbb{R}^n \to \mathbb{R}^n$ **be defined by** $g(x) = |x|^{a-1}x,\ |x| < 1,$ **and** $g(x) = |x|^{b-1}x\ ,\ |x| \ge 1$. **Does there exist a constant** c, **with** $c \to 1$, **when** $a \to 1, b \to 1$ **such that** $|g(x) - g(y)| \le c|g(x) - g(z)|$ **for all** $x, y, 0 < |x| \le |y|, z = (|x| + |x - y|)x/|x|$**?**



1.11 Balls

Write $B_d(x_0, r) = \{x \in X : d(x_0, x) < r\}$ **and** $\overline{B}_d(x_0, r) = \{x \in X : d(x_0, x) \le r\}$.

1.12 Problems ([Vu-IWQCMA05])

Suppose that X is locally convex.

1. **Are balls convex (radius $t > 0$)?**
2. **Are balls convex for a small radius t?**
3. **Are the boundaries nice/smooth?**

1.13 Ball inclusion problem

Suppose that $(X, d_j), j = 1, 2$, determine the same (e.g. euclidean) topology. Then

$$B_{d_2}(x_0, r) \subset B_{d_1}(x_0, s) \subset B_{d_2}(x_0, t)$$

for some $r, s, t > 0$. For a fixed $r > 0$, find the best radii s and t . (Naturally, we could replace d_2 e.g. by the euclidean metric.)

1.14 Theorem

$\tau = \{B_d(x,r) : x \in X, r > 0\} \cup \emptyset \cup X$ **defines a topology.**

1.15 Remarks

1. **This is the natural topology of a metric space.**
2. $\overline{B}_d(x_0, r)$ **is closed.**
3. $(\mathbb{Z}, d), d(x,y) = |x-y|$ **is a metric space.**

   $B_d(0,1) = \{0\}$, $\overline{B}_d(0,1) = \{-1,0,1\}$. **Hence** $clos(B_d(x_0,r))$ **need not be** $\overline{B}_d(x_0,r)$. **Also**

   $$\text{diam}(B_d(0,1)) = 0 < \text{diam}(\overline{B}_d(0,1)) = 2\,.$$

4. **Balls need not be connected (cf. below).**
5. **It is possible that** $B_d(x_0,r)$ **is connected but** $clos(B_d(x_0,r))$ **is not. The closure may contain isolated points.**

1.16 Euclidean balls

In $\mathbb{R}^n$: balls are $B^n(x,r)$, spheres $\partial B^n(x,r) = S^{n-1}(x,r)$

1.17 Paths

A continuous map $\gamma :\ \Delta \to X, \Delta \subset \mathbb{R}$, is called a path. The length of $\gamma, \ell(\gamma)$, is

$$\ell(\gamma) = \sup \left\{ \sum_{i=1}^{n} d(\gamma(x_{i-1}), \gamma(x_i)) : \{x_0, , ..., x_n\} \text{ is a subdivision of } \Delta \right\}$$

We say a path is rectifiable if $\ell(\gamma) < \infty$. A rectifiable path $\gamma : \Delta \to X$ has a parameterization in terms of arc length $\gamma^o : [0, \ell(\gamma)] \to X$.

1.18 Exercise

$\ell([0,1]) = \infty$ **if** $d(x,y) = |x-y|^{1/2}$.

1.19 Definition

G is connected if for all $x, y \in G$ there exists a path $\gamma : [0,1] \to G$ such that $\gamma(0) = x, \gamma(1) = y$. We write Γ_{xy} for the set of all paths joining x with y in G. A domain is an open connected set.

1.20 Inner metric of a connected set $G \subset X$

$$d(x,y) = \inf\{\ell(\gamma) : \gamma \in \Gamma_{xy}\}, \quad x, y \in G.$$

1.21 Geodesics

A path $\gamma : [0,1] \to G$ where G is a domain, is a geodesic joining $\gamma(0)$ and $\gamma(1)$ if $\ell(\gamma) = d(\gamma(0), \gamma(t)) + d(\gamma(t), \gamma(1))$ for all $t \in [0,1]$.

1.22 Remarks

1. **In $(\mathbb{R}^n, |\cdot|)$ the segment $[x,y] = \{z \in \mathbb{R}^n : z = \lambda x + (1-\lambda)y, \lambda \in [0,1]\}$ is a geodesic**
2. **Let $G = B^2 \setminus \{0\}$ and d be the inner metric of G. There is no geodesics joining $-1/2$ and $1/2$ in (G,d).**

1.23 Path integrals

For a locally rectifiable path $\gamma : \Delta \to X$ and a continuous $f : \gamma\Delta \to \mathbb{R}$ the path integral is defined in two steps.

[I] **If γ is rectifiable, we set**

$$\int_\gamma f\,ds = \int_0^{\ell(\gamma)} f(\gamma^o(t))|(\gamma^o)'(t)|\,dt\,.$$

[II] **If γ is locally rectifiable, we set**

$$\int_\gamma f\,ds = \sup\left\{\int_\beta f\,ds : \ell(\beta) < \infty\,, \quad \beta \subset \gamma\right\}\,.$$

1.24 Weighted length

Let $G \subset X$ **be a domain and** $w : G \to (0, \infty)$ **continuous. Define**

$$d_w(x,y) = \inf\{\ell_w(\gamma) : \gamma \in \Gamma_{xy},\ \ell(\gamma) < \infty\}, \ell_w(\gamma) = \int_\gamma w(\gamma(z))\,|dz|\,.$$

1.25 Exercise

Show that d_w **defines a metric on** G.

1.26 Examples

1. **If** $G \subset X$ **is a domain, set** $w(z) = 1/d(z, \partial G)$**. The quasihyperbolic metric is a special case of the weighted metric,**

$$k_G(x, y) = \inf\{\ell_w(\gamma) : \ell_w(\gamma) = \int_\gamma w(\gamma(z))\,|dz|\,\}\,.$$

2. **If** $G = \mathbb{H}^n = \{x \in \mathbb{R}^n : x_n > 0\}$ **then the quasihyperbolic metric coincides with the usual hyperbolic metric. Often the notation** $\rho_{\mathbb{H}^n}$ **is used.**
3. **The hyperbolic metric of the unit ball** $\mathbb{B}^n$ **is a weighted metric with the weight function** $w(x) = 2/(1 - |x|^2)$ **. Often the notation** $\rho_{\mathbb{B}^n}$ **is used.**

All these three metrics have geodesics.

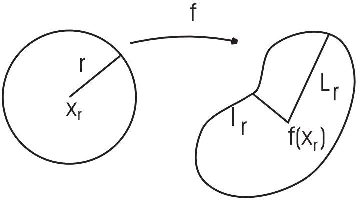
1.27 Special cases

1. **(1)** $\rho_{\mathbb{H}^n}(se_n, te_n) = |\log(s/t)|, \quad s, t > 0,$
2. **(2)** $\rho_n(0, se_n) = \int_0^s \frac{2\,dt}{1-t^2} = \log\frac{1+s}{1-s} = 2\,\mathbf{arth}\,s \quad s \in (0,1)$
3. **(3)** $k_n(0, se_n) = \int_0^s \frac{dt}{1-t} = \log\frac{1}{1-s} \quad s \in (0,1)$

1.28 Conformal mapping

If $G_1, G_2 \subset \mathbb{R}^n$ **are domains and** $f\colon G_1 \to G_2$ **is a diffeomorphism with** $|f'(x)h| = \underbrace{|f'(x)|}_{\text{operator n.}} \underbrace{|h|}_{\text{vector n.}}$ **we call** f **a *conformal map*. We use this definition also in the case** $G_1, G_2 \subset \overline{\mathbb{R}^n}$ **by excluding the two points** $\{\infty, f^{-1}(\infty)\}$.

1.29 Linear dilatation



For a homeomorphism $f : (X, d_1) \to (Y, d_2)$ **and** $x_0 \in X$ **define**

$$H(x_0, f, r) = \frac{L_r}{l_r}\,;\; H(x_0, f) = \limsup_{r\to 0} H(x_0, f, r)$$

1.30 Quasiconformal (qc) maps

A homeomorphism $f : G \to G', G, G' \subset \mathbb{R}^n$ is qc if there exists a constant $K(f) \geq 1$ such that $\forall x_0 \in G \; H(x_0, f) \leq K$.
N.B. Observe that for a conformal map $K(f) = 1$.

1.31 The Möbius group $\mathcal{GM}(\overline{\mathbb{R}^n})$

The group of Möbius transformations in $\overline{\mathbb{R}^n}$ is generated by transformations of two types

1. **inversions in $S^{n-1}(a, r) = \{z \in \mathbb{R}^n : |a - z| = r\}$**

$$x \mapsto a + \frac{r^2(x-a)}{|x-a|^2},$$

2. **reflections in hyperplane $P(a, t) = \{x \in \mathbb{R}^n : x \cdot a = t\}$**

$$x \mapsto x - 2(x \cdot a - t)\frac{a}{|a|^2}.$$

If $G \subset \overline{\mathbb{R}^n}$ we denote by $\mathcal{GM}(G)$ the group of all Möbius transformations with $fG = G$.

1.32 Plane versus space

1. **For $n = 2$ Möbius transformations are of the form $\frac{az+b}{cz+d}, ad - bc \neq 0$.**
2. **Recall that for $n = 2$ there are many conformal maps (Riemann mapping Theorem., Schwarz-Christoffel formula). For the case $n \geq 3$ conformal maps are Möbius transformations, by Liouville's theorem (suitable smoothness required).**
3. **Therefore conformal invariance for the case $n \geq 3$ is very different from the plane case $n = 2$.**

The stereographic projection $\pi : \overline{\mathbb{R}}^n \to S^n((1/2)e_{n+1}, 1/2)$ is defined by a Möbius transformation:

$$\pi(x) = e_{n+1} + \frac{x - e_{n+1}}{|x - e_{n+1}|^2}, x \in \mathbb{R}^n, \pi(\infty) = e_{n+1} .$$

1.33 Absolute (cross) ratio

For distinct points $a, b, c, d \in \overline{\mathbb{R}^n}$ **the absolute ratio is**

$$|a, b, c, d| = \frac{q(a,c)q(b,d)}{q(a,b)q(c,d)}, \quad q(x,y) = \frac{|x-y|}{\sqrt{1+|x|^2}\sqrt{1+|y|^2}}.$$

The most important property is Möbius invariance:
f **Möbius** $\Rightarrow |fa, fb, fc, fd| = |a, b, c, d|$**.**
(Permutations of a, b, c, d **lead to 6 different values of the absolute ratio. Sometimes in the literature these other values are used.)**

1.34 Examples

In most examples below, the metric spaces will have additional structure. In particular, we will study metric spaces (X, d) where the group Γ of automorphisms of X acts transitively (i.e. given $x, y \in X$ there exists $h \in \Gamma$ such that $hx = y$. We say that the metric d is quasiinvariant under the action of Γ if there exists $C \in [1, \infty)$ such that $d(hx, hy)/d(x, y) \in [1/C, C]$ for all $x, y \in X$, $x \neq y$, and all $h \in \Gamma$. If $C = 1$, then we say that d is invariant.

1. **The Euclidean space $\mathbb{R}^n$ equipped with the usual metric $|x - y| = (\sum_{j=1}^n (x_j - y_j)^2)^{1/2}$, Γ is the group of translations.**
2. **The unit sphere $S^n = \{z \in \mathbb{R}^{n+1} : |z| = 1\}$ equipped with the metric of $\mathbb{R}^{n+1}$ and Γ is the set of rotations of S^n .**

3. **Let $G \subset \mathbb{R}^n, G \neq \mathbb{R}^n$, for $x, y \in G$ set**

$$j_G(x,y) = \log\left(1 + \frac{|x-y|}{\min\{d(x,\partial G), d(y,\partial G)\}}\right).$$

Then one can prove that j_G is a quasiinvariant metric under the Möbius selfmaps of G, see e.g. [Se-99]. In fact, there exists a constant $C > 1$ such that for the unit ball $\mathbb{B}$ of $\mathbb{R}^n$

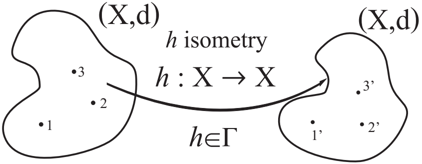
$$1/C \leq \rho_{\mathbb{B}}(x,y)/j_{\mathbb{B}}(x,y) \leq C$$

for all $x, y \in \mathbb{B}$. Here $\rho_{\mathbb{B}}$ is the hyperbolic metric of $\mathbb{B}$.

1.35 F.Klein's Erlangen Program (KEP) 1872 for geometry

- use isometries ("rigid motions") to study geometry
- Γ is the group of isometries
- two configurations are considered equivalent if they can be mapped onto each other by an element of Γ
- the basic "models" of geometry are
  - (a) Euclidean geometry of $\mathbb{R}^n$
  - (b) hyperbolic geometry of the unit ball B^n in $\mathbb{R}^n$
  - (c) spherical geometry (Riemann sphere)

The main examples of Γ are subgroups of Möbius transformations of $\overline{\mathbb{R}}^n = \mathbb{R}^n \cup \{\infty\}$. KEP had a profound influence not only on geometry but also on Geometric Function Theory (GFT).



1.36 Example: Rigid motions and invariant metrics in GFT

X	Γ	**metric**
G	$\mathcal{M}(G)$	ρ_G **hyperbolic metric**, $G = B^n, \mathbb{H}^n$
$\overline{\mathbb{R}}^n$	**Iso**$(\overline{\mathbb{R}}^n)$	q **chordal metric**
$\mathbb{R}^n$	**transl.**	$\lvert \cdot \rvert$ **Euclidean metric**
D	**conf.autom.**	h_D **hyp.metric of** $D \subset \mathbb{C}$

1.37 Beyond Erlangen, dictionary of the new world

We wish to study mappings in subdomains of $\mathbb{R}^n$. We must go beyond KEP in order to get a rich class of mappings. We are searching for a new kind of GFT.

Conformal map	$\rightarrow$	”Quasiconformal”
Conformal map	$\rightarrow$	”Metric-preserving maps”
Analytic	$\rightarrow$	”Quasiregular”
Invariance	$\rightarrow$	”Quasi-invariance”
Unit ball	$\rightarrow$	”Classes of domains”
Metric	$\rightarrow$	”Deformed metric”
World	$\rightarrow$	”Quasiworld”
Smooth	$\rightarrow$	"Nonsmooth"
Hyperbolic	$\rightarrow$	"Neohyperbolic"

A neohyperbolic metric is also called a hyperbolic type metric.

1.38 Metrics in particular domains: uniform, quasidisks

Sometimes inequalities between hyperbolic type metrics are used to characterize several subdomains in $\mathbb{R}^n$ such as uniform domains, quasidisks (images of $\mathbb{B}^2$ under a qc map of $\mathbb{R}^2$), φ-uniform domains, etc. For instance,

1. **[GO-79] A domain $G \subsetneq \mathbb{R}^n$ is uniform if and only if there exist constants $c, d > 0$ such that**

   $$k_G(x,y) \le c\, j_G(x,y) + d \text{ for all } x, y \in G\,.$$

   It was shown in [Vu-85] that we can replace (c,d) with $(c_2, 0)$.
2. **[Ge-99] A simply connected domain D is a quasidisk if and only if there is a constant $c > 0$ such that**

   $$h_D(z_1,z_2) \le c\, j_D(z_1,z_2) \quad \text{for all } z_1, z_2 \in D\,.$$
3. **Later in 2000, Gehring and Hag [GH-00] obtained the following simple characterization: A simply connected domain D is a quasidisk if and only if there is a constant c such that**

   $$h_D(z_1,z_2) \le c\, a_D(z_1,z_2) \quad \text{for all } z_1, z_2 \in D\,.$$

1.39 Open problem

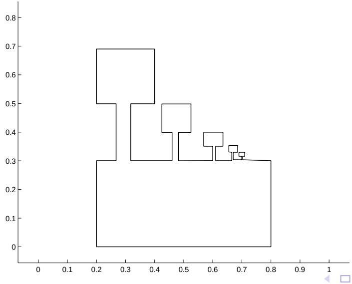
For which pairs of metrics (d_1, d_2) the condition: $\exists C \geq 1$ such that $\forall x, y \in G$

$$d_2(x,y) \leq C d_1(x,y) \text{ or } \leq \omega(d_1(x,y))$$

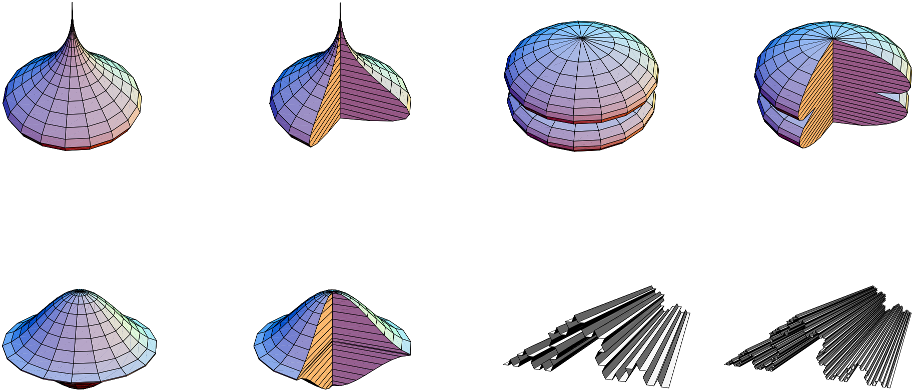
leads to a nice class of domains (ω is a function)? Usually we are interested in "hyperbolic-type metrics".

1.40 Non-uniform domains

Shrinking bottleneck domain. Rooms and corridors example.



1.41 Some non-smooth domains



2 Classical geometries

2.1 Weighted metric, geodesics and examples

1. **Let $G \subset \mathbb{R}^n$ be a domain and $w\colon G \to (0,\infty)$ a function such that for every rectifiable curve γ in G the integral**

$$\ell_w(\gamma) = \int_\gamma w\, ds$$

is defined. We call $\ell_w(\gamma)$ the w-length of γ.

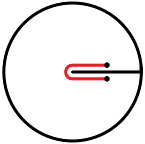
2. **Fix $a, b \in G$ and consider Γ_{ab}, the collection of all rectifiable curves in G joining a and b. The w-length minimizing property**

$$m_w(a,b) = \inf_{\gamma \in \Gamma_{ab}} \ell_w(\gamma),$$

defines the weighted metric in G. If a length minimizing curve exists, it is called a geodesic segment.

3. **When $w \equiv 1$ is the Euclidean distance in a convex subdomain $G \subset \mathbb{R}^n$. The geodesics are the Euclidean segments.**

4. **In the special case when $w \equiv 1$ in the non-convex set $G = B^2 \setminus [0,1)$ geodesics do not exist (consider the points $a = \frac{1}{2} + \frac{i}{10}$ and $b = \overline{a}$)**

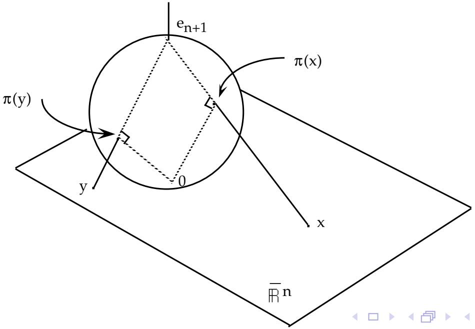


5. **If $w(x) = 1/d(x, \partial G)$, then m_w is the quasihyperbolic metric. Now geodesics exist (Gehring-Osgood 1979). Note: $w(x) = 1/d(x, \partial G)$ is like a "penalty-function", the geodesic segments try to keep away from the boundary.**
6. **If $G = \mathbb{H}^n = \{x \in \mathbb{R}^n \colon x_n > 0\}$ and $w(x) = 1/x_n$ then m_w is the usual hyperbolic metric.**

7. **If (X, d) is a metric space and $h\colon [0,\infty) \to [0,\infty)$ is a homeomorphism such that $h(t)/t$ is decreasing then $(X, h \circ d)$ is also a metric space.**
8. **Stereographic projection defines the chordal distance by**

$$q(x,y) = |\pi x - \pi y| = \frac{|x-y|}{\sqrt{1+|x|^2}\sqrt{1+|y|^2}}$$

for $x, y \in \overline{\mathbb{R}}^n = \mathbb{R}^n \cup \{\infty\}$



2.2 Comparison of metric balls (ball inclusion problem)

For $r, s > 0$ we obtain the formula

$$\rho(re_n, se_n) = \left| \int_s^r \frac{dt}{t} \right| = \left| \log \frac{r}{s} \right| . \qquad \text{(I)}$$

For $f \in \mathcal{GM}(\mathbb{H}^n)$ we have the invariance property:

$$\rho(x, y) = \rho(f(x), f(y)) \; \forall x, y \in \mathbb{H}^n. \qquad \text{(II)}$$

For $a \in \mathbb{H}^n$ and $M > 0$ the *hyperbolic ball* $\{ x \in \mathbb{H}^n : \rho(a, x) < M \}$ is denoted by $D(a, M)$. It is well known that $D(a, M) = B^n(z, r)$ for some z and r (this also follows from (II)!).

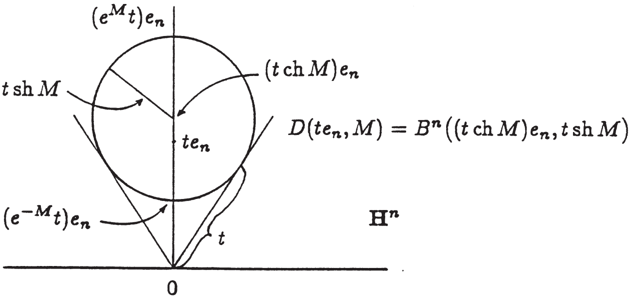
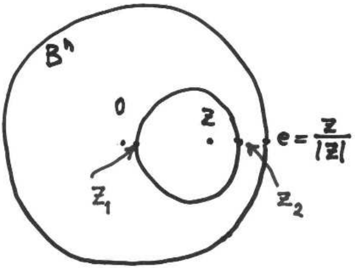


Figure: ***The hyperbolic ball*** $D(te_n, M) \subset \mathbb{H}^n$ ***as a Euclidean ball.***

This fact together with the observation that $\lambda te_n, (t/\lambda)e_n \in \partial D(te_n, M)$**,** $\lambda = e^M$ **(cf. (I)), yields**

$$\begin{cases} D(te_n, M) = B^n\big((t\cosh M)e_n, t\sinh M\big)\ , \\ B^n(te_n, rt) \subset D(te_n, M) \subset B^n(te_n, Rt)\ , \\ r = 1 - e^{-M}\ , \quad R = e^M - 1\ . \end{cases} \tag{III}$$



Consider now the balls $D(z, M)$ of $(\mathbb{B}, \rho)$. As in the case of $\mathbb{H}^n$ we have $D(z, M) = B^n(y, r)$ for some $y \in \mathbb{B}$ and $r > 0$. Making use of this fact, we shall find y and r. Let L_z be a Euclidean line through 0 and z and $\{z_1, z_2\} = L_z \cap \partial D(z, M)$, $|z_1| \leq |z_2|$. We may assume that $z \neq 0$ since with obvious changes the following argument works for $z = 0$ as well. Let $e = z/|z|$ and $z_1 = se$, $z_2 = ue$, $u \in (0, 1)$, $s \in (-u, u)$. Then it follows that

$$
\begin{aligned}
\rho(z_1, z) &= \log\left(\frac{1+|z|}{1-|z|} \cdot \frac{1-s}{1+s}\right) = M\,, \\
\rho(z_2, z) &= \log\left(\frac{1+u}{1-u} \cdot \frac{1-|z|}{1+|z|}\right) = M
\end{aligned}
$$

Solving these for s and u and using the fact that

$$
D(z, M) = B^n\left(\tfrac{1}{2}(z_1 + z_2),\ \tfrac{1}{2}|u - s|\right)
$$

one obtains the following formulae:

$$
\begin{cases}
D(x, M) = B^n(y, r) \\[2ex]
y = \dfrac{x(1-t^2)}{1-|x|^2 t^2}\,, \quad r = \dfrac{(1-|x|^2)t}{1-|x|^2 t^2}\,, \quad t = \tanh \tfrac{1}{2} M\,,
\end{cases}
\qquad \text{(IV)}
$$

and

$$\begin{cases} B^n\big(x,\ a(1-|x|)\big) \subset D(x,M) \subset B^n\big(x,\ A(1-|x|)\big)\ , \\[2ex] a = \dfrac{t(1+|x|)}{1+|x|t}\ ,\quad A = \dfrac{t(1+|x|)}{1-|x|t}\ ,\quad t = \tanh \tfrac{1}{2}M\ . \end{cases}$$

A special case of (IV):

$$D(0,M) = B_\rho(0,M) = B^n(\tanh \frac{1}{2}M)\ .$$

2.3 Hyperbolic metric of the unit ball B^n

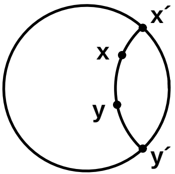
Four definitions of the hyperbolic metric ρ_{B^n}

1. $\rho_{B^n} = m_w$, $w(x) = \frac{2}{1-|x|^2}$,
2. $\sinh^2 \frac{\rho_{B^n}(x,y)}{2} = \frac{|x-y|^2}{(1-|x|^2)(1-|y|^2)}$, **([Be-82]),**
3. $\rho_{B^n}(x,y) = \sup\{\log|a,x,y,d| : a,d \in \partial B^n\}$,
4. $\rho_{B^n}(x,y) = \log|x_*,x,y,y_*|$.

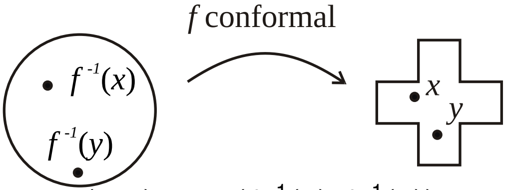
N.B. Invariance under $\mathcal{GM}(B^n)$.

2.4 The hyperbolic line through x, y

The hyperbolic geodesics between x, y in the unit ball are subarcs of the circular arcs joining x and y orthogonal to ∂B^n.

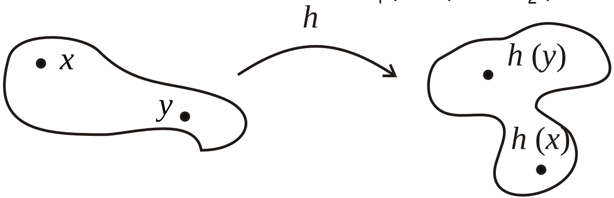


2.5 Hyperbolic metric of $G = f(B^n)$, f conformal



$$\rho(x, y) = \rho_{B^n}(f^{-1}(x), f^{-1}(y)).$$

The case $G_k = f_k(B^n)$ when f_k is conformal.
Let $h\colon G_1 \to G_2$ be conformal. Then $\rho_{G_1}(x, y) = \rho_{G_2}(hx, hy)$.



For $n = 2$ one can generalize the hyperbolic metric, using covering transformations, to a domain $G \subset \overline{\mathbb{R}}^2$ with card$(\overline{\mathbb{R}}^2 \setminus G) \geq 3$ **[KL-07].**

2.6 Spherical metric

$$w(x) = \frac{1}{1+|x|^2}, \quad x, y \in \overline{\mathbb{R}}^n .$$

The length minimizing property defines the spherical metric $\sigma(x,y)$. This spherical metric is equivalent to the chordal metric q. In fact, the two relationships

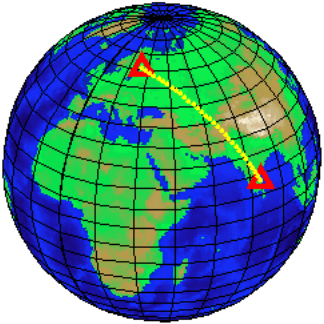
$$\sigma(x,y) = 2 \arcsin q(x,y) \quad \textbf{and} \quad 1 \le \frac{\sigma(x,y)}{q(x,y)} \le \pi = 4 \arctan 1$$

hold for all distinct $x, y \in \overline{\mathbb{R}}^n$.

2.7 Spherical geodesics

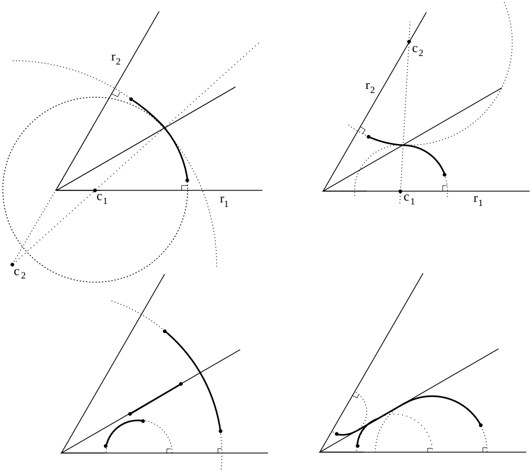
A spherical geodesic joining x and y is the smallest circular arc of the greatest circle joining x and y on the Riemann sphere.

Distance = 6505 kms

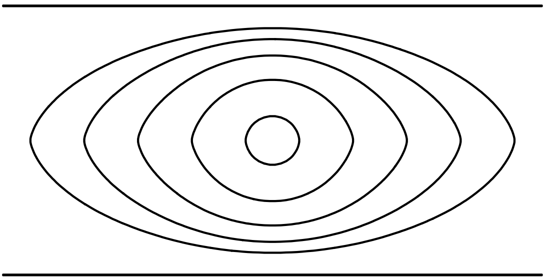


2.8 Quasihyperbolic geodesics [GO-79],[Mar-85]

(1) Since $k_{H^n} = \rho_{H^n}$, quasihyperbolic geodesics are hyperbolic geodesics in half-spaces.
(2) Sectorial (angular) domains [Lin-05]



(3) Infinite strip: Quasihyperbolic balls centered on "the middle line" have smooth boundaries.



(4) For every domain G, the quasihyperbolic geodesics are smooth [Mar-85].

3 Hyperbolic type geometries

3.1 The distance ratio metric j_G

For $x, y \in G$ the *distance ratio metric* j_G is defined [Vu-85] by

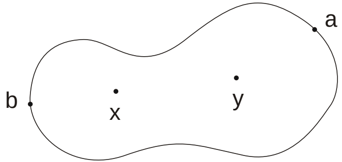
$$j_G(x,y) = \log\left(1 + \frac{|x-y|}{\min\{d(x), d(y)\}}\right).$$

In a slightly different form, this metric was studied in [GO-79]. We collect the following well-known facts:

1. **Inner metric of the j_G metric is the quasihyperbolic metric k_G.**
2. $k_G(x,y) \geq j_G(x,y), \forall x, y \in G$
3. $k_G(x,y) \leq 2j_G(x,y)$ **when** $j_G(x,y) < \log(3/2)$.
4. **Both k_G and j_G define Euclidean topology.**
5. j_G **is not geodesic; the balls** $B_j(z,M) = \{x \in G : j_G(z,x) < M\}$ **may be disconnected for large M. The closure of $B_j(z,M)$ may contain an isolated point.**

3.2 Apollonian metric of $G \subsetneq \mathbb{R}^n$

$$\alpha_G(x, y) = \sup\{\log |a, x, y, b| : a, b \in \partial G\}.$$



- α_G **agrees with** ρ_G**, if** G **equals** B^n **and** H^n**.**
- $\alpha_{hG}(hx, hy) = \alpha_G(x, y)$ **for** $h \in \mathcal{GM}(\mathbb{R}^n)$
- α_G **is a pseudometric if** ∂G **is "degenerate"**
- **The new history of** α_G **begins with Beardon 1998 and continues with P. Seittenranta 1998, Z. Ibragimov 2002, P. Hästö 2003, S. Sahoo 2008.**
- **The old history: D. Barbilian 1934.**

Facts

1. The well-known sharp relations $\alpha_G \leq 2j_G$ and $\alpha_G \leq 2k_G$ are due to Beardon [Be-98].
2. α_G is not geodesic.
3. Inner metric of the Apollonian metric is called the Apollonian inner metric and it is denoted by $\tilde{\alpha}_G$ (see [Ha-03, Ha-04, HPS-06]).
4. We have $\alpha_G \leq \tilde{\alpha}_G \leq 2k_G$.
5. $\tilde{\alpha}_G$-geodesic exists between any pair of points in $G \subsetneq \mathbb{R}^n$ if G^c is not contained in a hyperplane [Ha-04].
6. $\tilde{\alpha}_G$ can be expressed as a weighted integral [Ha-04].

3.3 Seittenranta's metric δ_G

For $x, y \in G \subsetneq \mathbb{R}^n$, Seittenranta's metric [Se-99] is defined by

$$\delta_G(x,y) = \sup_{a,b\in\partial G} \log\{1 + |a,x,b,y|\} .$$

Facts

1. **The function δ_G is a metric.**
2. **δ_G agrees with ρ_G, if G equals B^n or H^n.**
3. **The inequality $j_G \le \delta_G \le \tilde{j}_G \le 2j_G$ holds for every open set $G \subsetneq \mathbb{R}^n$, where the metric $\tilde{j}_G$ is a metric defined by**

$$\tilde{j}_G(x,y) = \log\left(1 + \frac{|x-y|}{d(x)}\right)\left(1 + \frac{|x-y|}{d(y)}\right) .$$

4. **It follows from the definitions that $\delta_{\mathbb{R}^n\setminus\{a\}} = j_{\mathbb{R}^n\setminus\{a\}}$ for all $a \in \mathbb{R}^n$.**
5. **$\alpha_G \le \delta_G \le \log(e^{\alpha_G} + 2) \le \alpha_G + 3$. The first two inequalities are best possible for δ_G in terms of α_G only [Se-99].**

6. **The inner metric of the metric δ_G is the so-called Ferrand metric [Fe-88] and it is defined by the weighted integral**

$$\sigma_G(x,y) = \inf_{\gamma\in\Gamma_{xy}} \int_\gamma w(z)\,|dz|\,,$$

where the weight function

$$w(z) = \sup_{a,b\in\partial G} \frac{|a-b|}{|z-a|\,|z-b|}, \quad z\in G\setminus\{\infty\}\,.$$

7. σ_G **is Möbius invariant.**
8. σ_{B^n} **and** σ_{H^n} **coincide with the hyperbolic metrics of** B^n **and** H^n **respectively.**
9. **In a simply connected planar domain** G**, with at least two boundary points,** $\rho_G \le \sigma_G \le 2\rho_G$**.**
10. $k_G \le \sigma_G \le 2k_G$ **for every** $G \subsetneq \mathbb{R}^n$**.**

3.4 Comparisons in B^n

If we compare the density functions of the hyperbolic and the quasihyperbolic metrics of B^n, we see that

$$\rho_{B^n}(x,y)/2 \le k_{B^n}(x,y) \le \rho_{B^n}(x,y)$$

for all $x, y \in B^n$. For the case of B^n we make use of an explicit formula [Be-82], [Vu-book, (2.18)] to the effect that for $x, y \in B^n$

$$\sinh \frac{\rho_{B^n}(x,y)}{2} = \frac{|x-y|}{t}\,, t = \sqrt{(1-|x|^2)(1-|y|^2)}\,. \qquad (1)$$

The following proposition gathers together several basic properties of the metrics k_G and j_G, see for instance [GP-76, Vu-book].

3.5 Proposition

1. **For a domain** $G \subset \mathbb{R}^n, x, y \in G$, **we have**

$$k_G(x,y) \geq \log\left(1 + \frac{L}{\min\{\delta(x),\delta(y)\}}\right) \geq j_G(x,y)\,,$$

where $L = \inf\{\ell(\gamma) : \gamma \in \Gamma(x,y)\}\,.$

2. **For** $x \in B^n$ **we have**

$$k_{B^n}(0,x) = j_{B^n}(0,x) = \log\frac{1}{1-|x|}\,.$$

3. **Moreover, for** $b \in S^{n-1}$ **and** $0 < r < s < 1$ **we have**

$$k_{B^n}(br,bs) = j_{B^n}(br,bs) = \log\frac{1-r}{1-s}\,.$$

Proposition...

4. **Let $G \subsetneq \mathbb{R}^n$ be any domain and $z_0 \in G$. Let $z \in \partial G$ be such that $\delta(z_0) = |z_0 - z|$. Then for any $u, v \in [z_0, z]$ we have**

$$k_G(u,v) = j_G(u,v) = \left|\log \frac{\delta(z_0) - |z_0 - u|}{\delta(z_0) - |z_0 - v|}\right| = \left|\log \frac{\delta(u)}{\delta(v)}\right| .$$

5. **For $x, y \in B^n$ we have**

$$j_{B^n}(x,y) \le \rho_{B^n}(x,y) \le 2 j_{B^n}(x,y)$$

with equality on the right hand side when $x = -y$.

Proof.

(1) Without loss of generality we may assume that $\delta(x) \le \delta(y)$. **Fix** $\gamma \in \Gamma(x,y)$ **with arc length parameterization** $\gamma : [0,u] \to G, \gamma(0) = x, \gamma(u) = y$

$$\ell_k(\gamma) = \int_0^u \frac{|\gamma'(t)|\,dt}{d(\gamma(t),\partial G)} \ge \int_0^u \frac{dt}{\delta(x)+t} = \log\frac{\delta(x)+u}{\delta(x)} \ge \log\left(1+\frac{|x-y|}{\delta(x)}\right) = j_G(x,y)\,.$$

(2) We see by (1) that

$$j_{B^n}(0,x) = \log\frac{1}{1-|x|} \le k_{B^n}(0,x) \le \int_{[0,x]} \frac{|dz|}{\delta(z)} = \log\frac{1}{1-|x|}$$

and hence $[0,x]$ **is the** k_{B^n}**-geodesic between** 0 **and** x **and the equality in (2) holds.**

Proof (Continued...)

The proof of (3) follows from (2) because the quasihyperbolic length is additive along a geodesic

$$k_{B^n}(0, bs) = k_{B^n}(0, br) + k_{B^n}(br, bs)\,.$$

The proof of (4) follows from (3).
The proof of (5) is given in [AVV-book, Lemma 7.56].

□

3.6 Lemma

1. **For** $0 < s < 1$ **and** $x, y \in B^n(s)$ **we have**

$$j_{B^n}(x,y) \le k_{B^n}(x,y) \le (1+s)\, j_{B^n}(x,y).$$

2. **Let** $G \subsetneq \mathbb{R}^n$ **be a domain,** $w \in G$, **and** $w_0 \in (\partial G) \cap S^{n-1}(w, \delta(w))$. **If** $s \in (0,1)$ **and** $x, y \in B^n(w, s\delta(w))$ **are such that** $\delta(x) = |x - w_0| \le \delta(y)$, **then we have**

$$k_G(x,y) \le (1+s) j_G(x,y)\,.$$

3. **Let** $s \in (0,1)$, $G = \mathbb{R}^n \setminus \{0\}$, $x, y, w \in G$ **with** $|x| \le |y|$ **and** $|x - w| < s\delta(w)$, $|y - w| < s\delta(w)$. **Then we have**

$$k_G(x,y) \le (1+s) j_G(x,y)\,.$$

Proof

(1) Fix $x, y \in B^n(s)$ and the geodesic γ of the hyperbolic metric joining them. Then $\gamma \subset B^n(s)$ and for all $w \in B^n(s)$ we have

$$\frac{1}{1-|w|} < \frac{1+s}{2}\frac{2}{1-|w|^2}\,.$$

Therefore by Proposition 3.5(5)

$$k_{B^n}(x,y) \le \int_\gamma \frac{|dw|}{1-|w|} \le \frac{1+s}{2}\int_\gamma \frac{2|dw|}{1-|w|^2}$$

$$\le \frac{1+s}{2}\rho_{B^n}(x,y) \le (1+s)j_{B^n}(x,y)\,.$$

for $x, y \in B^n(s)$. The first inequality follows from Proposition 3.5(1).

Proof.

For the proof of (2) set $B = B^n(w, \delta(w))$. Then by part (1)

$$k_G(x,y) \le k_B(x,y) \le (1+s) j_B(x,y) = (1+s) j_G(x,y) \,.$$

The proof of (3) follows from the proof of (2). □

3.7 Remark

(1) Lemma 3.6 (1) and (3) improve [Vu-book, Lemma 3.7(2)] for the cases of B^n and $\mathbb{R}^n \setminus \{0\}$. We have been unable to prove a similar statement for a general domain.
(2) The proof of Proposition 3.5 shows that the diameter $(-e, e), e \in S^{n-1}$, of B^n is a geodesic of k_{B^n} and hence the quasihyperbolic distance is additive on a diameter. At the same time we see that the j metric is additive on a radius of the unit ball but not on the full diameter because for $x \in B^n \setminus \{0\}$

$$j_{B^n}(-x, x) < j_{B^n}(-x, 0) + j_{B^n}(0, x) .$$

Our next goal is to compare the Euclidean and the quasihyperbolic metric in a domain and we recall in the next lemma a sharp inequality for the hyperbolic metric of the unit ball proved in [Vu-book, (2.27)].

Lemma 3.8

For $x, y \in B^n$ let t be as in (1). Then

$$\tanh^2 \frac{\rho_{B^n}(x,y)}{2} = \frac{|x-y|^2}{|x-y|^2 + t^2},$$

$$|x-y| \leqslant 2 \tanh \frac{\rho_{B^n}(x,y)}{4} = \frac{2|x-y|}{\sqrt{|x-y|^2+t^2}+t},$$

where equality holds for $x = -y$.

Earle and Harris [EH-09] provided several applications of this inequality and extended this inequality to other metrics such as the Carathéodory metric. Several remarks about Lemma 3.8 are in order. Notice that Lemma 3.8 gives a sharp bound for the modulus of continuity

$$id : (B^n, \rho_{B^n}) \to (B^n, |\cdot|).$$

3.9 QC Schwarz lemma

If $f : \mathbb{B}^n \to \mathbb{B}^n$ is K-qc (NOT DEFINED!), then for all $x, y \in \mathbb{B}^n$

$$\rho_{\mathbb{B}^n}(f(x), f(y)) \le 2\,\mathrm{arth}\,\varphi_{K,n}(\mathrm{th}\frac{1}{2}\rho_{\mathbb{B}^n}(x, y))$$

For $n = 2$ the result is sharp for each $K \geq 1$, see [LV-book, p. 65 (3.6)]. The particular case $K = 1$ yields a classical Schwarz lemma.

As a preliminary step we record Jung's Theorem [Ber-87, Theorem 11.5.8] which gives a sharp bound for the radius of a Euclidean ball containing a given bounded domain.

Lemma 3.10

Let $G \subset \mathbb{R}^n$ be a domain with $\mathrm{diam}\, G < \infty$. Then there exists $z \in \mathbb{R}^n$ such that $G \subset B^n(z, r)$, where $r \leq \sqrt{n/(2n+2)}\,\mathrm{diam}\, G$.

Lemma 3.11

1. *If x, y are on a diameter of B^n and $w = |x-y|\, e_1/2$, then we have*

$$k_{B^n}(x,y) \geq k_{B^n}(-w,w) = 2\, k_{B^n}(0,w) = 2 \log \frac{2}{2-|x-y|} \geq |x-y|\,,$$

where the first inequality becomes equality when $y = -x$.

2. *If $x, y \in B^n$ are arbitrary and $w = |x-y|\, e_1/2$, then*

$$k_{B^n}(x,y) \geq k_{B^n}(-w,w) = 2\, k_{B^n}(0,w) = 2 \log \frac{2}{2-|x-y|} \geq |x-y|\,,$$

where the first inequality becomes equality when $y = -x$.

Lemma 3.12 (Continued...)

3. *Let* $G \subsetneq \mathbb{R}^n$ *be a domain with* $\operatorname{diam} G < \infty$ *and* $r = \sqrt{n/(2n+2)}\,\text{diam}\, G$*. Then we have*

$$k_G(x,y) \ge 2 \log \frac{2}{2-t} \ge t = |x-y|/r\,,$$

for all distinct $x, y \in G$ *with equality in the first step when* $G = B^n(z,r)$ *and* $z = (x+y)/2$.

Proof

In the proof of (1) and (2), without loss of generality, we assume that $|x| \geq |y|$.

1. **If $0 \in [x, y]$, by Proposition 3.5(2) we have**

$$k_{B^n}(x,y) = k_{B^n}(x,0) + k_{B^n}(0,y) = \log \frac{1}{(1-|x|)(1-|y|)},$$

and hence

$$k_{B^n}(-w,w) = 2\log \frac{1}{1-|w|}.$$

We need to prove that

$$(1-|w|)^2 \geq (1-|x|)(1-|y|).$$

It suffices to show that

$$\frac{|x|+|y|}{2} = \frac{|x-y|}{2} = |w| \leq 1 - \sqrt{(1-|x|)(1-|y|)},$$

which is equivalent to $(|x|-|y|)^2 \geq 0$.
If $y \in [x,0]$**, then the proof goes in a similar way. Indeed, we note that** $|x|-|y| = |x-y| = 2|w|$**. Then by Proposition 3.5(3) we have**

$$k_{B^n}(x,y) = \log \frac{1-|y|}{1-|x|}\,.$$

It is enough to show that

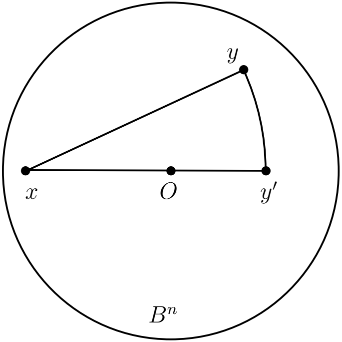
$$(1-|w|)^2 \geq \frac{1-|x|}{1-|y|} = 1 + \frac{|y|-|x|}{1-|y|}\,.$$

Substituting the value of $|w|$ **and then squaring we see that**

$$(|x|-|y|)\left(1-\frac{1}{1-|y|}\right) \leq \left(\frac{|x|-|y|}{2}\right)^2,$$

which is trivial as the left hand term is ≤ 0**. Equality holds if** $y=-x$**.**

2. **Choose $y' \in B^n$ such that $|x-y| = |x-y'| = 2|w|$ with x and y' on a diameter of B^n (see figure below).**



Then

$$k_{B^n}(x,y) \geq k_{B^n}(x,y') \geq k_{B^n}(-w,w)\,,$$

where the first inequality holds trivially and the second one holds by (1).

3. **Since G is a bounded domain, by Lemma 3.10, there exists $z \in \mathbb{R}^n$ such that $G \subset B^n(z,r)$. Denote $B := B^n(z,r)$. Then the domain monotonicity property gives**

$$k_G(x,y) \geq k_B(x,y)\,.$$

Without loss of generality we may now assume that $z = 0$. Choose $u, v \in B$ in such a way that $u = -v$ and $|u - v| = 2|u| = |x - y|$. Hence by (2) we have

$$k_G(x,y) \geq k_B(x,y) \geq k_B(-u,u) = 2\log\frac{r}{r-|u|}\,.$$

This completes the proof.

Corollary 3.13

1. *For every* $x, y \in B^n$ *we have*

$$|x-y| \le 2(1-\exp(-k_{B^n}(x,y)/2)) \le k_{B^n}(x,y)\,,$$

where the first inequality becomes equality when $y=-x$.

2. *If* $G \subsetneq \mathbb{R}^n$ *is a domain with* $\operatorname{diam} G < \infty$ *and* $r = \sqrt{n/(2n+2)}\operatorname{diam} G$, *then we have*

$$|x-y|/r \le 2(1-\exp(-k_G(x,y)/2)) \le k_G(x,y)$$

for all distinct $x, y \in G$ *with equality in the first step when* $G = B^n(z,r)$ *and* $z=(x+y)/2$.

A counterpart of Lemma 3.11 for the metric j_G is discussed below.

Lemma 3.14

1. *If x, y are on a diameter of B^n and $w = |x - y| e_1/2$, then we have*

$$j_{B^n}(x,y) \ge j_{B^n}(-w,w) = \log \frac{2+t}{2-t} \ge t = |x-y|\,,$$

where the first inequality becomes equality when $y = -x$.

2. *If $x, y \in B^n$ are arbitrary and $w = |x - y| e_1/2$, then*

$$j_{B^n}(x,y) \ge j_{B^n}(-w,w) = \log \frac{2+t}{2-t} = 2\operatorname{artanh}(t/2) \ge t = |x-y|\,,$$

where the first inequality becomes equality when $y = -x$.

3. *Let $G \subsetneq \mathbb{R}^n$ be a domain with $\operatorname{diam} G < \infty$ and $r = \sqrt{n/(2n+2)}\operatorname{diam} G$. Then we have*

$$j_G(x,y) \ge \log \frac{2+t}{2-t} \ge t = |x-y|/r\,,$$

for all distinct $x, y \in G$ with equality in the first step when $G = B^n(z,r)$ and $z = (x+y)/2$.

Proof.

In the proof of (1) and (2), without loss of generality, we may assume that $|x| \geq |y|$.

1. **If $0 \in [x, y]$, we have**

$$j_{B^n}(x,y) = \log\left(1 + \frac{|x-y|}{1-|x|}\right) = \log\frac{1+|y|}{1-|x|}$$

and hence

$$j_{B^n}(-w,w) = \log\frac{1+|w|}{1-|w|} = \log\frac{2+|x-y|}{2-|x-y|} \geq |x-y|\,.$$

The inequality $j_{B^n}(x,y) \geq j_{B^n}(-w,w)$ is clear due the fact that $2|w| = |x| + |y|$ and $|x| \geq |y|$. If $y \in [x, 0]$, a similar reasoning gives the conclusion.

Proof (Continued...)

2. **Choose $y' \in B^n$ such that $|x-y| = |x-y'| = 2|w|$ with x and y' on a diameter of B^n. Then**

$$j_{B^n}(x,y) = j_{B^n}(x,y') \ge j_{B^n}(-w,w)\,,$$

   where the lower bound holds by (1).

3. **The proof is very similar to the proof of Lemma 3.11(3).**

This completes the proof. □

Corollary 3.15

1. *For every* $x, y \in B^n$ *we have*

$$|x-y| \le 2\tanh(j_{B^n}(x,y)/2) \le j_{B^n}(x,y)\,,$$

where the first inequality becomes equality when $y=-x$.

2. *If* $G \subsetneq \mathbb{R}^n$ *is a domain with* $\operatorname{diam} G < \infty$ *and* $r=\sqrt{n/(2n+2)}\operatorname{diam} G$, *then we have*

$$|x-y|/r \le 2\tanh(j_G(x,y)/2) \le j_G(x,y)\,,$$

for all distinct $x, y \in G$ *with equality in the first step when* $G = B^n(z,r)$ *and* $z=(x+y)/2$.

Remark 3.16

Let us denote the spherical chordal metric in $\overline{\mathbb{R}}^n$ *by* $q(x,y)$. *Starting with the sharp inequality [AVV-book, 7.17 (3), p. 378]*

$$|x-y| \geq \frac{2q(x,y)}{1+\sqrt{1-q(x,y)^2}}$$

we deduce that

$$q(x,y) \leq \frac{|x-y|}{1+(|x-y|/2)^2}$$

with equality for $y=-x$. *Therefore, we see that the identity mapping*

$$id : (\overline{\mathbb{R}}^n, |\cdot|) \to (\overline{\mathbb{R}}^n, q)$$

has the sharp modulus of continuity $\omega(t) = t/(1+(t/2)^2)$ *for* $t \in (0,2)$.

3.17 Existence of k_G geodesics

1. **In the case of $\mathbb{R}^n \setminus \{0\}$ Martin and Osgood (see [MO-86]) have determined the geodesics. Their result states that given $x, y \in \mathbb{R}^n \setminus \{0\}$, $|x| \le |y|$, the geodesic segment can be obtained as follows: let φ be the angle between the segments $[0,x]$ and $[0,y]$, $0 < \varphi < \pi$. The triple $0, x, y$ clearly determines a 2-dimensional plane Σ, and the geodesic segment connecting x to y is the logarithmic spiral in Σ with polar equation**

$$r(\omega) = |x| \exp\left(\frac{\omega}{\varphi} \log \frac{|y|}{|x|}\right) .$$

In this punctured space the quasihyperbolic distance is given by the formula

$$k_{\mathbb{R}^n \setminus \{0\}}(x,y) = \sqrt{\varphi^2 + \log^2 \frac{|x|}{|y|}} .$$

1. **[Lin-05] Let $\varphi \in (0, \pi]$ and $x, y \in S_\varphi = \{(r, \theta) \in \mathbb{R}^2 : 0 < \theta < \varphi\}$, the angular domain. Then the quasihyperbolic geodesic segment is a curve consisting of line segments and circular arcs orthogonal to the boundary. If $\varphi \in (\pi, 2\pi)$, then the geodesic segment is a curve consisting of line segments, logarithmic spirals and circular arcs orthogonal to the boundary.**
2. **[Lin-05] In the punctured ball $\mathbb{B}^n \setminus \{0\}$, the quasihyperbolic geodesic segment is a curve consisting of logarithmic spirals and geodesic segments of the quasihyperbolic metric of B^n.**

3.18 Diameter problems

There exists a domain $G \subsetneq \mathbb{R}^n$ and $x \in G$ such that $j(\partial B_j(x,M)) \neq 2M$ for all $M > 0$. Indeed, let $G = B^n$. Choose $x \in (0, e_1)$ and consider the j-sphere $\partial B_j(0,M)$ for $M = j_G(x,0)$. Now, $B_j(0,M)$ is a Euclidean ball with radius $|x| = 1 - e^{-M}$. The diameter of the j-sphere $\partial B_j(0,M)$ is

$$j_G(x,-x) = \log\left(1 + \frac{|2x|}{d(x)}\right) = \log\left(1 + \frac{2-2e^{-M}}{e^{-M}}\right) = \log(2e^M - 1)\,.$$

We are interested in knowing whether $j_G(x,-x) = 2M$ holds, equivalently in this case, $(e^M - 1)^2 = 0$ which is not true for any $M > 0$. Therefore, we always have $j_G(x,-x) < 2M$ and the diameter of $\partial B_j(0,M)$ is less than twice the radius M.

Open problem 3.19 ([Vu-IWQCMA05], [Kle-09])

Does there exist a number $M_0 > 0$ such that for all $M \in (0, M_0]$ we have $k(\partial B_k(x, M)) = 2M$.

For a convex domain G, it is known by Martio and Väisälä [MV-08] that $k(\partial B_k(x, M)) = 2M$.
The same question, but for the hyperbolic metric of a twice punctured plane, was solved in the negative by Beardon and Minda during the ROMFIN2009 meeting [BM-09].

4 Particular Cases, Examples

4.1 Bounded and convex domains

1. **[Se-99, Theorem 4.1] If $G \subsetneq \mathbb{R}^n$ is bounded and $d(\partial G)$ represents diameter of ∂G, then we have**

$$\delta_G(x,y) \geq \log\left(1 + \frac{d(\partial G)}{d(\partial G) - \max\{d(x), d(y)\}} \frac{|x-y|}{\min\{d(x), d(y)\}}\right)$$

for all $x, y \in G$.

2. **In [Be-98], it has been proved that if G is a simply connected domain bounded plane domain such that the inequalities $\alpha_G \leq \rho_G$ or $j_G \leq \rho_G$ holds, then G is convex.**
3. **[Se-99, Theorem 4.2] If $G \subsetneq \mathbb{R}^n$ be a convex domain, then $j_G \leq \alpha_G$.**

4.2 The Apollonian and its inner metric

In this section, we present some materials from [HPS-06]. In order to state the results in a succinct manner we define some relations on the set of metrics in G.

Definition 4.3

Let d and d' be metrics on G.

1. *We write $d \lesssim d'$ if there exists a constant $K > 0$ such that $d \le Kd'$.*
2. *We write $d \approx d'$ if $d \lesssim d'$ and $d \gtrsim d'$.*
3. *We write $d \ll d'$ if $d \lesssim d'$ and $d \not\gtrsim d'$.*
4. *We write $d \lessgtr d'$ if $d \not\lesssim d'$ and $d \not\gtrsim d'$.*

Let us first of all note that the following inequalities hold in every domain $G \subsetneq \mathbb{R}^n$:

$$\alpha_G \lesssim j_G \lesssim k_G \quad \textbf{and} \quad \alpha_G \lesssim \tilde{\alpha}_G \lesssim k_G. \tag{0}$$

The first two are from [Be-98, Theorem 3.2] and the second two from [Ha-03, Remark 5.2 and Corollary 5.4]. We see that of the four metrics to be considered, the Apollonian is the smallest and the quasihyperbolic is the largest.

#	Inequality	A	B	#	Inequality	A	B
1.	$\alpha \approx j \approx \tilde{\alpha} \approx k$	+	+	**7.**	$\alpha \approx j \ll \tilde{\alpha} \ll k$	-	-
2.	$\alpha \ll j \approx \tilde{\alpha} \approx k$	-	-	**8.**	$\alpha \ll j \ll \tilde{\alpha} \ll k$	-	-
3.	$\alpha \approx j \approx \tilde{\alpha} \ll k$	-	-	**9.**	$\alpha \approx \tilde{\alpha} \ll j \approx k$	-	+
4.	$\alpha \ll j \approx \tilde{\alpha} \ll k$	-	-	**10.**	$\alpha \ll \tilde{\alpha} \ll j \approx k$	-	+
5.	$\alpha \approx j \ll \tilde{\alpha} \approx k$	+	+	**11.**	$\alpha \approx \tilde{\alpha} \ll j \ll k$	-	?
6.	$\alpha \ll j \ll \tilde{\alpha} \approx k$	+	+	**12.**	$\alpha \ll \tilde{\alpha} \ll j \ll k$	-	?

Table: Inequalities between the metrics α_G, j_G, $\tilde{\alpha}_G$ and k_G. The subscripts are omitted for clarity with the understanding that every metric is defined in the same domain. The A-column refers to whether the inequality can occur in simply connected planar domains, the B-column to whether it can occur in proper subdomains of $\mathbb{R}^n$.

We will undertake a systematic study of which of the inequalities in (0) can hold in the strong form with $\ll$ and which of the relations $j_G \ll \tilde{\alpha}_G$, $j_G \approx \tilde{\alpha}_G$ and $j_G \gg \tilde{\alpha}_G$ can hold. Thus we are led to twelve inequalities, which are given along with the results in Table 1, where we have indicated in column A whether the inequality can hold in simply connected planar domains and in column B whether it can hold in an arbitrary proper subdomains of $\mathbb{R}^n$. From the table we see that most of the cases cannot occur, which means that there are many restrictions on which inequalities can occur together. For instance, we deduce from items 1–4 that $j_G \approx \tilde{\alpha}_G$ implies that $\alpha_G \approx k_G$ and from items 9–12 that the inequality $\tilde{\alpha}_G \ll j_G$ cannot occur in simply connected planar domains.
Items 11 and 12 in Table 1, were solved recently in [HPWS-09] for general domains of $\mathbb{R}^n$ and the answers are negative. The main tool to solve this question was the concept of uniform domains which is described in the next section.

4.4 Definitions of uniform domains

In this section we shall consider several versions of uniform domains, in terms of hyperbolic type metrics, which are equivalent to the uniform domains originally defined in the sense of Martio and Sarvas [MS-79].
The following form is interesting because it gives a direct comparison between the quasihyperbolic and the *j*-metrics.

Definition 4.5

A domain $G \subsetneq \mathbb{R}^n$ is called uniform, if there exists a number $A \geq 1$ such that

$$k_G(x,y) \leq A\, j_G(x,y) \tag{1}$$

for all $x, y \in G$. Furthermore, the best possible number

$$A_G := \inf\{A \geq 1 : A \text{ satisfies } (1)\}$$

is called the uniformity constant of G.

We now collect constants of uniformity in specific domains [Lin-05].

1. **For the domain $\mathbb{R}^n \setminus \{0\}$, the uniformity constant is given by**

$$A_{\mathbb{R}^n \setminus \{0\}} = \pi / \log 3 \approx 2.8596\,.$$

2. **Constant of uniformity in the punctured ball $B^n \setminus \{0\}$ is same as that in $\mathbb{R}^n \setminus \{0\}$.**
3. **For the angular domain S_φ, the uniformity constant is given by**

$$A_{S_\varphi} = \frac{1}{\sin \frac{\varphi}{2}} + 1$$

when $\varphi \in (0, \pi]$.
Using the case of small angles we also get bounds for the case of large angles $\varphi \in (\pi, 2\pi)$. However, these are not sharp. Indeed, we have

$$\max\left\{2, \frac{2\log\tan(\varphi/4) + \varphi - \pi}{\log(1 - 2\cos(\varphi/2))}\right\} \le A_{S_\varphi} \le 4\left(\frac{\varphi}{2\pi - \varphi}\right)^2 \left(\frac{1}{\sin(\varphi/2)} + 1\right).$$

Since $j_G \le \delta_G \le 2j_G$ and $k_G \le \sigma_G \le 2k_G$, in the sense of Seittenranta [Se-99], we have the following alternative definition

Definition 4.6

A domain $G \subsetneq \mathbb{R}^n$ is called c-uniform, $c \ge 1$, if

$$\sigma_G(x,y) \le c\,\delta_G(x,y)$$

for all $x, y \in G$.

Remarks 4.7

1. *Definitions 4.5 and 4.6 have the following connection: if a domain G is uniform in the sense of Definition 4.5 with constant c, then it is $2c$-uniform in the sense of Definition 4.6. Conversely, if a domain G is c-uniform in the sense of Definition 4.6 then it is uniform in the sense of Definition 4.5 with constant $2c$.*
2. *The domains B^n and H^n are 1-uniform, whereas the best possible constant c for B^n and H^n in Definition 4.5 is 2.*

In [HPWS-09] the approach to solve items 11 and 12 of Table 1, generates another concept of uniformity in terms of $\tilde{\alpha}_G$ and j_G as follows:

Definition 4.8

A domain $G \subsetneq \mathbb{R}^n$ is called C-uniform, $C \geq 1$, if

$$\tilde{\alpha}_G(x,y) \leq C\, j_G(x,y)$$

for all $x, y \in G$.

Remarks 4.9

1. *Since $j_G \leq \delta_G \leq 2j_G$, in a similar fashion, it is also interesting to define uniformity in terms of $\tilde{\alpha}_G$ and δ_G.*
2. *Constants of uniformity in the sense of Definition 4.8 are not known, however it is not difficult to obtain the constants C for B^n and H^n.*

Open problem 4.10

What can we say about sharp constants of uniformity in the sense of Definitions 4.6 and 4.8 for the domains $B^n \setminus \{0\}$, $\mathbb{R}^n \setminus \{0\}$ and S_φ?

4.11 φ-uniform domains

As indicated in Section 1, the notion of φ-uniformity comes from that of uniformity. In this section we mainly concentrate on the recent work [KSV-09] and [HKSV-09].

Examples of φ-uniform domains

1. **Uniform domains are φ-uniform as indicated before.**
2. **Consider domains G satisfying the following property [Vu-85, 2.50]: there exists a constant $C \geq 1$ such that each pair of points $x, y \in G$ can be joined by a rectifiable path $\gamma \in G$ with**

   $$\ell(\gamma) \leq C\,|x-y| \text{ and } d(\gamma, \partial G) \geq \min\{\delta(x), \delta(y)\}/C\,.$$

   Then G is φ-uniform with $\varphi(t) = C^2 t$.
3. **In particular, every convex domain is φ-uniform with $\varphi(t) = t$.**
4. **However, in general, convex domains need not be uniform (for example parallel strips).**
5. **There exist φ-uniform domains with "arbitrary" $\varphi(t)$**

Characterization of φ-uniform domains

Theorem 4.12

The identity mapping id : $(G, j_G) \to (G, k_G)$ is uniformly continuous if and only if G is φ-uniform.

Proof.

Sufficiency part is trivial. Indeed, for $x, y \in G$ we have

$$k_G(x,y) \le \varphi(\exp(j_G(x,y)) - 1) = \omega(j_G(x,y))$$

where $\omega(t) = \varphi(e^t - 1)$. Conversely, define $\varphi : (0,\infty) \to (0,\infty)$ by

$$\varphi(t) = \sup\{k_G(x,y) : j_G(x,y) \le t\} .$$

By assumption $\varphi(t)$ exists. Also, $\varphi(t)$ is continuous, strictly increasing and fixes the origin. Since $\log(1+t) \le t$ for all $t \ge 0$, it follows that G is φ-uniform. □

Complement of φ-uniform domains

- **Because simply connected uniform domains in plane are quasidisks [MS-79], it follows that the complement of such a uniform domain also is uniform.**
- **Since the half-strip defined by** $S = \{(x,y) \in \mathbb{R}^2 :\ x > 0,\ -1 < y < 1\}$ **is convex, by the above discussion we observe that it is** φ**-uniform with** $\varphi(t) = t$**. On the other hand, by considering the points** $z_n = (n, -2)$ **and** $w_n = (n, 2)$ **we see that** $G := \mathbb{R}^2 \setminus \overline{S}$ **is not a** φ**-uniform domain. Indeed, we have** $j_G(z_n, w_n) = \log 5$ **and for some** $m \in \mathbb{R} \cap J_G[z_n, w_n]$

$$k_G(z_n, w_n) \geq k_G(m, w_n) \geq \log\left(1 + \frac{|m - w_n|}{\delta(w_n)}\right) \geq \log(1+n) \to \infty \quad \textbf{as } n \to \infty.$$

- **Consider the domain**

$$D = \left\{(x,y) \in \mathbb{R}^2 : -\exp(-1-x) < y < \exp(-1-x),\ x > 0\right\}.$$

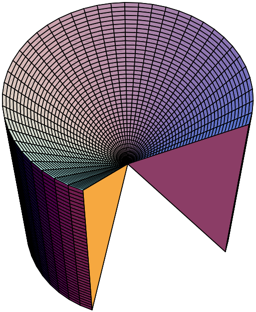
It is clear by previous investigation that D is φ-uniform with $\varphi(t) = 4t$. We next show that its complementary domain $G := \mathbb{R}^2 \setminus \overline{D}$ is not φ-uniform. We see that the points $z_n = (n, -e^{-n})$ and $w_n = (n, e^{-n})$ are in G, and $j_G(z_n, w_n) = \log 3$. On the other hand, let $m \in J_G[z_n, w_n] \cap \mathbb{R}$. Then

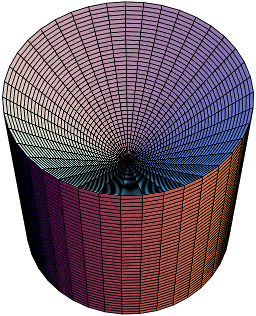
$$\begin{aligned} k_G(z_n, w_n) \geq k_G(z_n, m) &\geq \log\left(1 + \frac{|z_n - m|}{e^{-n}}\right) \\ &\geq \log(1 + ne^n) > n \to \infty \ \textbf{ as } n \to \infty\,. \end{aligned}$$

- **(Question!) Are there any bounded φ-uniform domains whose complement is not φ-uniform ?**

Let T be the triangle with vertices $(1,-1)$, $(0,0)$ and $(1,1)$. Consider the domain D bounded by the surface of revolution generated by revolving T about the vertical axis. Then we see that D is φ-uniform. Indeed, let $x, y \in D$ be arbitrary. Without loss of generality we assume that $|x| \geq |y|$. Consider the path $\gamma = [x, x'] \cup C$ joining x and y, where $x' \in S^1(|y|)$ is chosen so that $|x' - x| = d(x, S^1(|y|))$; and C is the smaller circular arc of $S^1(|y|)$ joining x' to y. For $z \in D$, we write $\delta(z) := d(z, \partial D)$.

Uniform domain $\mathbb{D}$, $\mathbb{R}^3 \setminus \mathbb{D}$ not φ-uniform for any φ





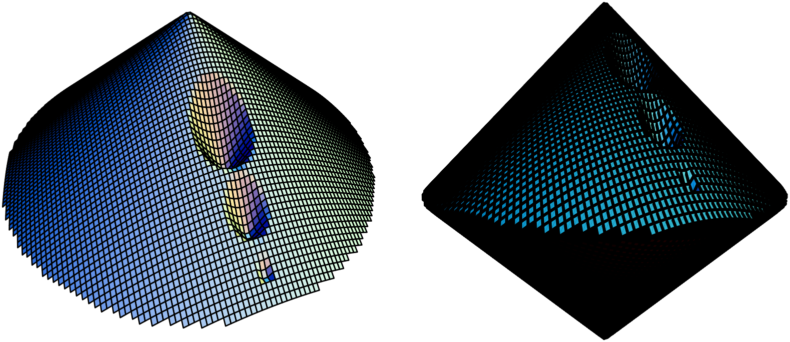


Figure: A double cone domain with two-sided drillings. This picture provides a schematic view of the simply-connected domain $D \subset \mathbb{R}^3$ constructed in [KSV-09]. The domain is uniform but its complement is not φ-uniform for any φ.

Then for all $x, y \in D$ we have

$$k_D(x,y) \le \int_\gamma \frac{|dz|}{\delta(z)} = \int_{[x,x']} \frac{|dz|}{\delta(z)} + \int_C \frac{|dz|}{\delta(z)}$$
$$\le \frac{|x-y|}{\min\{\delta(x),\delta(y)\}} + \int_C \frac{|dz|}{\delta(z)}$$
$$\le \left(1+\frac{\pi}{2}\right)\frac{|x-y|}{\min\{\delta(x),\delta(y)\}},$$

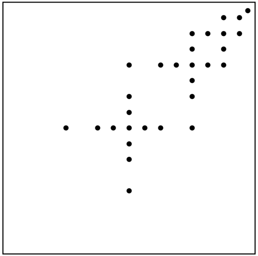
where the last inequality follows by the fact that $\ell(C) \le \pi|x-y|/2$. On the other hand, its complement $G = \mathbb{R}^3 \setminus \overline{D}$ is not φ-uniform. Because for the point $z_t = te_2 \in G$, $0 < t < 1$, we have $j_G(-z_t, z_t) = \log(1+2\sqrt{2})$; and by a similar argument as in the previous example we have

$$k_G(-z_t, z_t) \ge \log\left(1+\frac{\sqrt{2}}{t}\right) \to \infty \quad \textbf{as } t \to 0\,.$$

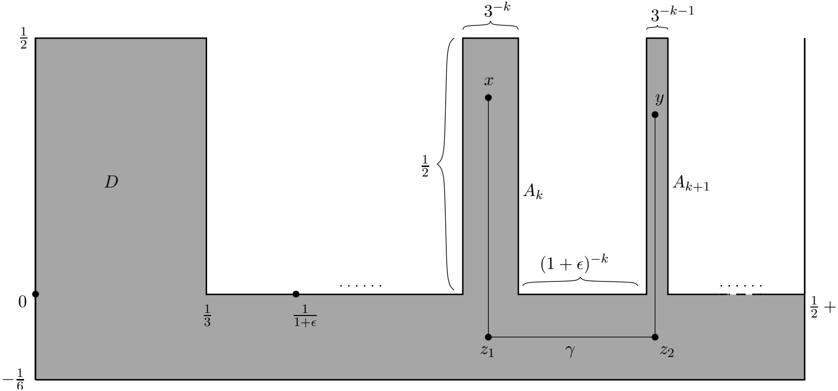
Quasiconvexity and φ-uniformity

- **A domain $G \subset \mathbb{R}^n$ is said to be *quasiconvex* if there exists a constant $c > 0$ such that any pair of points $x, y \in G$ can be joined by a rectifiable path $\gamma \subset G$ satisfying $\ell(\gamma) \le c\,|x - y|$.**
- **We see from the above examples that the complementary domains are not quasiconvex.**
- **There exist quasiconvex domains which are not φ-uniform and also conversely (for example, see the figure below).**

A quasiconvex domain which is not φ-uniform



A φ-uniform plane domain which is not quasiconvex



Properties of φ-uniform domains

1. **Let $G \subsetneq \mathbb{R}^n$ be a φ_1-uniform domain and $z_0 \in G$. Then $G \setminus \{z_0\}$ is φ-uniform for some φ depending on φ_1 only.**
2. **Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be an L-bilipschitz mapping, that is**

$$|x - y|/L \leq |f(x) - f(y)| \leq L|x - y|$$

for all $x, y \in \mathbb{R}^n$. If $G \subsetneq \mathbb{R}^n$ is φ-uniform, then $f(G)$ is φ_1-uniform with $\varphi_1(t) = L^2\varphi(L^2 t)$.

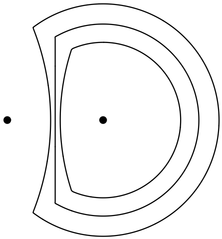
3. **Let $z_0 \in \mathbb{R}^n$ and $R > 0$ be arbitrary. Denote by h an inversion in $S^{n-1}(z_0, R)$. For $0 < m < M$, if $G \subset B^n(z_0, M) \setminus \overline{B}^n(z_0, m)$ is a φ-uniform domain, then $h(G)$ is φ_1-uniform with $\varphi_1(t) = (M/m)^2\varphi(M^2 t/m^2)$.**
4. **Suppose that $G \subsetneq R^n$ is a φ-uniform domain and f is a quasiconformal map of R^n which maps G onto $G' \subsetneq R^n$. Then G' is φ_1-uniform for some φ_1 as well.**
5. **If the identity map $id : (G, j_G) \to (G, k_G)$ is η-QS, then G is φ-uniform for some φ depending on η only.**

4.13 Convexity problem [Vu-IWQCMA05]

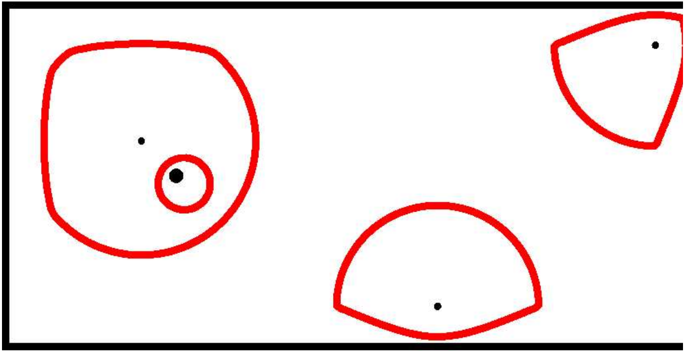
Fix a domain $G \subsetneq \mathbb{R}^n$ and neohyperbolic metric m in a collection of metrics (e.g. quasihyperbolic, Apollonian, j_G, hyperbolic metric of a plane domain etc.). Does there exist constant $T_0 > 0$ such that the ball $B_m(x, T) = \{z \in G \colon m(x, z) < T\}$, is convex (in Euclidean geometry) for all $T \in (0, T_0)$?

Theorem 4.14 ([Kle-08])

For a domain $G \subsetneq \mathbb{R}^n$ and $x \in G$ the j-balls $B_j(x, M)$ are convex if and only if $M \in (0, \log 2]$.



Diversity of shapes of j-disks, Klen [Kle-08]



Domain G is a rectangle with one puncture. The red curves are boundaries of j disks of constant radius with centers marked with a dot.

j-disks with same center, varying radii

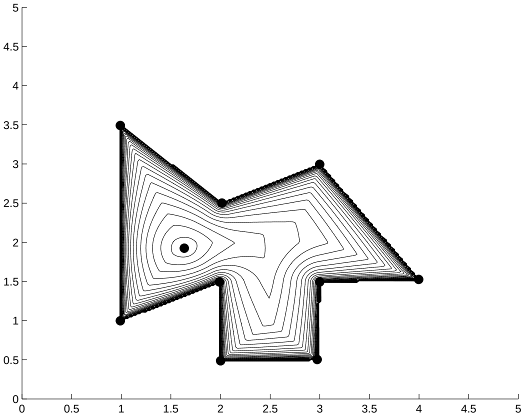


Figure: ***Boundaries (nonsmooth!) of*** j***-disks*** $B_{j_{\mathbb{R}^2\setminus\{0\}}}(x,M)$ ***with radii*** $M=-0.1+\log 2$, $M=\log 2$ ***and*** $M=0.1+\log 2$.

Theorem 4.15 ([Kle-07], [MO-86])

For $x\in\mathbb{R}^2\setminus\{0\}$ *the quasihyperbolic disk* $B_k(x,M)$ *is strictly convex for* $M\in(0,1]$ *and it is not convex for* $M>1$.

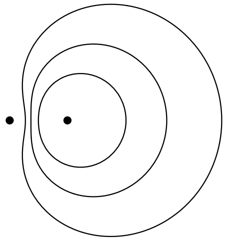


Figure: ***Boundaries of quasihyperbolic disks*** $B_{k_{\mathbb{R}^2\setminus\{0\}}}(x,M)$ ***with radii*** $M=0.7$***,*** $M=1.0$ ***and*** $M=1.3$***.***

An example of an unsmooth boundary of $B_k(x,r)$.

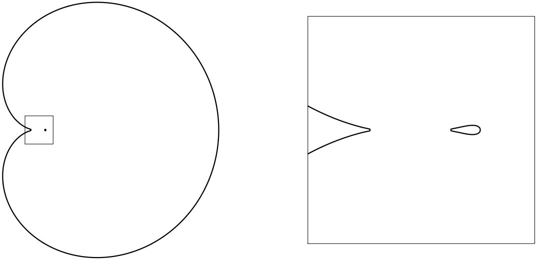


Figure: ***Boundary of*** $B_{k_{\mathbb{R}^2\setminus\{0\}}}(1,3.2)$***.***

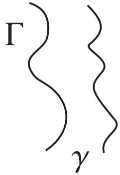
5 Conformal invariants μ_G, λ_G^{-1}

5.1 The modulus of a curve family

Definition 5.2

The modulus of a curve family Γ *of curves in* $G \subset \mathbb{R}^n$ *is defined as* $M(\Gamma) = \inf_{\rho \in F(\Gamma)} \int_G \rho^n \, dm$ *where*

$$F(\Gamma) = \{\rho : G \to \mathbb{R} \cup \{\infty\} : \rho \geq 0, \rho \text{ is Borel}, \int_\gamma \rho ds \geq 1 \; \forall \gamma \in \Gamma\}$$



Ahlfors-Beurling 1950, Väisälä 1961
Kühnau handbook 2002-2005

5.3 Conformal invariance

$M(\Gamma) = M(g\Gamma)$ **if g is conformal.**

$$\Delta(E, F; G) = \{\text{curves in G joining E and F}\}$$

Remark 5.4

Let $G, G' \subset \mathbb{R}^n$ be domains and $f : G \to G'$ a homeomorphism. f is K-quasiconformal (in the sense of Väisälä) if for all $\Gamma \subset G$:

$$M(\Gamma)/K \leq M(f\Gamma) \leq KM(\Gamma)\,.$$

5.5 Conformally invariant extremal problem: Grötzsch

The Grötzsch and Teichmüller rings arise from extremal problems of the following type, which were first posed for the case of the plane: Among all ring domains which separate two given closed sets E_1 and E_2, $E_1 \cap E_2 = \emptyset$, find one whose modulus has the greatest value.

Let E_1 be a continuum and E_2 consist of two points not separated by E_1. By the conformal invariance of the modulus one may then suppose that $E_1 = S^1$(circle) and $E_2 = \{0, r\}$, $0 < r < 1$. Then the extremal problem is solved by the bounded Grötzsch ring $R(\mathbb{R}^2 \setminus B^2, [0, r])$.

N.B. $R(E, F)$ stands for the ring domain with the complementary components E, F.

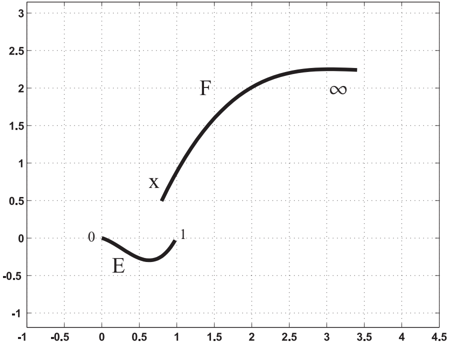


Figure: ***The extremal problem of Teichmüller***

5.6 Conformally invariant extremal problem

Teichmüller's problem asks to determine for $x \in \mathbb{R}^n \setminus \{0, e_1\}$, $n \geq 2$, in terms of well-known special functions,

$$p(x) = \inf_{E,F} \mathsf{M}(\triangle(E, F)),$$

where the infimum is taken over all pairs of continua E and F in $\overline{\mathbb{R}}^n$ with $0, e_1 \in E$, $x, \infty \in F$.
For a proper subdomain G of $\overline{\mathbb{R}}^n$ and for all $x, y \in G$ define

$$\mu_G(x, y) = \inf_{C_{xy}} \mathsf{M}\big(\Delta(C_{xy}, \partial G; G)\big)$$

where the infimum is taken over all continua C_{xy} such that $C_{xy} = \gamma[0, 1]$ and γ is a curve with $\gamma(0) = x$ and $\gamma(1) = y$. It is clear that μ_G is a conformal invariant. It is easy to show that μ_G is a metric if $\operatorname{cap} \partial G > 0$. If $\operatorname{cap} \partial G > 0$, we call μ_G the *modulus metric* or *conformal metric* of G. ($G = B^n \Leftrightarrow$ Grötzsch)

5.7 Conformal invariants

If G is a proper subdomain of $\overline{\mathbb{R}^n}$, then for $x, y \in G$ with $x \neq y$ we define (following J. Ferrand)

$$\lambda_G(x, y) = \inf_{C_x, C_y} \mathsf{M}(\Delta(C_x, C_y; G))$$

where $C_z = \gamma_z[0, 1)$ and $\gamma_z \colon [0, 1) \to G$ is a curve such that $z \in |\gamma_z|$ and $\gamma_z(t) \to \partial G$ when $t \to 1$, $z = x,\ y$. Clearly, λ_G is invariant under conformal mappings of G. That is, $\lambda_{fG}(f(x), f(y)) = \lambda_G(x, y)$, if $f \colon G \to fG$ is conformal and $x, y \in G$ are distinct. (For $n = 2, \partial G = \{0, \infty\} \Leftrightarrow$ Teichmüller.)

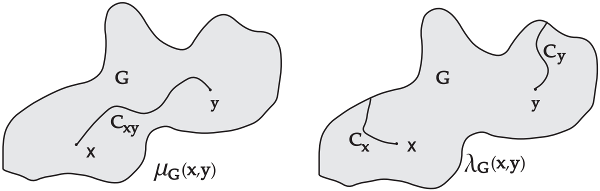


Figure: *The conformal invariants λ_G and μ_G .*

5.8 Ferrand's metric

J. Ferrand has proved that $\lambda_G(x,y)^{1/(1-n)}$ **defines a metric [Fe-97]. For the case** $G = B^n$, **with** $\rho(x,y) = \rho_{B^n}(x,y)$ **we have**

$$\lambda_{B^n}(x,y) = 2^{n-1}\tau_n(1/\sinh^2(\rho(x,y)/2))\,,$$

$$\mu_{B^n}(x,y) = \tau_n(\sinh^2(\rho(x,y)/2))\,.$$

For $n = 2$ **we have**

$$\tau_2(t) = \pi/\mu(1/\sqrt{1+t}),\;\; \mu(r) = \frac{\pi}{2}\,\frac{K(\sqrt{1-r^2}\,)}{K(r)}\,,$$

where

$$K(r) = \int_0^1 [(1-x^2)(1-r^2x^2)]^{-1/2}dx$$

for $0 < r < 1$**. In particular, we have**

$$\mu_{B^2}(x,y)\lambda_{B^2}(x,y) = 4\,.$$

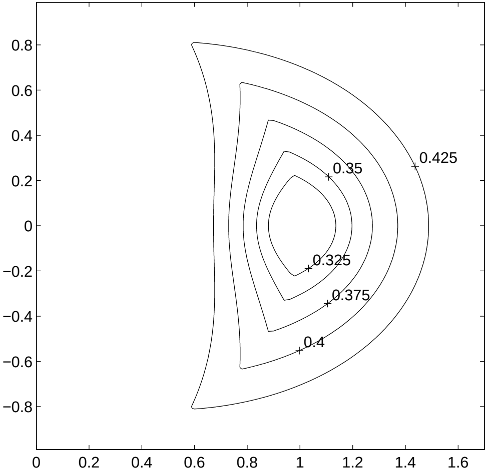


Figure: **Some spheres** $\{y \in \mathbb{R}^2 : \lambda_G(1,y)^{-1} = c\}$, $G = \mathbb{R}^2 \setminus \{0\}$ **of the Ferrand metric.**

5.9 Generalized Lipschitz problem (GLP)

Consider a category of maps $\mathcal{F}$ (e.g. qc, bilip, isometries, even identity map), a class of domains $\mathcal{D}$ (including quasidisks, uniform domains, . . .) and neohyperbolic metrics $\mathcal{M}$.

Problem 5.10

Suppose that $f : (X, d_1) \to (Y, d_2)$ is lipschitz. Is it true that f is qc?

N.B. Here $f \in \mathcal{F}$, $X, Y \in \mathcal{D}$, $d_j \in \mathcal{M}$ are allowed to vary independently. Therefore this problem alone yields several dozens of problems for moderately small categories $\mathcal{F}$, $\mathcal{D}$, $\mathcal{M}$.
J. Ferrand [LF-73] posed this question for the metric $\lambda_G^{1/(1-n)}$ and it was solved in the negative in [FMV-91].

5.11 Ramifications of Lipschitz's problem

We can ask similar questions for many combinations of the metrics letting d_j be one of the neohyperbolic metrics we have seen so far.
There are numerous ramifications of this problem. E.g. for $X = Y = B^n$ we can ask to determine the modulus of continuity of the identity mapping from $(B^n, \rho) \to (B^n, |.|)$. We have the sharp nontrivial inequality for $x, y \in B^n$

$$|x - y| \le 2\tanh(\rho(x, y)/4) .$$

A particular case of the above problem is to characterize isometries $f : (X, d_1) \to (Y, d_2)$. This interesting special case is largely open, but thanks to the recent work of Hästö, Ibragimov, Lindén in some cases the solution is known.
The study of these problems offers topics for PhD theses...

For example, the phd thesis of R. Klén [Kle-09] focused mainly on similar problems for the metrics $m \in \{q, k, j\}$ where they together well-defined. The following is one such

Theorem 5.12

For $R > 1$, consider the annulus $G = A(1/R, R) := \{z \in \mathbb{R}^n : \ 1/R < |z| < R\}$. Let $m_G \in \{k_G, j_G\}$. Then there exists a constant $C(R) \geq 1$ such that $C(R) \to 1$ as $R \to \infty$ and

$$m_G(x, y) \leq c(R)\, m_{\mathbb{R}^n \setminus \{0\}}(x, y)$$

for all $x, y \in A(1/\sqrt{R}, \sqrt{R})$.

6 Comparison of geometries[Vu-85],[Se-99],[Man-09]

6.1 Relations among j_G, k_G, λ_G and μ_G

The the following table we shall list some relations among the metrics j_G, k_G, λ_G and μ_G. If a and b are positive functions in $G \times G$ then we write $a \lessdot b$ if and only if there exists a strictly increasing continuous function $\zeta : (0, \infty) \to (0, \infty)$ such that $a(x, y) \le \zeta(b(x, y))$ for all $x, y \in G$ and $\zeta(t) \to 0$ as $t \to 0$.

	j_G	k_G	μ_G	λ_G
j_G	$=$	$\le$	$\lessdot$	$j_G \lessdot 1/\lambda_G$
k_G	$\lessdot$ φ-unif.	$=$	$\lessdot$ ∂G conn., φ-unif.	$k_G \lessdot 1/\lambda_G$ φ-unif.
μ_G	$\lessdot$ φ-unif.	$\lessdot$	$=$	$\mu_G \lessdot 1/\lambda_G$ φ-unif.
λ_G	$\lambda_G \lessdot j_G$	$\lambda_G \lessdot k_G$ φ-unif.	$\lambda_G \lessdot 1/\mu_G$ ∂G conn., φ-unif.	$=$

6.2 Open problems

Assume that $G \subset \mathbb{R}^n$ is a proper subdomain. For what follows, we will be interested mainly in the cases when the domain is a member of some well-known class of domains. Some examples are uniform domains, QED-domains, domains with uniformly perfect (in the sense of Pommerenke [Su-03]) boundaries and quasiballs, i.e. domains G of the form $G = f\mathbb{B}^n$ for quasiconformal $f\colon \mathbb{R}^n \to \mathbb{R}^n$. We denote the class of domains with $\mathcal{D}$. Let us consider collection of metrics $\mathcal{M} = \{\alpha_G, h_G, j_G, k_G, \lambda_G^{1/(1-n)}, \mu_G, q, |\cdot|\}$ where h_G refers to the hyperbolic metric when $n = 2$. Interesting categories of mappings, we denote them by $\mathcal{C}$, are Hölder, Lipschitz, isometries, quasiisometries and identity mappings.
The problems that we list below are just examples. There are a great many variations, by letting the domain, mapping and metric independently vary over the categories $\mathcal{D}$, $\mathcal{C}$, and $\mathcal{M}$.

6.3 Convexity of balls and smoothness of spheres

Fix $m \in \mathcal{M}$. Does there exist constant $T_0 > 0$ such that $D_m(x, T) = \{z \in G\colon m(x,z) < T\}$, is convex (in Euclidean geometry) for all $T \in (0, T_0)$? Is $\partial G_m(x, T)$ smooth for $T < T_0$? For instance, in the case $m = k_G$ both of these problems seem to be open. In passing, we remark that it follows from (4.4) and Theorem 4.7 (2) that when the radius tends to 0, quasihyperbolic balls become more and more round. The quasihyperbolic metric is used as a tool for many applications, but very little about the metric itself is known. See the theses [Mar-85] and [Lin-05] and also [Lin-iwqcma05].

6.4 Lipschitz-constant of identity mapping

For $x, y \in \mathbb{B}, x \neq y$, the following inequality holds [Vu-book, (2.27)]

$$|x-y| \le 2\tanh \frac{\rho_{\mathbb{B}}(x,y)}{4} < \frac{\rho_{\mathbb{B}}(x,y)}{2}.$$

We may now regard this result as an inequality for the modulus of continuity of $id\colon (\mathbb{B}, \rho_{\mathbb{B}}) \to (\mathbb{B}, |\cdot|)$. Instead of considering the identity mapping we could now take any mapping in our category of mappings and consider the problem of estimating the modulus of continuity between any two metric spaces in our category of metric spaces, see [Vu-85], [Se-99]. We list several particular cases of our problem.

For $G = \mathbb{R}^n \setminus \{0\}$ does there exist constants A or B such that for all $x, y \in G$

$$q(x, y) \le Ak_G(x, y),$$

and

$$q(x, y) \le B\lambda_G^{1/(1-n)}(x, y)\,?$$

For $G = \mathbb{C} \setminus \{0, 1\}$ does there exist constant C such that for all $x, y \in G$

$$q(x, y) \le Ch_G(x, y)\,,$$

For $G = \mathbb{R}^n \setminus \{0\}$ does there exist a constant E such that for all $x, y \in G$

$$\lambda_G^{1/(1-n)}(x, y) \le Ej_G(x, y)\,?$$

6.5 Characterization of isometries and quasiisometries

Given two metric spaces in our category of spaces, does there exist a quasiisometry, mapping the one space onto the other space? Again, we could consider, in place of quasiisometries, any other map in our category of maps.
What is the modulus of continuity of $id\colon (G,\mu_G) \to (G,\lambda_G^{1/(1-n)})$?
Is a quasiisometry $f\colon (G,\lambda_G^{1/(1-n)}) \to (fG,\lambda_{fG}^{1/(1-n)})$ quasiconformal?
J. Lelong-Ferrand raised this question in [LF-73] and the question was answered in the negative in [FMV-91]. There it was also shown that the answer is affirmative under the stronger requirement that $f\colon (D,\lambda_D^{1/(1-n)}) \to (fD,\lambda_{fD}^{1/(1-n)})$ be uniformly continuous for all subdomains D of G. However, it is not known what the isometries are.
Are isometries $f\colon (G,\alpha_G) \to (fG,\alpha_{fG})$ Möbius transformations? (see Beardon [Be-98], Hästö and Ibragimov [HI-05] and also [Ha-iwqcma05].

6.6 Conformal invariants

The conformal invariant $p(x)$ is relatively well-known. See [HV-03] for further information. However, much less is known about the invariants μ_G and λ_G. For domains whose boundaries are uniformly perfect (in the sense of Pommerenke), there are some inequalities for μ_G in terms of j_G, see [Vu-87] and [JV-96]. Some results for λ_G when $G = \mathbb{B} \setminus \{0\}$, were proved in [Hei-01] and [BV-00]. But even the basic question of finding a formula for $\lambda_{\mathbf{B}^2 \setminus \{0\}}(x, y)$ is open.
Some of these problems may be hard, some are very easy. Because of the very general setup, it would require some effort even to single out the interesting combinations of domains in $\mathcal{D}$, mappings in $\mathcal{C}$, and metrics in $\mathcal{M}$.

6.7 KMV generalization

(I am not sure about it!)

7 Addendum

Theorem 7.1

For all $x, y \in \mathbb{R}^n \setminus \{0\}$

(i) $2q(x,y) \le k(x,y) \le \frac{\pi}{\log 3} j(x,y)$,

(ii) $q(x,y)\log 3 \le j(x,y) \le k(x,y)$.

The constant in the first inequality of (i) is the best possible and the second inequality of (i) holds with equality for $x = -y$. *The first inequality of (ii) holds with equality for* $x = -y$, $|x| = 1$, *and the second inequality of (ii) holds with equality for* $\measuredangle(x, 0, y) = 0$.

Theorem 7.2 (Law of Cosines)

Let $x, y, z \in \mathbb{R}^2 \setminus \{0\}$.
(i) For the quasihyperbolic triangle $\triangle_k(x, y, z)$

$$k(x, y)^2 = k(x, z)^2 + k(y, z)^2 - 2k(x, z)k(y, z)\cos \measuredangle_k(y, z, x).$$

(ii) For the quasihyperbolic trigon $\triangle_k^*(x, y, z)$

$$k(x, y)^2 = k(x, z)^2 + k(y, z)^2 - 2k(y, z)k(z, x)\cos \measuredangle_k(y, z, x) - 4\pi(\pi - \alpha),$$

where $\alpha = \measuredangle(x, 0, y)$.

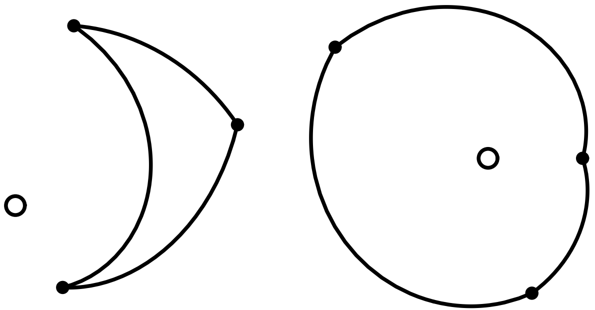


Figure: An example of a quasihyperbolic triangle (left) and a quasihyperbolic trigon (right).

Lemma 7.3

Let $x, y, z \in \mathbb{H}^2$ be distinct points. Then

$$k_{\mathbb{H}^2}(x,y)^2 \geq k_{\mathbb{H}^2}(x,z)^2 + k_{\mathbb{H}^2}(y,z)^2 - 2k_{\mathbb{H}^2}(y,z)k_{\mathbb{H}^2}(x,z)\cos\gamma,$$

where γ is the Euclidean angle between geodesics $J_k[z,x]$ and $J_k[z,y]$.

Theorem 7.4

Let $\triangle_k(x,y,z)$ be a quasihyperbolic triangle. Then the quasihyperbolic area of $\triangle_k(x,y,z)$ is

$$\sqrt{s(s-k(x,y))\,(s-k(y,z))\,(s-k(z,x))},$$

where $s = \big(k(x,y) + k(y,z) + k(z,x)\big)/2$.

Theorem 7.5

Let $m \in \{k, j\}$, $R > 1$ and $G = A(1/R, R)$. Then there exists a constant $c(R) \geq 1$ such that $c(R) \to 1$ as $R \to \infty$ and

$$m_G(x, y) \leq c(R) m_{\mathbb{R}^n \setminus \{0\}}(x, y)$$

for all $x, y \in A(1/\sqrt{R}, \sqrt{R})$.

If a metric space is geodesic, then all metric balls are connected. For nongeodesic metric spaces the connectivity of metric balls depends on the setting. For example, spherical balls are always connected while *j*-balls need not be connected [Kle-08, Remark 4.9 (2)]. We construct next such a domain that for any $m \in \mathbb{N}$ the *j*-ball has exactly m components.

Let us first consider the planar case $n = 2$. The generalization to $n > 3$ is straightforward. We denote by m the number of components of the *j*-ball we want to construct. We assume first $m \geq 9$ and denote the $(m-1)$th roots of unity by $\mathbf{e}_1, \ldots, \mathbf{e}_{m-1}$. Let $E_p = \{z \in \mathbb{R}^2 \colon |z| \leq 2,\ \measuredangle(z, \mathbf{e}_p, 2\mathbf{e}_p) \leq \pi/(m-1)\}$ for all $p = 1, \ldots, m-1$ and

$$G_m = \mathbb{R}^2 \setminus \bigcup_{p=1}^{m-1} E_p. \tag{2}$$

The set G_{12} is illustrated in Figure 3.

Lemma 7.6

For $m \geq 9$ and G_m as in (2) the j-ball $B_j(0, \log 4)$ has exactly m components.

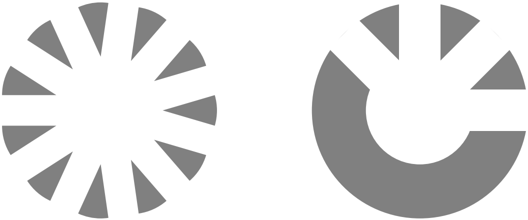


Figure: Examples of the domain G_{12} (left) and G_5. The gray area represents the complement of the domain.

Lemma 7.7

Let $G \subset \mathbb{R}^n$ be a domain, $x \in G$, and $r > 0$. Then for each connected component D of $B_j(x, r)$ we have

$$\mathrm{diam}_k(D) \leq c(r, n).$$

8 Recent Progress

Theorem 8.1

For a domain $G \subsetneq \mathbb{R}^n$ and $x \in G$ the j-metric ball $B_j(x, r)$ is close-to-convex, if $r \in (0, \log(1+\sqrt{3})]$.
For $y \in \mathbb{R}^n \setminus \{0\}$ the quasihyperbolic ball $B_k(y, r)$ is close-to-convex, if $r \in (0, \lambda]$, where λ has a numerical approximation $\lambda \approx 2.97169$.
Moreover, the constants $\log(1+\sqrt{3})$ and λ are sharp in the case $n = 2$.

Theorem 8.2

Let G be a proper subdomain of $\mathbb{R}^n$, $x \in G$ and denote $r_q(x) = \min\{1/\sqrt{1+|x|^2}, |x|/\sqrt{1+|x|^2}\}$.

1) *Let $G = \mathbb{R}^n \setminus \{0\}$ and $r \in (0, \pi/2)$. Then*

$$B_j(x, m_1) \subset B_k(x, r),$$

where

$$m_1 = \log\left(1 + 2\sin\frac{r}{2}\right).$$

2) *For $G = \mathbb{R}^n \setminus \{0\}$ and $r \in (0, r_q(x))$ we have*

$$B_j(x, m_2) \subset B_q(x, r) \quad and \quad B_k(x, m_2) \subset B_q(x, r),$$

where

$$m_2 = \log\left(1 + \frac{2r^2}{\sqrt{1-r^2}}\right).$$

Theorem...

3) **Let $G = \mathbb{H}^n$ and $r > 0$. Then**

$$B_j(x, m_3) \subset B_k(x, r),$$

where

$$m_3 = \log\left(1 + \sqrt{2}\sqrt{\cosh r - 1}\right).$$

4) **For $G = \mathbb{H}^n$, $r \in (0, r_q(x))$ and x with $x_1 = x_2 = \cdots = x_{n-1} = 0$ we have**

$$B_j(x, m_4) \subset B_q(x, r) \quad \text{and} \quad B_k(x, m_5) \subset B_q(x, r),$$

where

$$m_4 = \log\left(1 + \frac{2r}{1 - r^2}\right), \quad m_5 = \log\left(1 + \frac{2r^2}{\sqrt{1 - r^2}}\right).$$

Theorem 8.3

Let $G = \mathbb{B}^n$, $x \in G$ and $r > 0$. Then

$$B_j(x, m_1) \subset B_\rho(x, r),$$

$$B_j(x, m_2) \subset B_k(x, r),$$

$$B_j(x, m_3) \subset B_q(x, r), \quad r < (1 - |x|)/\sqrt{2(1 + |x|^2)}$$

where

$$m_1 = \log\left(1 + 2\sinh\frac{r}{2}\right),$$

$$m_2 = \log\left(1 + 2\sinh\frac{r}{4}\right),$$

$$m_3 = \log\left(1 + \frac{r}{\sqrt{1 - r^2}}\right).$$

Circles passing through x, y with centers in $P_{(x+y)/2}(x-y)$

Let $x, y \in \mathbb{R}^n$ **and** $0 < \alpha < \pi$**. Let** $P_{xy} = P_m(x-y)$ **with** $m = (x+y)/2$**. Let** $C = S(z,r) \subset \mathbb{R}^n$ **be a circle centered at** z **with radius** $r = |x-y|/(2\sin\alpha)$ **such that** $x, y \in C$**. Denote**

$$\mathcal{C}_{xy}^{\alpha} = \{C = S(z,r) : z \in P_{xy},\ d(z,[x,y]) = \frac{|x-y|}{2\tan\alpha},\ r = \frac{|x-y|}{2\sin\alpha}\},$$

where $x \neq y$**.**

α-envelope

We define the α*-envelope* **of the pair** (x, y) **to be**

$$E_{xy}^{\alpha} = [x,y] \cup \left(\bigcup \{\mathrm{comp}_{\alpha}(C) : C \in \mathcal{C}_{xy}^{t},\ \alpha < t < \pi\} \right)$$

if $0 < \alpha < \pi$**,** $E_{xy}^{0} = \mathbb{R}^n \setminus (ray(x, x-y) \cup ray(y, y-x))$ **and** $E_{xy}^{\pi} = [x,y]$**.**

The visual angle metric

Let $G \subsetneq \mathbb{R}^n$ and $x, y \in G$. We define a distance function v_G by

$$v_G(x,y) = \sup\left\{\alpha : E_{xy}^{\alpha} \cap \partial G \neq \emptyset\right\}.$$

The function $v_G \colon G \times G \to [0,\pi]$ is a similarity invariant pseudometric for every domain $G \subsetneq \mathbb{R}^n$. It is a metric unless ∂G is a proper subset of a line and will be called *the visual angle metric*.

Theorem 8.4

For $G \in \{\mathbb{B}^n, \mathbb{H}^n\}$ *and* $x, y \in G$, *let* $\rho^*_G(x,y) = \arctan(\mathrm{sh}\frac{\rho_G(x,y)}{2})$. *Then*

$$\rho^*_G(x,y) \le v_G(x,y) \le 2\rho^*_G(x,y).$$

The left-hand side of the inequality is sharp and the constant 2 in the right-hand side of the inequality is best possible.

Theorem 8.5

Let $f : \mathbb{B}^n \to \mathbb{B}^n$ *be a Möbius transformation. Then*

$$\sup_{\substack{f \in \mathcal{GM}(\mathbb{B}^n), \\ x \neq y \in \mathbb{B}^n}} \frac{v_{\mathbb{B}^n}(f(x), f(y))}{v_{\mathbb{B}^n}(x,y)} = 2.$$

Theorem 8.6

Let $f : \mathbb{H}^2 \to \mathbb{B}^2$ *be a Möbius transformation. Then for all* $x\,, y \in \mathbb{H}^2$

$$v_{\mathbb{H}^2}(x,y)/2 \le v_{\mathbb{B}^2}(f(x),(y)) \le 2v_{\mathbb{H}^2}(x,y),$$

and the constants $1/2$ *and* 2 *are both best possible.*

Theorem 8.7

Let $a\,, b\,, c\,, d \in \mathbb{R}$ *and* $ad - bc = 1$ *and* $c \neq 0$*. Let* $f : \mathbb{H}^2 \to \mathbb{H}^2$ *be a Möbius transformation with* $f(z) = \frac{az+b}{cz+d}$*. Then*

$$\sup_{x \neq y \in \mathbb{H}^2} \frac{v_{\mathbb{H}^2}(f(x),f(y))}{v_{\mathbb{H}^2}(x,y)} = 2.$$

Theorem 8.8

Let $f : \mathbb{B}^n \to \mathbb{B}^n$ *be an L-bilipschitz map with respect to the visual angle metric. Then* $H(x, f) \le 16L^2$ *for all* $x \in \mathbb{B}^n$.

Theorem 8.9

Let $f : \mathbb{B}^n \to \mathbb{B}^n$ *be an L-bilipschitz map with respect to the visual angle metric. Then* f *is a* $4L$*-bilipschitz map with respect to the hyperbolic metric.*

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