Matematiikan ja tilastotieteen laitos
Funktionaalianalyysin perusteet
Harjoitus 1
12.9.2013

1. Fix $x, y > 0, p > 0$, and set $S_p(x,y) = (x^p + y^p)^{1/p}$. Show that $0 < p_1 < p_2$ implies $S_{p_1}(x,y) \geq S_{p_2}(x,y)$.

2. Let (X, d_1) and (Y, d_2) be metric spaces and $f : (X, d_1) \to (Y, d_2)$ uniformly continuous. Show that there exists an increasing function $\omega : [0, t_0) \to [0, t_1), \omega(0) = 0$, such that $d_2(f(x), f(x)) \leq \omega(d_1(x,y))$. The function ω is called the modulus of continuity of f .

3. Show that the function $\sigma : (0, \infty) \to (0, \infty), \sigma(x,y) = |\log(x/y)|$ defines a metric on $X = (0, \infty)$. Let $d(x,y) = |x - y|$ and $f : X \to X$ be $f(x) = 1/x$. Show that f is homeomorphism. (a) Is $f : (X, d) \to (X, d)$ uniformly continuous? (b) Is $f : (X, \sigma) \to (X, \sigma)$ uniformly continuous?

4. (a) Give an example of a metric space and its ball such that the diameter of a ball with radius $r > 0$ is strictly smaller than $2r$.
   (b) Give an example of a metric space and its open ball such that the closure of the open ball is not the corresponding closed ball.

5. Let $g(x,y)$ be the time a walker needs to go from the place x to another place y in Turku. Does this define a metric? Why/ why not?

6. Let W be a set and d_1, d_2 two metrics on it. Is it true that the linear combination $ad_1 + bd_2, a, b > 0$ is a metric, too? What about the product $d_1 d_2$?