Inequalities for the generalized trigonometric and hyperbolic functions

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Outline

Definitions

- Analytic point of view
- Geometric point of view

Inequalities

This talk is based on the joint work:

- R. Klén, M. Vuorinen, and X.-H. Zhang: *Inequalities for the generalized trigonometric and hyperbolic functions.* Manuscript.

1. Definitions

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- Analytic point of view
- Geometric point of view
- Integral operator
- Eigenfunctions for the $p-$Laplacian
- Approximation theory

Analytic point of view

- Classical sine function:

$$\arcsin(x) = \int_0^x \frac{1}{(1-t^2)^{1/2}} dt, \quad 0 \leq x \leq 1$$

$$\frac{\pi}{2} = \arcsin 1 = \int_0^1 \frac{1}{(1-t^2)^{1/2}} dt$$

Analytic point of view

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$$\frac{\pi}{2} = \arcsin 1 = \int_0^1 \frac{1}{(1-t^2)^{1/2}}dt$$

- Generalized sine function ($1 < p < \infty$):

$$\arcsin_p(x) = \int_0^x \frac{1}{(1-t^p)^{1/p}}dt, \quad 0 \le x \le 1$$

$$\frac{\pi_p}{2} = \arcsin_p(1) = \int_0^1 \frac{1}{(1-t^p)^{1/p}}dt$$

$$\sin_p = \text{inverse of } \arcsin_p \text{ on } [0, \pi_p/2]$$

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    - $$\cos_p(x) = (1 - \sin_p(x)^p)^{1/p}, \quad x \in [0, \pi_p/2]$$
    - $$|\sin_p(x)|^p + |\cos_p(x)|^p = 1, \quad x \in \mathbb{R}$$
    - $$\frac{d}{dx}\cos_p(x) = -\cos_p(x)^{2-p}\sin_p(x)^{p-1}$$

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π_p

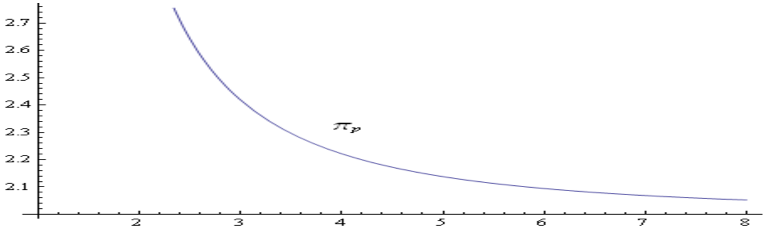

Figure: $\lim_{p\to 1} \pi_p = \infty$, $\lim_{p\to\infty} \pi_p = 2$

$\sin_p(x)$ and $\cos_p(x)$

- Generalized tangent function:

$$\tan_p(x) = \frac{\sin_p(x)}{\cos_p(x)}, \quad x \in \mathbb{R} \setminus \{k\pi_p + \pi_p/2 : \; k \in \mathbb{Z}\}$$

  -

$$\frac{d}{dx}\tan_p(x) = 1 + |\tan_p(x)|^p, \quad x \in (-\pi_p/2, \pi_p/2)$$

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Generalized hyperbolic functions

- $$\operatorname{arcsinh}_p(x) = \begin{cases} \int_0^x \frac{1}{(1+t^p)^{1/p}} dt, & x \in [0, \infty) \\ -\operatorname{arcsinh}_p(-x), & x < 0 \end{cases}$$

$$\cosh_p(x) = \frac{d}{dx} \sinh_p(x)$$

$$\tanh_p(x) = \frac{\sinh_p(x)}{\cosh_p(x)}$$

Generalized hyperbolic functions

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  $$\cosh_p(x) = \frac{d}{dx} \sinh_p(x)$$
  $$\tanh_p(x) = \frac{\sinh_p(x)}{\cosh_p(x)}$$

- $$\cosh_p(x)^p - |\sinh_p(x)|^p = 1$$
  $$\frac{d}{dx} \cosh_p(x) = \cosh_p(x)^{2-p} \sinh_p(x)^{p-1}$$
  $$\frac{d}{dx} \tanh_p(x) = 1 - |\tanh_p(x)|^p$$

Plane $\mathbb{R}^2$ with the l_2 metric

- Circle:

$$S_r = \{(x, y) \in \mathbb{R}^2 : \; x^2 + y^2 = r^2\}$$

- polar coordinates:

$$\begin{cases} x &= r\cos(\phi) \\ y &= r\sin(\phi) \end{cases}$$

- $$x^2 + y^2 = r^2 \Rightarrow \cos(\phi)^2 + \sin(\phi)^2 = 1$$

$$\phi = \arctan(y/x), \quad x, y > 0$$

Plane $\mathbb{R}^2$ with the l_p metric

p–circle:

$$S_r^p = \{(x, y) \in \mathbb{R}^2 : \; x^p + y^p = r^p\}$$

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- We expect the following identities:

$$\begin{cases} x &= r\cos_p(\phi) \\ y &= r\sin_p(\phi) \end{cases}$$

$$|\cos_p(\phi)|^p + |\sin_p(\phi)|^p = 1$$

- We expect the following identities:

$$\begin{cases} x = r\cos_p(\phi) \\ y = r\sin_p(\phi) \end{cases}$$

$$|\cos_p(\phi)|^p + |\sin_p(\phi)|^p = 1$$

- Then we have

$$\cos_p(\phi) = (1 - \sin_p(\phi)^p)^{1/p}, \quad x, y > 0$$

with additional natural conditions:

$$\frac{d}{d\phi}\sin_p(\phi) = \cos_p(\phi)$$

$$\phi = 0 \quad \text{when } (x, y) = (0, 1)$$

Solving the previous differential equation, we get

$$\arcsin_p(x) = \int_0^x \frac{1}{(1-t^p)^{1/p}} dt, \quad 0 \le x \le 1$$

The Mitrinović-Adamović inequality

- For $p \in (1, \infty)$, the function

$$f(x) = \frac{\log(\sin_p(x)/x)}{\log \cos_p(x)}$$

is strictly decreasing from $(0, \pi_p/2)$ onto $(0, 1/(1+p))$.

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- In particular, for all $p \in (1, \infty)$ and $x \in (0, \pi_p/2)$,

$$\cos_p(x)^\alpha < \frac{\sin_p(x)}{x} < 1 \tag{2.1}$$

with the best constant $\alpha = 1/(1+p)$.

The Mitrinović-Adamović inequality

The Lazarević inequality

- For $p \in (1, \infty)$, the function

$$f(x) = \frac{\log(\sinh_p(x)/x)}{\log \cosh_p(x)}$$

is strictly increasing from $(0, \infty)$ onto $(1/(1+p), 1)$.

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The Lazarević inequality

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$$f(x) = \frac{\log(\sinh_p(x)/x)}{\log \cosh_p(x)}$$

is strictly increasing from $(0, \infty)$ onto $(1/(1+p), 1)$.

- In particular, for all $p \in (1, \infty)$ and $x \in (0, \infty)$,

$$\cosh_p(x)^\alpha < \frac{\sinh_p(x)}{x} < \cosh_p(x)^\beta \tag{2.2}$$

with the best constants $\alpha = 1/(1+p)$ and $\beta = 1$.

The Lazarević inequality

Huygens-type inequalities (1)

Let $p > 1$. Then the following inequalities hold

$$(2.3) \quad (p+1)\frac{\sin_p(x)}{x} + \frac{1}{\cos_p(x)} > p+2 \quad \textit{for} \quad x \in (0, \pi_p/2),$$

and

$$(2.4) \quad (p+1)\frac{\sinh_p(x)}{x} + \frac{1}{\cosh_p(x)} > p+2 \quad \textit{for} \quad x > 0.$$

Proof: *AG inequality:* $ta + (1-t)b > a^t b^{1-t}$.
Put $t = (p+1)/(p+2)$, $a = \sin_p(x)/x$, $b = 1/\cos_p(x)$
Combine the left side of (2.1).

$$(p+1)\frac{\sin_p(x)}{x} + \frac{1}{\cos_p(x)} >$$

$$(p+2)\left(\frac{\sin_p(x)}{x}\right)^{(p+1)/(p+2)}\left(\frac{1}{\cos_p(x)}\right)^{1/(p+2)} > p+2.$$

□

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Huygens type inequalities (2)

For $p > 1$, the following inequalities hold

$$\frac{p\sin_p(x)}{x} + \frac{\tan_p(x)}{x} > 1 + p, \quad 0 < x < \frac{\pi_p}{2}, \tag{2.5}$$

and

$$\frac{p\sinh_p(x)}{x} + \frac{\tanh_p(x)}{x} > 1 + p, \quad x > 0. \tag{2.6}$$

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Wilker-type inequality

- For $p > 1$ and $x > 0$,

$$\left(\frac{\sinh_p(x)}{x}\right)^p + \frac{\tanh_p(x)}{x} > 2. \tag{2.7}$$

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Wilker-type inequality

- For $p > 1$ and $x > 0$,

$$\left(\frac{\sinh_p(x)}{x}\right)^p + \frac{\tanh_p(x)}{x} > 2. \tag{2.7}$$

- **Proof:** *Bernoulli inequality: for* $a > 1$ *and* $t > 0$, $(1 + t)^a > 1 + at$.
Setting $t = \sinh_p(x)/x - 1$ *and* $a = p$, *and then combining the inequality (2.6), we have*

$$\left(\frac{\sinh_p(x)}{x}\right)^p > 1 + p\left(\frac{\sinh_p(x)}{x} - 1\right) > 2 - \frac{\tanh_p(x)}{x}.$$

□

Cusa-Huygens-type inequalities (1)

For $x \in (0, \pi_p/2]$,

$$\frac{\sin_p(x)}{x} < \frac{\cos_p(x) + p}{1 + p} \leq \frac{\cos_p(x) + 2}{3}, \ if \ p \in (1, 2]$$

$$\frac{\sin_p(x)}{x} > \frac{p - 1 + \cos_p(x)}{p} \geq \frac{1 + \cos_p(x)}{2} \ if \ p \in [2, \infty).$$

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Cusa-Huygens-type inequalities (1)

Cusa-Huygens-type inequalities (1)

Cusa-Huygens-type inequalities (2)

For all $x > 0$,

$$\frac{\sinh_p(x)}{x} < \frac{\cosh_p(x) + p}{1 + p}, \quad \textit{if } p \in (1, 2],$$

$$\frac{\sinh_p(x)}{x} < \frac{\cosh_p(x) + 2}{3}, \quad \textit{if } p \in [2, \infty).$$

Cusa-Huygens-type inequalities (2)

Cusa-Huygens-type inequalities (2)

Cusa-Huygens-type inequalities (3)

For $p \in [2, \infty)$ and $x \in (0, \pi_p/2)$,

$$\frac{\sinh_p(x)}{x} < \frac{3}{2 + \cos_p(x)}.$$

Tool of proof: l'Hôpital Monotone Rule [AVV]

Let $-\infty < a < b < \infty$, and let $f, g : [a, b] \to \mathbb{R}$ be continuous functions that are differentiable on (a, b), with $f(a) = g(a) = 0$ or $f(b) = g(b) = 0$. Assume that $g'(x) \neq 0$ for each $x \in (a, b)$. If f'/g' is increasing (decreasing) on (a, b), then so is f/g.

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