Matematiikan ja tilastotieteen laitos
Funktionaalianalyysin perusteet
Harjoitus 3
26.9.2013

1. (Viimeksi käsittelemättä jäänyt) An inversion in the sphere $S^{n-1}(a,r) \subset \mathbb{R}^n$ is defined by
$$f(x) = a + \frac{r^2(x-a)}{|x-a|^2}, f(a) = \infty, f(\infty) = a.$$
(a) Show that for $x, y \in \mathbb{R}^n \setminus \{a\}$
$$|f(x) - f(y)| = \frac{r^2|x-y|}{|x-a||y-a|}.$$
(b) Next show that the formula
$$m(x,y) = |f(x) - f(y)|$$
defines a metric.
(c) Show also that for $a = e_n, r = 1$ we have for all $x \in \mathbb{R}^{n-1}$
$$|f(x) - f(y)| = \frac{|x-y|}{\sqrt{1+|x|^2}\sqrt{1+|y|^2}}.$$

2. Use the Gram-Schmidt algorithm to find an orthonormal basis for $Sp\{1, x, x^2\}$ in $L^2[-1,1]$.

3. If $A = \{\{x_n\} \in \ell^2 : x_{2n} = 0 \text{ for all } n \in \mathbb{N}\}$, find $A^\perp$.

4. Let X be an inner product space and let $A \subset X$. Show that $A^\perp = \overline{A}^\perp$.

5. Let X and Y be linear subspaces of a Hilbert space H. Recall that $X + Y = \{x + y : x \in X, y \in Y\}$. Prove that $(X+Y)^\perp = X^\perp \cap Y^\perp$.

6. Recall that analytic functions are open maps. We say that a map $f : G \to G', G, G' \subset \mathbb{C}$, is closed if it maps all sets F closed in G to sets closed in G'. Show that the function $exp(z)$ is not a closed map from the left half plane $\{z : Rez < 0\}$ onto $\mathbb{B}^2 \setminus \{0\}$, where $\mathbb{B}^2$ is the unit disk $\{z \in \mathbb{C} : |z| < 1\}$.

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