On the quasisymmetry of quasiconformal mappings and its applications

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Heinonen's open problem

Heinonen's result: Theorem 6.1 in [1]

Theorem A Suppose that

(1) both domains D and D' in $\mathbb{R}^n$ are bounded;

(2) $f : D \to D'$ is a K-quasiconformal mapping;

(3) D is φ-broad;

(4) $A \subset D$ is such that $f(A)$ is b-LLC_2 with respect to $\delta_{D'}$ in D'.

Then the restriction $f|_A : A \to f(A)$ is weakly H-quasisymmetric in the metrics δ_D and $\delta_{D'}$.

Reference

[1] J. HEINONEN, Quasiconformal mappings onto John domains, *Rev. Math. Iber.*, **5** (1989), 97–123.

Heinonen's open problem

A remark

This is a generalization of a result of Väisälä Theorem 2.20 in [2].

Reference

[2] J. VÄISÄLÄ, Quasiconformal maps of cylindrical domains, *Acta Math.*, **162** (1989), 201–225.

Heinonen's open problem

Heinonen's result: Lemma 8.3 in [3]

Theorem B Suppose that

(1) both domains D and D' in $\mathbb{R}^n$ are bounded;

(2) $f : D \to D'$ is a K-quasiconformal mapping;

(3) D is φ-broad;

(4) $A \subset D$ is arcwise connected and $f^{-1}|_{A'} : A' \to A$ is weakly H-quasisymmetric in the metrics $\delta_{D'}$ and δ_D.

Then $f(A) = A'$ is b-LLC_2 with respect to $\delta_{D'}$ in D'.

Reference

[3] J. Heinonen, Quasiconformal distortion on arcs, *J. Analyse Math.*, **63** (1994), 19–53.

Heinonen's open problem

The following result can be easily got from Theorems *A* and *B*.

A corollary

Theorem C Suppose that

(1) both domains D and D' in $\mathbb{R}^n$ are bounded;

(2) $f : D \to D'$ is a K-quasiconformal mapping;

(3) D is φ-broad.

Then the following statements are equivalent:

(1) $A \subset D$ is arcwise connected and $f^{-1}|_{A'} : A' \to A$ is weakly H-quasisymmetric in the metrics $\delta_{D'}$ and δ_D;

(2) $f(A) = A'$ is b-LLC_2 with respect to $\delta_{D'}$ in D'.

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Heinonen's open problem

Definition of quasisymmetric mappings

Quasisymmetric mappings: Let (X, d) and (X', d') be two metric spaces, and let $\eta : [0, \infty) \to [0, \infty)$ be a homeomorphism. An embedding $f : X \to X'$ is *η-quasisymmetric*, or briefly η-QS, in the metrics d and d' if $d(a, x) \le td(a, y)$ implies

$$d'(a', x') \le \eta(t) d'(a', y')$$

for all a, x, $y \in X$, where $a' = f(a)$, $x' = f(x)$ and $y' = f(y)$.

Heinonen's open problem

Definition of weakly quasisymmetric mappings

Weakly quasisymmetric mappings: If there is a constant $\nu \geq 1$ such that $d(a,x) \leq d(a,y)$ implies

$$d'(a',x') \leq \nu d'(a',y'),$$

then f is said to be *weakly ν-quasisymmetric*, or briefly weakly ν-QS, in the metrics d and d'.

A relation

Obviously, “quasisymmetry" implies “weak quasisymmetry".

Heinonen's open problem

In [1], Heinonen asked the following problem:

Heinonen's open problem

Whether is the word “weakly" in the conclusion “$f|_A : A \to f(A)$ being weakly H-QS in the metrics δ_D and $\delta_{D'}$" in Theorem A is redundant or not?

See the paragraph next to the statement of Theorem 6.5 in [1].

Main result

On Heinoinen's problem, our result is as follows.

The main result: Theorem $HPRW_1$

Theorem $HPRW_1$: Suppose that

(1) D and D' are bounded domains in $\mathbb{R}^n$, and D is φ-broad;

(2) $f : D \to D'$ is K-quasiconformal;

(3) $A \subset D$ is arcwise connected.

Then the following statements are equivalent:

(1) $f(A)$ is b-LLC_2 with respect to $\delta_{D'}$ in D';

(2) The restriction $f|_A : A \to f(A)$ is η-QS in the metrics δ_D and $\delta_{D'}$ with η depending only on the data

$$\mu = \mu\left(n, K, b, \varphi, \frac{\delta_D(A)}{d_D(x_0)}, \frac{\delta_{D'}(f(A))}{d_{D'}(f(x_0))}\right).$$

Main result

Some remarks on Theorem $HPRW_1$

(1) Theorem $HPRW_1$ shows that the answer to Heinonen's problem mentioned as above is affirmative when the set A is arcwise connected.

(2) Obviously, Theorem $HPRW_1$ is a generalization of Theorem C;

(3) Theorem $HPRW_1$ is a generalization of Theorem 6.6 in [1]. In fact, Theorem $HPRW_1$ shows that the conditions "A being BT" and "D' being BT" in [1, Theorem 6.6] are redundant.

Main result

Theorem 6.6 in [1]

Theorem D: Suppose that

(1) D and D' are bounded domains in $\mathbb{R}^n$;

(2) $f : D \to D'$ is K-quasiconformal;

(3) $A \subset D$ is arcwise connected, b_1-LLC$_2$ with respect to δ_D and b_2-BT in D;

(4) D' is φ-broad and b_3-BT.

Then $f : A \to f(A)$ is η-QS in the metrics δ_D and $\delta_{D'}$ with η depending only on the data

$$\mu = \mu\left(n, K, b_1, b_2, b_3, \varphi, \frac{\delta_D(A)}{d_D(x_0)}, \frac{\delta_{D'}(f(A))}{d_{D'}(f(x_0))}\right).$$

Main result

The sketch of the proof of Theorem $HPRW_1$

We first prove the following lemma.

Lemma A: Suppose that

(1) D and D' are bounded, and D is φ-broad;

(2) $f : D \to D'$ is K-quasiconformal;

(3) $A \subset D$ is arcwise connected such that $f|_A : A \to A'$ is weakly H-QS in the metrics δ_D and $\delta_{D'}$.

For all $z_1, z_2, z_3 \in A$, if $\delta_D(z_1, z_3) \leq c\delta_D(z_1, z_2)$, then

$$\delta_{D'}(z_1', z_3') \leq \mu_1 \delta_{D'}(z_1', z_2'),$$

where μ_1 is a constant.

Main result

The sketch of the proof of Theorem $HPRW_1$

By Lemma A, the main lemma follows.

Lemma B: Suppose that

(1) D and D' are bounded, and D is φ-broad;

(2) $f : D \to D'$ is K-quasiconformal;

(3) $A \subset D$ is arcwise connected such that $f|_A : A \to A'$ is weakly H-QS in the metrics δ_D and $\delta_{D'}$.

Then $\delta_D(a, x) \le \delta_D(a, y)$ implies

$$\frac{\delta_{D'}(a', x')}{\delta_{D'}(a', y')} \le \psi\Big(\frac{\delta_D(a, x)}{\delta_D(a, y)}\Big)$$

for all a, x, $y \in A$, where $\psi : (0, 1] \to (0, +\infty)$ is an increasing homeomorphism.

Main result

The sketch of the proof of Theorem $HPRW_1$

Based on Lemma B, we can construct a homeomorphism from $[0, \infty)$ to $[0, \infty)$ which is the required. The proof of Theorem $HPRW_1$ is finished.

Main result

The next result easily follows from Theorem $HPRW_1$.

The main result: Theorem $HPRW_2$

Theorem $HPRW_2$: Suppose that

(1) $f : D \to D'$ is a K-quasiconformal mapping onto a φ-broad D';

(2) A is an arcwise connected subset of D.

Then the following statements are equivalent:

(1) A is c-LLC_2 with respect to δ_D in D;

(2) $f|_A : A \to A'$ is weakly H-QS in the metrics δ_D and $\delta_{D'}$;

(3) $f|_A : A \to f(A)$ is η-QS in the metrics δ_D and $\delta_{D'}$.

Application I: The equivalence of John domains

Definition of uniform domains

Uniform domains: A domain D in $\mathbb{R}^n$ is said to be *c-uniform* if there exists a constant c with the property that each pair of points z_1, z_2 in D can be joined by a rectifiable arc γ in D satisfying

(1) $\min_{j=1,2} \ell(\gamma[z_j, z]) \leq c\, d_D(z)$ for all $z \in \gamma$, and

(2) $\ell(\gamma) \leq c\, |z_1 - z_2|$,

where $\ell(\gamma)$ denotes the arc length of γ, $\gamma[z_j, z]$ the part of γ between z_j and z, and $d_D(z)$ is the distance from z to the boundary ∂D of D.

Application I: The equivalence of John domains

Definition (1) of John domains

John domains: A domain D in $\mathbb{R}^n$ is said to be a *c-John domain* if it satisfies the condition (1) in the definition of uniform domains, but not necessarily (2).

Definition of Carrot property

A domain D in $\mathbb{R}^n$ is said to have the *c-carrot property* with center $x_0 \in \overline{D}$ if there exists a constant c with the property that for each point z_1 in A, z_1 and x_0 can be joined by a rectifiable arc γ in D satisfying

$$\ell(\gamma[z_1, z]) \leq c\, d_D(z)$$

for all $z \in \gamma$.

Application I: The equivalence of John domains

Definition (2) of John domains

A domain D in $\overline{\mathbb{R}}^n$ is said to be a *c-John domain with center x_0 in $\overline{D}$* if it has the c-carrot property with center $x_0 \in \overline{D}$.

Equivalence of the definitions for John domains

Definitions (1) and (2) for John domains stated as above are quantitatively equivalent for bounded domains.

Application I: The equivalence of John domains

In [1], Heinonen studied the quasiconformal mappings of the unit ball $\mathbb{B}$ in $\mathbb{R}^n$ onto John domains D in $\mathbb{R}^n$. The main aim of the paper of Heinonen [1] was to provide nine equivalent conditions for D to be John. In fact, by using Theorem A, Heinonen proved the following.

The equivalence of John domains: Henonen's result

Theorem E: Suppose that

(1) $f: \mathbb{B} \to D$ is a K-quasiconformal mapping, where D is bounded;

(2) $f: \overline{\mathbb{B}} \to \overline{D}$ is continuous.

Then the following statements are equivalent.

Application I: The equivalence of John domains

The equivalence of John domains: Henonen's result

(1) D is a b-John domain with center $f(0)$;

(2) D is φ-broad;

(3) $f : \mathbb{B} \to (D, \delta_D)$ is η-QS;

(4) For all $x \in \mathbb{B}$ and each $I(x) \in \Phi(x)$, $diam\big(f(I(x))\big) \leq b_1 d_D(f(x))$;

(5) For all $w \in \mathbb{S}$ and $x \in [0, w]$, $diam(f[x, w]) \leq b_2 d_{D'}(f(x))$;

(6) For all $w \in \mathbb{S}$ and $0 \leq \rho \leq r < 1$,

$$a_f(rw)(1 - r)^{1-\alpha} \leq b_3 a_f(\rho w)(1 - \rho)^{1-\alpha};$$

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Application I: The equivalence of John domains

The equivalence of John domains: Henonen's result

(7) $\frac{diam(f(I))}{diam(f(J))} \leq b_4\left(\frac{diam(I)}{diam(Q)}\right)^{\alpha}$ for all boundary caps $I \subset J \subset \mathbb{S}$;

(8) D is b_5-LLC_2;

(9) D is b_6-LLC_2 with respect to δ_D;

(10) $f : \mathbb{B} \to (D, \delta_D)$ is weakly H-QS.

The constants b, b_1, b_2, b_3, b_4, b_5, b_6, α, H and the functions φ, η depend only on each other and the data

$$v = v\left(c, n, k, \frac{diam(D)}{d_D(f(0))}\right).$$

Application I: The equivalence of John domains

Heinonen’s remarks on [1]

In [1], Heinonen specially pointed out that the requirement “D is quasiconformally equivalent to $\mathbb{B}$" in Theorem E cannot be replaced e.g. by “D is homeomorphic to $\mathbb{B}$" or “D is a John domain".

Application I: The equivalence of John domains

The equivalence of John domains: Theorem $HPRW_3$

Theorem $HPRW_3$: Suppose that

(1) D and D' are bounded domains in $\mathbb{R}^n$ and D is c-uniform;
(2) $f: D \to D'$ is a K-quasiconformal mapping and $f: \overline{D} \to \overline{D'}$ is continuous.

Then the following statements are equivalent.

(1) D' is a b-John domain with center $f(x_0)$;
(2) D' is φ-broad;
(3) $f: (D, \delta_D) \to (D', \delta_{D'})$ is η-QS;

Application I: The equivalence of John domains

The equivalence of John domains: Theorem $HPRW_3$

(4) For $x \in D$ and each $I(x) \in \Phi(x)$, $diam\big(f(I(x))\big) \leq b_1 d_{D'}(f(x))$;

(5) For $x, w \in D$, if $|x - w| \leq 8cd_D(x)$, then $\delta_{D'}(f(x), f(w)) \leq b_2 d_{D'}(f(x))$;

(6) For $x, w \in D$, if $|x - w| \leq 8cd_D(x)$ and $d_D(w) \leq 2cd_D(x)$, then $a_f(w) \leq b_3 a_f(x)\left(\frac{d(x)}{d(w)}\right)^{1-\alpha}$;

(7) $\frac{diam(f(P))}{diam(f(Q))} \leq b_4\left(\frac{diam(P)}{diam(Q)}\right)^{\alpha}$ for all continua $P \subset Q \subset \partial D$;

(8) D' is b_5-LLC_2;

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Application I: The equivalence of John domains

The equivalence of John domains: Theorem $HPRW_3$

(9) D' is b_6-LLC_2 with respect to $\delta_{D'}$;

(10) $f : (D, \delta_D) \to (D', \delta_{D'})$ is weakly H-QS.
The constants b, b_1, b_2, b_3, b_4, b_5, b_6, α and the functions φ, η depend only on each other and the data

$$v = v\left(c, n, k, \frac{diam(D)}{d_D(x_0)}, \frac{diam(D')}{d_{D'}(f(x_0))}\right).$$

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Application I: The equivalence of John domains

Remarks on Theorem $HPRW_3$

(1) The ball "$\mathbb{B}$" in the requirement "D being quasiconformally equivalent to $\mathbb{B}$" in Theorem E is replaced by the one "D being a uniform domain". We remark that every ball in $\mathbb{R}^n$ is uniform.

(2) Theorem $HPRW_3$ is a generalization of Theorem 1 in Pommerenke's paper [4].

Reference

[4] CH. POMMERENKE, One-sided smoothness conditions and conformal mapping, *J. London Math. Soc.*, **26** (1982), 77–88.

Application II: The Hölder continuity of quasiconformal mappings

Definition of Hölder continuity

A mapping f of a set A in a metric space (X_1, d_1) into another metric space (X_2, d_2) is said to be *Hölder continuous* with exponent $\alpha \in (0, 1]$ at a point x in A if there is a constant M such that

$$d_2(f(x), f(y)) \leq M d_1(x, y)^{\alpha}$$

for all y in A.

Further, if the above inequality holds for all points x and y in A with fixed M and α, then we say that f is *uniformly Hölder continuous* with exponent α in A or that f belongs to the Lipschitz class in A with exponent α. We use the notation $\mathrm{Lip}_{\alpha}(A)$ to denote this class.

Application II: The Hölder continuity of quasiconformal mappings

Näkki and Palka's result: [5, Theorem 10]

Theorem F: Suppose $f:\ \mathbb{B}\to D$ is a K-quasiconformal mapping. If D is bounded and c-uniform, then f belongs to $\mathrm{Lip}_\alpha(D)$ and f^{-1} belongs to $\mathrm{Lip}_\beta(\mathbb{B})$, where the constants $\alpha\le 1$ and $\beta\le 1$ depend only on the outer dilation of f, the uniformality coefficient c of D and the dimension n.

Reference

[5] R. Näkki and B. Palka, Lipschitz conditions and quasiconformal mappings, *Indiana. Univ. Math. J.*, **29** (1980), 41–66.

Application II: The Hölder continuity of quasiconformal mappings

Our results: Theorem $HPRW_4$

Theorem $HPRW_4$: Suppose that

(1) both D and D' are bounded domains in $\mathbb{R}^n$;

(2) D is a c-uniform domain and D' is a c_1-John domain;

(3) $f : D \to D'$ is a K-quasiconformal mapping.

Then f belongs to $\mathrm{Lip}_\alpha(D)$, where $\alpha = \alpha(c, c_1, K, n) \le 1$.

Application II: The Hölder continuity of quasiconformal mappings

Our results: Theorem $HPRW_5$

Theorem $HPRW_5$: Suppose that

(1) both D and D' are bounded domains in $\mathbb{R}^n$;

(2) D is a c-uniform domain and D' is a c_1-uniform domain;

(3) $f : D \to D'$ is a K-quasiconformal mapping.

Then f belongs to $\mathrm{Lip}_\alpha(D)$ and f^{-1} belongs to $\mathrm{Lip}_\alpha(D')$, where $\alpha = \alpha(c, c_1, K, n) \leq 1$.

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Application II: The Hölder continuity of quasiconformal mappings

Remarks

(1) Theorem $HPRW_4$ shows that for the assertion "$f \in \mathrm{Lip}_\alpha(D)$", the ball "$\mathbb{B}$" in the assumption of Theorem F can be replaced by "a John domain".

(2) Theorem $HPRW_5$ shows that the ball "$\mathbb{B}$" in the assumption of Theorem F can be replaced by "a uniform domain".

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THANK YOU