**Matematiikan ja tilastotieteen laitos**
**Funktionaalianalyysin perusteet**
**Harjoitus 6**
**17.10.2013**

1. Let $X$ be a normed linear space and let $P, Q \in B(X)$. Show that the linear transformation $T : B(X) \to B(X)$ defined by $T(R) = PRQ, R \in B(X)$, is bounded.

2. Let $T : \ell^2 \to \ell^2$ be the bounded linear operator defined by
$$T(x_1, x_2, x_3, x_4, \dots) = (0, 4x_1, x_2, 4x_3, x_4, \dots).$$
(a) Find $T^2$. (b) Hence find $||T^2||$ and compare this with $||T||^2$.

3. Let $c = \{c_n\} \in \ell^\infty$ and let $T_c \in B(\ell^2)$ be defined by $T_c(\{x_n\}) = \{c_n\, x_n\}$.
   (a) If $\inf\{|c_n| : n \in \mathbb{N}\} > 0$ and $d_n = 1/c_n$ show that $d = \{d_n\} \in \ell^\infty$ and that $T_c T_d = T_d T_c = I$.
   (b) If $\lambda \notin \overline{\{c_n : n \in \mathbb{N}\}}$ show that $T_c - \lambda I$ is invertible.

4. Let $c = \{c_n\} \in \ell^\infty$ and let $T_c \in B(\ell^\infty)$ be defined by $T_c(\{x_n\}) = \{c_n\, x_n\}$. If $c_n = 1/n$ show that $T_c$ is not invertible.

5. Let $X$ be a Banach space and suppose that $\{T_n\}$ is a sequence of invertible operators in $B(X)$ which converges to $T \in B(X)$. Suppose also that $||T_n^{-1}|| < 1$ for all $n \in \mathbb{N}$. Show that $T$ is invertible.

6. Let $U = \{u = \{u_n\} \in \ell^2 : u_n \neq 0 \text{ for only finitely many } n\}$. For $n \in \mathbb{N}$, define $T_n \in B(U, \mathbb{F})$ by $T_n(u) = n\, u_n$. Show that the set $\{||T_n|| : n \in \mathbb{N}\}$ is not bounded. Deduce that the hypothesis in Theorem 4.52 that $U$ be complete is necessary.