On Existence of Solutions for Differential Equations with Impulses

Xuxin Yang

Department of Mathematics ,
Hunan First Normal University

17.8.2012

Abstract

First we give an introduction to theory of impulsive differential equations.

Next we study the existence of solutions to the Duffing equation with impulses. By means of the Poincaré-Birkhoff fixed point theorem under given conditions, we obtain the sufficient condition of existence of infinitely many solutions. Our results generalize those of T.R. Ding. An example is presented to demonstrate applications of our main result.

This presentation is based on the paper:
Existence of Periodic Solutions for the Duffing Equation with Impulses by Xuxin Yang, Weibing Wang and Jianhua Shen , Mathematical Problems in Engineering Volume 2012, Article ID 903653, 13 pages doi:10.1155/2012/903653.

References

1. V. Lakshmikantham, D. Bainov, P. Simeonov, Theory of impulsive differential equations, World Scientific, Singapore, 1989.
2. W.Y. Ding, *Fixed point of twist mappings and periodic solutions of ordinary differential equations*, Acta Math. Sinica 25 (1981), 227–235.
3. T.R. Ding, *An infinite class of periodic solutions of periodically perturbed Duffing equations at resonance*, Proc. Amer. Math. Soc. 86(1) (1982), 47–54.
4. X. Ming, S.P. Wu and J.Q. Liu, *Periodic solutions of the 1-dimensional p-Laplace equation*, J. Math. Anal. Appl. 325 (2007), 879–888.
5. A. Ambrosetti, and P. Rabinowitz, *Dual variational methods in critical point theory*, J. Funct. Anal. 14 (1973),349–381.
6. W. Ding, J. Ran and M. Han, *Periodic boundary value problems for the first order impulsive functional differential equations*, Appl. Math. Comput. 165 (2005), 443–456.

References

7. W.Y. Ding, *A generalization of the Poincaré-Birkhoff theorem*, Proc. Amer. Math. Soc. 88 (1983), 341–346.
8. J. Franks, *Generalization of the Poincaré-Birkhoff theorem*, Ann. of Math. 128 (1988), 139–151.
9. H. Jacobowitz, *Periodic solutions of* $x'' + f(t,x) = 0$ *via the Poincaré-Birkhoff theorem*, J. Differential Equations, 20 (1976), 37–52.

Introduction

Theory of impulsive differential equations 1/2

- Many processes in the nature change their state abruptly. These processes which are subject to short-time perturbations whose duration is negligible in comparison with the duration of the process, and it is natural to assume that perturbations act instantaneously, that is, in the form of impulses.

Theory of impulsive differential equations 1/2

- Many processes in the nature change their state abruptly. These processes which are subject to short-time perturbations whose duration is negligible in comparison with the duration of the process, and it is natural to assume that perturbations act instantaneously, that is, in the form of impulses.
- For example, many biological phenomena involving thresholds, certain models in medicine and biology, optimal control models in economics and frequency modulated systems, do exhibit impulsive effects.

Theory of impulsive differential equations 1/2

- Many processes in the nature change their state abruptly. These processes which are subject to short-time perturbations whose duration is negligible in comparison with the duration of the process, and it is natural to assume that perturbations act instantaneously, that is, in the form of impulses.
- For example, many biological phenomena involving thresholds, certain models in medicine and biology, optimal control models in economics and frequency modulated systems, do exhibit impulsive effects.
- Thus impulsive differential equations are a natural description of evolution phenomena observed in several real world problems.

Theory of impulsive differential equations 2/2

- The theory of impulsive differential equations is much richer than the corresponding theory of differential equations without impulse effects.

Theory of impulsive differential equations 2/2

- The theory of impulsive differential equations is much richer than the corresponding theory of differential equations without impulse effects.
- For example, initial value problems may not, in general, have any solutions at all even when the corresponding differential equation is smooth enough, fundamental properties such as continuous dependence relative to initial data may be violated, and qualitative properties like stability may need a suitable new interpretation.

Theory of impulsive differential equations 2/2

- The theory of impulsive differential equations is much richer than the corresponding theory of differential equations without impulse effects.
- For example, initial value problems may not, in general, have any solutions at all even when the corresponding differential equation is smooth enough, fundamental properties such as continuous dependence relative to initial data may be violated, and qualitative properties like stability may need a suitable new interpretation.
- Moreover, a simple impulsive differential equation may exhibit several new phenomena such as rhythmical beating, merging of solutions, and noncontinuability of solutions.

Theory of impulsive differential equations 2/2

- The theory of impulsive differential equations is much richer than the corresponding theory of differential equations without impulse effects.
- For example, initial value problems may not, in general, have any solutions at all even when the corresponding differential equation is smooth enough, fundamental properties such as continuous dependence relative to initial data may be violated, and qualitative properties like stability may need a suitable new interpretation.
- Moreover, a simple impulsive differential equation may exhibit several new phenomena such as rhythmical beating, merging of solutions, and noncontinuability of solutions.
- Consequently, the theory of impulsive differential equations is interesting in itself and it is likely to assume greater importance in the near future as new applications to various fields arise.

Background and motivation

Impulsive differential equations serve as basic models to study the dynamics of processes that are subject to sudden changes in their states. Thus impulsive differential equations, that is, differential equations involving impulse effects, appear as a natural description of observed evolution phenomena of several real world problems [1].

Duffing equation

The Duffing equation is a non-linear second-order ODE which is used in physics to model oscillators. It is an example of a system exhibiting chaotic behavior. The equation is given by

$$x'' + \delta x' + \alpha x + \beta x^3 = \gamma \cos(\omega t), \tag{1}$$

where $x = x(t)$ is the displacement at time t, and numbers $\delta, \alpha, \beta, \gamma$ and ω are constants.

Duffing equation

The Duffing equation is a non-linear second-order ODE which is used in physics to model oscillators. It is an example of a system exhibiting chaotic behavior. The equation is given by

$$x'' + \delta x' + \alpha x + \beta x^3 = \gamma \cos(\omega t), \tag{1}$$

where $x = x(t)$ is the displacement at time t, and numbers $\delta, \alpha, \beta, \gamma$ and ω are constants.

We study the Duffing type model given by W.Y. Ding [2]

$$x'' + g(x) = f(x, t). \tag{2}$$

Clearly this equation contains the Duffing equation (1) without friction (i.e. $\delta = 0$) as a special case. In [3], T.R. Ding consider the equation (2) in the case where $g \in C(\mathbb{R}, \mathbb{R})$ is superlinear at infinity

$$\lim_{x \to \infty} \frac{g(x)}{x} = +\infty,$$

and the function $f \in C(\mathbb{R} \times \mathbb{R}, \mathbb{R})$ is T-periodic in t.

By using the Poincaré-Birkhoff fixed point theorem, Ding showed that there exists infinitely many periodic solutions to this equation.

By using the Poincaré-Birkhoff fixed point theorem, Ding showed that there exists infinitely many periodic solutions to this equation.

By using a similar technique, Ming, Wu and Liu [4] gave results concerning existence of infinitely many periodic solutions to the p-Laplace equation

$$\left(|x'|^{p-2}x'\right)' + g(x) = f(t,x), \qquad p > 1, \tag{3}$$

where $g \in C(\mathbb{R}, \mathbb{R})$ is p-sublinear in the sense

$$\lim_{|x| \to 0} \frac{g(x)}{|x|^{p-2}x} = +\infty, \tag{4}$$

and $f \in C(\mathbb{R} \times \mathbb{R}, \mathbb{R})$ is 1-periodic in t. For this problem, only partial results are known. For example, the conjecture is true if g is even and superlinear at infinity and $f \equiv 0$ [5]. For $f \neq 0$, more restrictions on g are required [6].

By developing ideas of [5,7], we extend this technique to the situation where x is allowed to have impulses at given points. We consider the periodic solutions to the Duffing equation with impulses

$$\begin{cases} x'' + g(x) = 0, & t \neq t_k, \\ x(t_k^+) = a_k x(t_k), \ \ k \in \mathbb{Z}, \\ x'(t_k^+) = b_k x'(t_k), \ \ k \in \mathbb{Z}, \end{cases} \tag{5}$$

By developing ideas of [5,7], we extend this technique to the situation where x is allowed to have impulses at given points. We consider the periodic solutions to the Duffing equation with impulses

$$\begin{cases} x'' + g(x) = 0, & t \neq t_k, \\ x(t_k^+) = a_k x(t_k), \ \ k \in \mathbb{Z}, \\ x'(t_k^+) = b_k x'(t_k), \ \ k \in \mathbb{Z}, \end{cases} \tag{5}$$

where $a_k > 0$, $a_k b_k = 1$, $a_{k+q} = a_k$, $t_{k+q} = t_k + T$, $0 < t_1 < t_2 < \cdots < t_q < T$, $g \in C(\mathbb{R}, \mathbb{R})$, and $\mathbb{Z}$ denotes the set of integers, $x(t_k^+)$ and $x'(t_k^+)$ are right limits of $x(t)$ and $x'(t)$ at $t = t_k$, respectively. Let $PC(\mathbb{R}) = \{x : \mathbb{R} \to \mathbb{R};\ x(t)$ is continuous everywhere except for t_k at which $x(t_k^+)$ and $x(t_k^-)$ exist and $x(t_k^-) = x(t_k),\ k \in \mathbb{Z}\}$; $PC^1(\mathbb{R}) = \{x \in PC(\mathbb{R}); x'(t)$ is continuous differentiable everywhere except for t_k at which $x'(t_k^+)$ and $x'(t_k^-)$ exist and $x'(t_k^-) = x'(t_k),\ k \in \mathbb{Z}\}$.

Existence of Periodic Solutions for the Duffing Equation with Impulses

Preliminary Lemmas

The Poincaré-Birkhoff fixed point theorem is a powerful tool in studying periodic solutions for the planar ODE's of the second order. There are several versions of this theorem, see [4, 6,8,9].
Let (r, θ) be a polar coordinates on $\mathbb{R}^2$ and $A = \{x \in \mathbb{R}^2 : r_1 \leq |x| \leq r_2\}$ an annulus on $\mathbb{R}^2$.
Definition 2.1 A mapping T is called as twist map if

$$T : (r, \theta) \to (h(r, \theta), \theta + l(r, \theta)),$$

where $h(r, \theta), l(r, \theta)$ are continuous on A, 2π-periodic in θ, and $l(r_1, \theta)l(r_2, \theta) < 0$.

Preliminary Lemmas

The Poincaré-Birkhoff fixed point theorem is a powerful tool in studying periodic solutions for the planar ODE's of the second order. There are several versions of this theorem, see [4, 6,8,9].
Let (r, θ) be a polar coordinates on $\mathbb{R}^2$ and
$A = \{x \in \mathbb{R}^2 : r_1 \leq |x| \leq r_2\}$ an annulus on $\mathbb{R}^2$.
Definition 2.1 A mapping T is called as twist map if

$$T : (r, \theta) \rightarrow (h(r, \theta), \theta + l(r, \theta)),$$

where $h(r, \theta), l(r, \theta)$ are continuous on A, 2π-periodic in θ, and $l(r_1, \theta)l(r_2, \theta) < 0$.

The proof of our main results is based on the following version of the Poincaré-Birkhoff theorem, due to W.Y. Ding [2] (see also [7]).

Preliminary Lemmas

Lemma 2.2 [2] Let $R > r > 0$. Suppose that $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ is an area preserving homeomorphism, such that T is a twist map in an annulus

$$A = \{x \in \mathbb{R}^2 : r \le |x| \le R\},$$

and $0 \in T(D)$, and $D = \{x : |x| < r\}$. Then T has at least two fixed points in A.

Preliminary Lemmas

Lemma 2.2 [2] Let $R > r > 0$. Suppose that $T : \mathbb{R}^2 \to \mathbb{R}^2$ is an area preserving homeomorphism, such that T is a twist map in an annulus

$$A = \{x \in \mathbb{R}^2 : r \le |x| \le R\},$$

and $0 \in T(D)$, and $D = \{x : |x| < r\}$. Then T has at least two fixed points in A.

For our convenience, we introduce the following condition:
(H_1) $g(x)x > 0$ for $x \neq 0$ and

$$\lim_{x \to 0} \frac{g(x)}{x} = +\infty. \tag{6}$$

Preliminary Lemmas

Lemma 2.3 Assume that $g \in C(\mathbb{R}, \mathbb{R})$ and (H_1) holds. Then for given $M > 0$, there exists a constant $\delta = \delta(M)$ such that for any $\varepsilon > 0$ there is a continuous function g_ε such that
(i)

$$|g_\varepsilon(x) - g(x)| \leq \varepsilon, \tag{7}$$

(ii) g_ε has a form $g_\varepsilon(x) = Mx$ near $x = 0$, i.e., there exists $\delta > 0$ such that

$$\frac{g_\varepsilon(x)}{x} \geq M \quad \text{for} \quad 0 < |x| \leq \delta, \tag{8}$$

and $g_\varepsilon \in C^1([-T, T] \backslash \{0\}, R)$, where $0 < T < +\infty$.
(iii) For a fixed $\lambda > 0$,

$$\lim_{x \to 0} \frac{G(\lambda x)}{G(x)} = \lambda^2, \tag{9}$$

where $G(x) = \int_0^x g_\varepsilon(s)ds$.

Preliminary Lemmas

Let $g \in C(\mathbb{R}, \mathbb{R})$ and satisfies the condition (H_1), and let g_ε be as Lemma 2.3. Consider the initial value problem for the following system:

$$\begin{cases} x' = -y, \\ y' = g_\varepsilon(x), \qquad t \neq t_k, \\ x(t_k^+) = a_k x(t_k), \\ y(t_k^+) = b_k y(t_k), \qquad k = 1, 2, \cdots, \\ x(0) = x_0, \qquad y(0) = y_0. \end{cases} \tag{10}$$

Preliminary Lemmas

Lemma 2.4 For any sufficiently small (x_0, y_0) the system (10) has a unique solution pair $x(t) = x(t, x_0, y_0)$, $y(t) = y(t, x_0, y_0)$. Moreover, the functions x, y continuously depend on x_0, y_0.

Main results

Our main result is the following theorem. This result generalizes Theorem 2.1 of T.R. Ding [3].

Theorem 3.1

Under the assumption (H_1), the Duffing equation (5) has an infinite sequence of solutions $\{x_n\}$ and $\|x_n\|_{PC^1[0,T]} \to 0$, *as* $n \to \infty$, where

$$\|x_n\|_{PC^1[0,T]} = \max\{\sup_{t\in[0,T]}\{|x_n(t)|\}, \sup_{t\in[0,T]}\{|x_n^{'}(t)|\}\}.$$

Main results

In order to prove the main result, we need the following lemmas. First we construct the twist map. Next we consider the system (10), our goal is to control the behavior of the norms $r(t) = \sqrt{x(t)^2 + y(t)^2}$ at the points $p(t) = \big(x(t), y(t)\big) \in \mathbb{R}^2$.

Lemma 3.2

Suppose that $p(t) = \big(x(t), y(t)\big)$, where $t \in [0, T]$ is a solution to the system (10). There exists a constant $R_0 > 0$ and functions $d_1, d_2, \varepsilon \colon (0, R_0] \to \mathbb{R}_+$ such that if $\varepsilon \leq \varepsilon(R)$, and $r(0) = R < R_0$, then

Main results

In order to prove the main result, we need the following lemmas. First we construct the twist map. Next we consider the system (10), our goal is to control the behavior of the norms $r(t) = \sqrt{x(t)^2 + y(t)^2}$ at the points $p(t) = \big(x(t), y(t)\big) \in \mathbb{R}^2$.

Lemma 3.2

Suppose that $p(t) = \big(x(t), y(t)\big)$, where $t \in [0, T]$ is a solution to the system (10). There exists a constant $R_0 > 0$ and functions $d_1, d_2, \varepsilon \colon (0, R_0] \to \mathbb{R}_+$ such that if $\varepsilon \leq \varepsilon(R)$, and $r(0) = R < R_0$, then

(1)

$$d_1(R) \leq r(t) \leq d_2(R),$$

Main results

In order to prove the main result, we need the following lemmas. First we construct the twist map. Next we consider the system (10), our goal is to control the behavior of the norms $r(t) = \sqrt{x(t)^2 + y(t)^2}$ at the points $p(t) = \big(x(t), y(t)\big) \in \mathbb{R}^2$.

Lemma 3.2

Suppose that $p(t) = \big(x(t), y(t)\big)$, where $t \in [0, T]$ is a solution to the system (10). There exists a constant $R_0 > 0$ and functions $d_1, d_2, \varepsilon \colon (0, R_0] \to \mathbb{R}_+$ such that if $\varepsilon \leq \varepsilon(R)$, and $r(0) = R < R_0$, then

(1)

$$d_1(R) \leq r(t) \leq d_2(R),$$

(2)

$$\lim_{R\to 0} d_1(R) = \lim_{R\to 0} d_2(R) = 0.$$

Proof (idea)

- Define the Liapunov function by

$$F(x,y) = \frac{1}{2}y^2 + \int_0^x g_\varepsilon(s)\,ds.$$

Proof (idea)

- Define the Liapunov function by

$$F(x,y) = \frac{1}{2}y^2 + \int_0^x g_\varepsilon(s)\,ds.$$

- From Lemma 2.3, we see that for a fixed $\lambda > 0$, there exist $c_1(\lambda) \geq 0, c_2(\lambda) \geq 0$ such that

$$c_2(\lambda)G(x) \leq G(\lambda x) \leq c_1(\lambda)G(x),$$

where $G(x) = \int_0^x g_\varepsilon(s)ds$, when $|x|$ is sufficiently small.

Proof (idea)

- Define the Liapunov function by

$$F(x,y) = \frac{1}{2}y^2 + \int_0^x g_\varepsilon(s)\,ds.$$

- From Lemma 2.3, we see that for a fixed $\lambda > 0$, there exist $c_1(\lambda) \geq 0, c_2(\lambda) \geq 0$ such that

$$c_2(\lambda)G(x) \leq G(\lambda x) \leq c_1(\lambda)G(x),$$

where $G(x) = \int_0^x g_\varepsilon(s)ds$, when $|x|$ is sufficiently small.
- By using an impulsive integer inequality, we obtain

$$F(p(0)) \prod_{0<t_k<t} \min\{b_k^2, c_2(a_k)\} \leq F(p(t_k)) \quad \leq \quad F(p(0)) \prod_{0<t_k<t} \max\{b_k^2, c_1(a_k)\}.$$

Proof (idea)

- Define the Liapunov function by
$$F(x,y) = \frac{1}{2}y^2 + \int_0^x g_\varepsilon(s)\,ds.$$
- From Lemma 2.3, we see that for a fixed $\lambda > 0$, there exist $c_1(\lambda) \geq 0, c_2(\lambda) \geq 0$ such that
$$c_2(\lambda)G(x) \leq G(\lambda x) \leq c_1(\lambda)G(x),$$
where $G(x) = \int_0^x g_\varepsilon(s)ds$, when $|x|$ is sufficiently small.
- By using an impulsive integer inequality, we obtain
$$F(p(0)) \prod_{0<t_k<t} \min\{b_k^2, c_2(a_k)\} \leq F(p(t_k)) \quad \leq \quad F(p(0)) \prod_{0<t_k<t} \max\{b_k^2, c_1(a_k)\}.$$
- Then here exist positive constants $d_1(R), d_2(R)$ such that
$$d_1(R) \leq r(t) \leq d_2(R).$$

Proof (idea)

- Define the Liapunov function by

$$F(x,y) = \frac{1}{2}y^2 + \int_0^x g_\varepsilon(s)\,ds.$$

- From Lemma 2.3, we see that for a fixed $\lambda > 0$, there exist $c_1(\lambda) \geq 0, c_2(\lambda) \geq 0$ such that

$$c_2(\lambda)G(x) \leq G(\lambda x) \leq c_1(\lambda)G(x),$$

where $G(x) = \int_0^x g_\varepsilon(s)ds$, when $|x|$ is sufficiently small.
- By using an impulsive integer inequality, we obtain

$$F(p(0)) \prod_{0<t_k<t} \min\{b_k^2, c_2(a_k)\} \leq F(p(t_k)) \quad \leq \quad F(p(0)) \prod_{0<t_k<t} \max\{b_k^2, c_1(a_k)\}.$$

- Then here exist positive constants $d_1(R), d_2(R)$ such that

$$d_1(R) \leq r(t) \leq d_2(R).$$

- Furthermore, we have

$$\lim_{R\to 0} d_1(R) = \lim_{R\to 0} d_2(R) = 0.$$

Main results

- Consider the following system:

$$\begin{cases} x' = -y, \\ y' = g(x), \qquad t \neq t_k, \\ x(t_k^+) = a_k x(t_k), \\ y(t_k^+) = b_k y(t_k), \qquad k = 1, 2, \ldots. \end{cases} \tag{11}$$

Main results

- Consider the following system:

$$\begin{cases} x' = -y, & \\ y' = g(x), & t \neq t_k, \\ x(t_k^+) = a_k x(t_k), & \\ y(t_k^+) = b_k y(t_k), & k = 1, 2, \ldots. \end{cases} \tag{11}$$

- Use the Poincaré map ϕ:

$$(x_0, y_0) \mapsto \big(x(T, x_0, y_0), y(T, x_0, y_0)\big) = (x_T, y_T), \tag{12}$$

where $(x(t, x_0, y_0), y(t, x_0, y_0)$ is the solution of (11) corresponding to the initial data (x_0, y_0).

Main results

Denote by $r(t)$, $\theta(t)$ the norm and the polar angle of $p(t) = \big(x(t), y(t)\big) \in \mathbb{R}^2$, $r(t) = \sqrt{x(t)^2 + y(t)^2}$, respectively. Then

Main results

Denote by $r(t)$, $\theta(t)$ the norm and the polar angle of $p(t) = \big(x(t), y(t)\big) \in \mathbb{R}^2$, $r(t) = \sqrt{x(t)^2 + y(t)^2}$, respectively. Then

$$\cos\theta(t) = \frac{x(t)}{\sqrt{x^2(t) + y^2(t)}}, \qquad \sin\theta(t) = \frac{y(t)}{\sqrt{x^2(t) + y^2(t)}}, \qquad t \neq t_k \tag{13}$$

and

$$\begin{cases} \cos\theta(t_k^+) = \frac{x(t_k^+)}{\sqrt{x(t_k^+)^2 + y(t_k^+)^2}} = \frac{a_k x(t_k)}{\sqrt{a_k^2 x(t_k)^2 + b_k^2 y(t_k)^2}}, \\ \sin\theta(t_k^+) = \frac{y(t_k^+)}{\sqrt{x(t_k^+)^2 + y(t_k^+)^2}} = \frac{b_k y(t_k)}{\sqrt{a_k^2 x(t_k)^2 + b_k^2 y(t_k)^2}}, \qquad k = 1, 2, \cdots . \end{cases} \tag{14}$$

Main results

Write $(r_0, \theta_0) = (r(0), \theta(0))$ and $(r_T, \theta_T) = (r(T), \theta(T))$. Then, the map ϕ can be expressed in the polar coordinates as

$$r_T = h(r_0, \theta_0), \qquad \theta_T = \theta_0 + l(r_0, \theta_0), \tag{15}$$

where h and l are continuous and 2π-periodic in θ_0. In order to apply the Poincaré-Birkhoff fixed point theorem, we need to estimate the difference $\theta_T - \theta_0$.

Main results

Lemma 3.3

For $R > 0$ there is a constant $K(R)$, and $r_0 = R$, then $\theta_T - \theta_0 = I(R, \theta_0) \leq K(R)$.

Main results

Lemma 3.4

There is $r < R$ such that

$$\theta_T - \theta_0 + 2m\pi = I(r, \theta_0) + 2m\pi > 0, \qquad \text{for all } \theta_0 \in [0, 2\pi].$$

So the map ϕ is the twist map in the annular region $A = \{(x, y) : r \leq \sqrt{x^2 + y^2} \leq R\}$.

Proof of Theorem 3.1 (idea)

Let ϕ be the map defined (12), or equivalently by (15). Define the mappings

$$\Phi_0 : (x_0, y_0) \to (x(t_1, x_0, y_0), y(t_1, x_0, y_0)) = (x_1, y_1),$$

$$\Phi_0^* : (x_1, y_1) \to (a_1 x(t_1, x_0, y_0), b_1 y(t_1, x_0, y_0)) = (x_1^*, y_1^*),$$

$$\Phi_i : (x_i^*, y_i^*) \to (x(t_{i+1}, x_0, y_0), y(t_{i+1}, x_0, y_0)) = (x_{i+1}, y_{i+1}),$$

$$\Phi_i^* : (x_{i+1}, y_{i+1}) \to (a_{i+1} x_{i+1}, b_{i+1} y_{i+1}) = (x_{i+1}^*, y_{i+1}^*), \quad i = 1, \ldots, q-1,$$

$$\Phi_q : (x_q^*, y_q^*) \to (x(T, x_0, y_0), y(T, x_0, y_0)).$$

Proof of Theorem 3.1 (idea)

When $a_k b_k = 1$, Φ_i $(0 \leq i \leq q)$, Φ_j^* $(0 \leq j \leq q-1)$ are area-preserving mappings. Since

$$\phi = \Phi_q \circ \Phi_{q-1}^* \circ \Phi_{q-1} \circ \cdots \circ \Phi_0^* \circ \Phi_0,$$

Proof of Theorem 3.1 (idea)

When $a_k b_k = 1$, Φ_i $(0 \leq i \leq q)$, Φ_j^* $(0 \leq j \leq q-1)$ are area-preserving mappings. Since

$$\phi = \Phi_q \circ \Phi_{q-1}^* \circ \Phi_{q-1} \circ \cdots \circ \Phi_0^* \circ \Phi_0,$$

ϕ is an area-preserving mapping. Obviously, $\phi(0,0) = (0,0) \in D = \{(x,y) : x^2 + y^2 < r^2\}$. Lemmas 3.3, 3.4 imply that ϕ is a twist map on the annulus $A = \{(x,y) : r^2 \leq x^2 + y^2 \leq R^2\}$ for sufficiently small ε. Now it follows from the result of Ding, Lemma 2.2 given in the introduction, that ϕ has at least two fixed points in A. Let $(x_{\varepsilon_i}(t), y_{\varepsilon_i}(t))$ be one of the corresponding periodic solutions of (11). By Lemma 3.2, we have

$$d_1(r) \leq \sqrt{x_{\varepsilon_i}^2(t) + y_{\varepsilon_i}^2(t)} \leq d_2(R). \tag{16}$$

Proof of Theorem 3.1 (idea)

By the Arzela-Ascoli theorem, a sequence of $\{x_{\varepsilon_i}(t), y_{\varepsilon_i}(t)\}$ converges to $(x(t), y(t))$ as $\varepsilon_i \to 0$. Then, $(x(t), y(t))$ satisfies (11), and $(x(t), y(t))$ is a periodic solution for the system (5) with $d_1(r) \leq \sqrt{x^2(t) + y^2(t)} \leq d_2(R)$. Since R is arbitrary, we obtain an infinite sequence of periodic solutions for system (5) with small amplitudes. □

Example

Example 4.1

Our main result expands the applied scope of the work of T.R. Ding [3] by allowing the equation to have impulses.

Example

Example 4.1

Our main result expands the applied scope of the work of T.R. Ding [3] by allowing the equation to have impulses.

Consider the equation

$$\begin{cases} x''(t) + \sqrt[3]{x(t)} = 0, & t \neq t_k, \\ x(t_k^+) = a_k x(t_k),\ x'(t_k^+) = b_k x'(t_k), & k \in \mathbb{Z}, \end{cases} \tag{17}$$

where $a_k > 0$, $a_k b_k = 1$, $a_{k+q} = a_k$, $t_{k+q} = t_k + T$, $0 < t_1 < t_2 < \cdots < t_q < T$.

Example

Example 4.1

Our main result expands the applied scope of the work of T.R. Ding [3] by allowing the equation to have impulses.

Consider the equation

$$\begin{cases} x''(t) + \sqrt[3]{x(t)} = 0, & t \neq t_k, \\ x(t_k^+) = a_k x(t_k),\ x'(t_k^+) = b_k x'(t_k), & k \in \mathbb{Z}, \end{cases} \tag{17}$$

where $a_k > 0$, $a_k b_k = 1$, $a_{k+q} = a_k$, $t_{k+q} = t_k + T$, $0 < t_1 < t_2 < \cdots < t_q < T$.

In fact, in (17), $g(x) = \sqrt[3]{x}$. Then, $xg(x) \geq 0$ and $\lim_{x \to 0} x^{-1} g(x) = +\infty$. The condition (H_1) is satisfied and by Theorem 3.1, equation (17) has an infinite sequence of periodic solutions.

One-dimensional Wave Equation

Consider the one-dimensional Wave Equation

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \qquad c^2 = \frac{T}{\rho} > 0, \\ \Delta u(x, t_k) = u(x, t_k^+) - u(x, t_k^-) = c_k u(x, t_k), \\ \Delta u_t(x, t_k) = u_t(x, t_k^+) - u_t(x, t_k^-) = d_k u_t(x, t_k) \end{cases} \tag{18}$$

which arises from the standard mathematical model for a vibrating string. Let $x = 0$ and $x = L$, yielding the boundary conditions $u(0, t) = 0$, $u(L, t) = 0$ for all t. Here c_k and d_k are constants, $0 = t_0 < t_1 < t_2 < \cdots < t_q < \cdots < +\infty$, $u(x, t_k^+)$ and $u_t(x, t_k^+)$ are right limits of $u(x, t)$ and $u_t(x, t)$ at $t = t_k$, respectively. Let $PC(\mathbb{R}) = \{u : \mathbb{R} \to \mathbb{R};\ u(x, t)$ is continuous everywhere except for t_k at which $u(x, t_k^+)$ and $u(x, t_k^-)$ exist and $u(x, t_k^-) = u(x, t_k),\ k \in \mathbb{Z}\}$; $PC^1(\mathbb{R}) = \{u \in PC(\mathbb{R}); u_t(x, t)$ is continuous differentiable everywhere except for t_k at which $u_t(x, t_k^+)$ and $u_t(x, t_k^-)$ exist and $u_t(x, t_k^-) = u_t(x, t_k),\ k \in \mathbb{Z}\}$.

Thank you for your attention!