**Matematiikan ja tilastotieteen laitos**
**Funktionaalianalyysin perusteet**
**Harjoitus 2**
**19.9.2013**

1. For each $n \in \mathbb{N}$ let $f_n : [0,1] \to \mathbb{R}$ be defined by $f_n(x) = x^n$. Find the norm of $f_n$ in the following cases:
   (a) in the normed space $C_{\mathbb{R}}([0,1])$;
   (b) in the normed space $L^1[0,1]$.

2. Let $X$ be a vector space with norm $||\cdot||_1$ and let $Y$ be a vector space with norm $||\cdot||_2$. Let $Z = X \times Y$ have the norm $||(x,y)|| = ||x||_1 + ||y||_2$. Let $\{(x_n, y_n)\}$ be a sequence in $Z$. (a) Show that $\{(x_n, y_n)\}$ converges to $(x,y)$ in $Z$ if and only if $\{x_n\}$ converges to $x$ in $X$ and $\{y_n\}$ converges in $y$ in $Y$. (b) Show that $\{(x_n, y_n)\}$ is Cauchy in $Z$ if and only if $\{x_n\}$ is Cauchy in $X$ and $\{y_n\}$ is Cauchy in $Y$.

3. Let $\mathcal{P}$ be the (infinite dimensional) vector space of polynomials defined on $[0,1]$. Since $\mathcal{P}$ is a linear subspace of $C_{\mathbb{F}}([0,1])$ it has a norm $||p||_1 = \sup\{|p(x)| : x \in [0,1]\}$, and since $\mathcal{P}$ is a linear subspace of $L^1[0,1]$ it has another norm $||p||_2 = \int_0^1 |p(x)|\, dx$. Show that $||p||_1$ and $||p||_2$ are not equivalent on $\mathcal{P}$.

4. If
$$S = \{\{x_n\} \in \ell^2 : \exists\ N \in \mathbb{N} \text{ such that } x_n = 0 \text{ for } n \geq N\},$$
so that $S$ is a linear subspace of $\ell^2$ consisting of sequences having only finitely many non-zero terms, show that $S$ is not closed.

5. Let $X$ be a normed linear space and, for any $x \in X$ and $r > 0$, let $T = \{y \in X : ||y - x|| \leq r\}$ and $S = \{y \in X : ||y - x|| < r\}$.
   (a) Show that $T$ is closed.
   (b) If $z \in T$ and $z_n = (r - 1/n)z$, for $n \in \mathbb{N}$, show that $\lim_{n\to\infty} z_n = z$ and hence show that $\overline{S} = T$.

6. An inversion in the sphere $S^{n-1}(a,r) \subset \mathbb{R}^n$ is defined by
$$f(x) = a + \frac{r^2(x-a)}{|x-a|^2}, f(a) = \infty, f(\infty) = a.$$
(a) Show that for $x, y \in \mathbb{R}^n \setminus \{a\}$
$$|f(x) - f(y)| = \frac{r^2|x-a|}{|x-a||y-a|}.$$
(b) Next show that the formula
$$m(x,y) = |f(x) - f(y)|$$
defines a metric.
   (c) Show also that for $a = e_n, r = 1$ we have
$$|f(x) - f(y)| = \frac{|x-y|}{\sqrt{1+|x|^2}\sqrt{1+|y|^2}}.$$

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