University of Turku / Department of Mathematics and Statistics
SCIENTIFIC COMPUTING
Exercise 03 / Solutions

1. Suppose that the coefficients of a quadratic polynomial are known (for instance, generate 10 polynomials with random coefficients). Find the roots with the command "roots" and write the coefficients and the the real and imaginary parts of the roots in a file. Also plot the graph of the function.

Solution:

```
% FILE d031.m begins.
close all
fid=fopen('h031.dat','w');
xx=0:0.01:3;
for m=1:10
  coef= 6*0.001*fix(1000*rand(1,3));
  rt=roots(coef);
  fprintf(fid,'\nKertoimet: ');
  fprintf(fid,' %12.6e',coef');
%TAPA1:
  fprintf(fid,'\nJuuret:');
  fprintf(fid,' %s', mat2str(rt,6));
%TAPA2:
%  fprintf(fid,'\nJuuret:\n');
%  fprintf(fid,'   Re(z)      Im(z) \n');
%  p=size(rt);
%  for pp=1:p
%    fprintf(fid,' %12.6e   %12.6e\n', real(rt(pp)), imag(rt(pp)));
%  end
  yy=polyval(coef,xx);
  plot(xx,yy,'LineWidth',2);
  hold on
end
grid
fclose(fid);
% FILE h031.m ends.
```

Output:

```
Kertoimet:  5.700000e+00 1.386000e+00 3.636000e+00
Juuret: [-0.121579+i*0.789375;-0.121579-i*0.789375]
Kertoimet:  2.910000e+00 5.346000e+00 4.572000e+00
Juuret: [-0.918557+i*0.85287;-0.918557-i*0.85287]
Kertoimet:  2.736000e+00 1.080000e-01 4.926000e+00
Juuret: [-0.0197368+i*1.34166;-0.0197368-i*1.34166]
Kertoimet:  2.664000e+00 3.690000e+00 4.746000e+00
Juuret: [-0.692568+i*1.141;-0.692568-i*1.141]
Kertoimet:  5.526000e+00 4.428000e+00 1.056000e+00
Juuret: [-0.400651+i*0.174857;-0.400651-i*0.174857]
Kertoimet:  2.430000e+00 5.610000e+00 5.496000e+00
Juuret: [-1.15432+i*0.963987;-1.15432-i*0.963987]
Kertoimet:  2.460000e+00 5.358000e+00 3.420000e-01
Juuret: [-2.11223;-0.0658188]
Kertoimet:  2.112000e+00 4.878000e+00 5.400000e-02
Juuret: [-2.29854;-0.0111237]
Kertoimet:  8.280000e-01 1.212000e+00 1.188000e+00
Juuret: [-0.731884+i*0.948224;-0.731884-i*0.948224]
Kertoimet:  3.618000e+00 1.632000e+00 1.188000e+00
Juuret: [-0.225539+i*0.526774;-0.225539-i*0.526774]
```

2. (a) Prove that $\int_a^b \int_c^d xy\,dx\,dy = (d^2 - c^2)(b^2 - a^2)/4$ for $b > a, d > c$.

(b) Use the MATLAB function doubleint0.m on the www-page to compute this integral when $(a, b, c, d) = (0, 3, 0, 2)$ and tabulate the difference exact value minus numerical value when the number $m[n]$ of subdivisions in the $x[y]$ direction has the value $m = 10 : 20 : 90, n = 10 : 20 : 90$. You may do this as follows

```
exact=  (d^2- c^2)*(b^2 -a^2)/4;
for m=10:20:90
for n=10:20:90
    numer=doubleint0(a,b,c,d,m,n);
    fprintf('  %12.3e', numer-exact);
end
fprintf('\n')
end
```

Solution:

```
% FILE doubleint0.m begins.
% Compute \int_{x_1}{x_2}  \int_{y_1}^{y_2} myf(x,y) dx dy
% This is based on the Riemann sum with m (n) steps in x (y) direction
function val=doubleint0(x1,x2,y1,y2,m,n)
a=x1; b=x2;
```

```
s=0.0;
for ii=0:(m-1)
      x=a+ii*(b-a)/m;
for jj=0:(n-1)
      dy=(y2-y1)/n;
      y=y1+jj*dy;
      s=s+myf(x,y)*dy*(b-a)/m;
end
end
val=s;
% fprintf('%12.5f\n', s)

function z= myf(x,y)
      z= x.*y;
% FILE doubleint0.m ends.
```

```
% FILE d032.m begins.
a=0; b=3; c=0;d=2;
exact=  (d^2- c^2)*(b^2 -a^2)/4;
for m=10:20:90
  for n=10:20:90
    numer=doubleint0(a,b,c,d,m,n);
    fprintf('  %12.3e', numer-exact);
  end
  fprintf('\n')
end
% REMARK: The built-in MATLAB function gives
% accuracy of the order 10E-16
% r2=dblquad('x.*y',a,b,c,d);
% fprintf('%12.3e\n', exact-r2)
% FILE d032.m ends.
```

Output:

-1.710e+00	-1.170e+00	-1.062e+00	-1.016e+00	-9.900e-01
-1.170e+00	-5.900e-01	-4.740e-01	-4.243e-01	-3.967e-01
-1.062e+00	-4.740e-01	-3.564e-01	-3.060e-01	-2.780e-01
-1.016e+00	-4.243e-01	-3.060e-01	-2.553e-01	-2.271e-01
-9.900e-01	-3.967e-01	-2.780e-01	-2.271e-01	-1.989e-01

3. The eigenvalues (=characteristic roots) of an $n \times n$ complex matrix $(a_{i,j})$ lie in the closed region of the z-plane consisting of all the disks

$$\{z \in \mathbb{C} : |a_{i,i} - z| \leq \sum_{j=1, j\neq i}^{n} |a_{i,j}|\}, j = 1, ..., n .$$

These are so called Gerschgorin disks. Check the validity of this statement as follows:

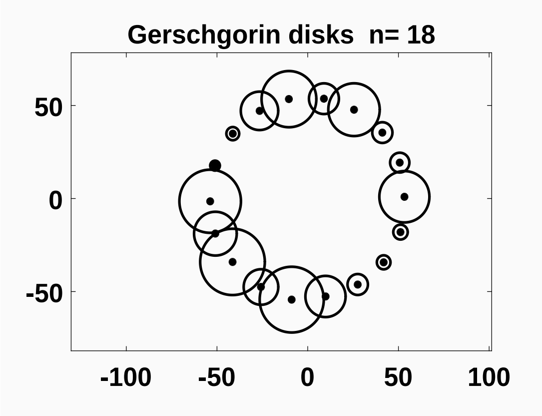
(a) For each n=3:3:18 generate a random complex $n \times n$ matrix and compute its eigenvalues.

(b) For each case plot the Gerschgorin disks and check visually that the statement holds.

Solution:

```
% FILE d033.m begins.
path(path,'../util')
close all
theta=0:0.05:6.3;
x=cos(theta);
y=sin(theta);

for n=3:3:18

%a=0.2*rand(n,n)-i*rand(n,n)/4;
a=((2/n)+rand(n,1)*(n-1)/n)*exp(i*2*pi*rand(1,n));
a= a+3*n*diag(exp(i*2*pi* ((1:n)+rand(1,n)/(1*n))/n));
myeig=eig(a);
figure
h=axes;
set(h,'FontWeight','bold','FontSize',20)
for j=1:n
myrad=sum(abs(a(j,:)))-1*abs(a(j,j))
% xx, yy are the x- and y-coordinates of the
% circle centered at a(j,j) with radius myrad
xx=real(a(j,j))+myrad*x;
yy=imag(a(j,j))+myrad*y;
pic=plot(xx,yy,'b-',...
real(myeig(j)), imag(myeig(j)),'r*');
set(pic,'LineWidth',2)
txt=[' n= ' num2str(n)];
title(['Gerschgorin disks ' txt],...
'FontWeight','bold','FontSize',20)
hold on
axis equal

end
widemarg(gcf)

if (n==6)
```

```
print -dps d033.ps
end
end

% FILE d033.m ends.
```



4. The fixed point method for numerical solution of $g(x) = x$ when $g : \mathbb{R} \to \mathbb{R}$ is based on the fixed point iteration $x_{n+1} = g(x_n)$. This converges for all $x_0 \in [a, b]$ if there exists $c \in (0, 1)$ such that $|g'(x)| < c$ for all $x_0 \in [a, b]$. We want to use this method to find the root of $f(x) = 1 - x - \sin x = 0$. Therefore we introduce an auxiliary function

$$g(x) = x + (1 - x - \sin x)/\lambda$$

and define the sequence $x_{n+1} = g(x_n)$. Show that $f(x) = 0$ is equivalent to $g(x) = x$ and if $\lim x_n = w$ then w satisfies the equation $1 - w - \sin w = 0$. (a) Plot the function $1 - x - \sin x$ and find a guess w for the root. Also choose a guess for the interval $[a, b] \ni w$ using w. (b) Differentiate g and find $\lambda \in (0, 3)$ such that $|g'(x)| < c$ for some $c \in (0, 1)$ and all $x \in [a, b]$. In this step we may need to replace $[a, b]$ by a smaller interval containing w. (c) Finally use the method with this fixed λ and $[a, b]$ and choose $x_0 \in [a, b]$ to start the iteration for g and find the root.

Solution:

```
% FILE d034.m begins.
function d034
close all
% Observe first that if x_n converges to z0 then
%  z0=z0+(1-z0-sin(z0))/lam implying that z0
```

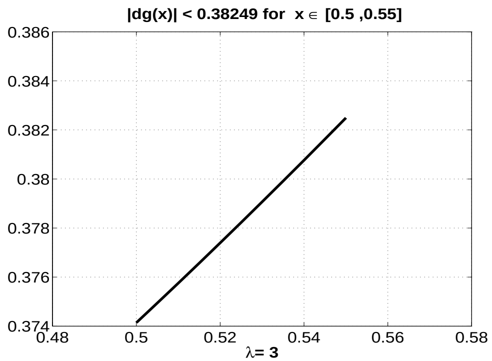


```
% satisfies the original equation f(x) = 0 for all lam non-zero
myf=inline('1-x-sin(x)');
% We plot f(x) to find an interval [a, b] for the
% iteration
x=0:0.05:1;   % Note: myf(0) >0 >myf(1)
y=myf(x);
a=max(x(y>0)); b=a+x(2)-x(1);
% We choose a nd b this way
myg=inline('x+ (1-x-sin(x))/lam ','x', 'lam');
mydg=inline('1-(1+cos(x))/lam ','x', 'lam');
% Can we find lam \neq 0 and c \in (0,1)
% such that |g'(x)| <c for all x in [a,b]?

mydata=[];
figure
axes('FontSize',[18],'FontWeight','bold');

for j=1:10          % In this loop we find the lambda that gives
                    % the smallest c
  lam=1+0.2*j;
  xx=a:((b-a)/10):b;
  yy=abs(mydg(xx,lam));
  myy=max(yy);
  plot(xx,yy,'LineWidth',3)
  ax=axis; ax(1)=ax(1)-0.02;
  axis(ax)
  grid on
  txt=['  |dg(x)| < '  num2str(max(myy)) ' for  x \in ['  ];
  txt=[txt  num2str(a) ' ,' num2str(b) ']'];
  title(txt,  'FontSize',18,  'FontWeight','bold')
  xlabel(['\lambda= ' num2str(lam) ],  'FontSize',18,'FontWeight','bold')
  pause(2)
  mydata=[mydata;  lam  myy ];
end
[myc,j]=min(mydata(:,2));    % j is the index of the smallest c
lam=  mydata(j,1);           % We choose lam(j)
fprintf('lambda= %8.4f\n', lam);
x=(a+b)/2;      % Start fixed point iteration at x
                % This loop is for fixed point iteration:
fprintf('  n       x_n           x_n- F(x_n) \n')
for j=1:10
x=myg(x,lam);
fprintf('%2d.  %16.12f   %12.4e\n',j,x, x-myg(x,lam));
```

```
end
% FILE d034.m ends.
```

Output:

```
lambda=   1.8000
  n       x_n             x_n- F(x_n)
 1.    0.510437219626    -5.5778e-04
 2.    0.510994996924     2.2433e-05
 3.    0.510972563529    -9.0062e-07
 4.    0.510973464152     3.6160e-08
 5.    0.510973427993    -1.4518e-09
 6.    0.510973429445     5.8288e-11
 7.    0.510973429386    -2.3402e-12
 8.    0.510973429389     9.3925e-14
 9.    0.510973429389    -3.7748e-15
10.    0.510973429389     1.1102e-16
```



5. A $m \times n$ matrix $A = (a_{ij})$ is called upper triangular if $a_{ij} = 0$ whenever $i > j$. Generate upper triangular 7×7 matrices and study experimentally whether (a) the product of two such matrices is again upper triangular, (b) an upper triangular matrix has an upper triangular matrix as the inverse (c) whether the determinant is always nonzero. Write a program to solve the upper triangular linear $n \times n$ system of equations.

Solution:

```
% FILE d035.m begins.
function d035
clear
```



```
m=4;
a1=uppertri(m,m);
b1=uppertri(m,m);
c1=a1*b1;
disp('Inverse of a triangular matrix is: ')
inv(a1)
rhs=rand(1,m);
x=trisolve(a1,rhs)
fprintf('Residual =|ax-b| = %12.4e\n',norm(a1*(x')-rhs'))

function a=uppertri(m,n)
a=rand(m,n);
for i=1:m
  for j=1:n
      if (i>j)
      a(i,j)=0.0;
      end
  end
end

function x=trisolve(a,b)
[d1,d2]=size(a);
if (d1 ~=d2)
  error('Must be square matrix for trisolve')
end
x=zeros(size(b));
x(d1)=b(d1)/a(d1,d1);
for j=(d1-1):(-1):1
  x(j) =(b(j) -sum( a(j,(j+1):d2 ).*x((j+1):d2)  ) )/a(j,j);
end
% FILE d035.m ends.
```

Output:

```
Inverse of a triangular matrix is:

ans =

    2.1320   -1.7969    2.2087   -2.6974
         0    1.1701   -1.6386    0.3366
         0         0    1.8719   -0.7843
         0         0         0    1.3670
```

```
x =

   -0.8043   -0.3010    0.0531    1.1842

Residual =|ax-b| =   1.5701e-16
```

6. Suppose that the random points a, b, c, d are located on the unit circle in such a way that b and c are on the smaller arc between a and d. Show by MATLAB tests that the point of intersection of the segments $[a, b]$ and $[c, d]$ (or of the lines containing these segments) is $(ab(c+d)-cd(a+b))/(ab-cd)$. N.B. that here the points are given as complex numbers.

Solution:

```
%FILE: d037.m begin

phi=0:0.01*pi:2*pi;
x=cos(phi); y=sin(phi);
clf
for pp=1:5
figure
 plot(x,y,'k-','LineWidth',2)
axis('equal')
hold on
widemarg(gcf)
a=exp(i*(1-0.1*rand)*pi);
d=exp(i*0.1*pi*rand);
%a=-1; d=1;
b=exp(i*0.75*pi*(1-pp/10));
c=exp(i*0.35*pi*(1+pp/8));
w=(a*b*(c+d)-c*d*(a+b))/(a*b-c*d);
wbar=(a*c*(b+d)-b*d*(a+c))/(a*c-b*d);
plot(real([a,b,w]), imag([a,b,w]),'b-',real(w),imag(w),'b*','LineWidth',2,...
'MarkerSize',15)
plot(real([d,b,w]), imag([d,b,w]),'b-')
plot(real([c,d,w]), imag([c,d,w]),'r-',real(w),imag(w),'ro','LineWidth',2,...
'MarkerSize',15)
plot(real([a,c,w]), imag([a,c,w]),'r-','LineWidth',2)
%plot(real(wbar),imag(wbar),'k-')

%real(w)-real(wbar) = if a=-1, d=1;
%tst=(w-wbar)*conj(a-d)+conj(w-wbar)*(a-d);
%fprintf('%12.4e\n',tst)
```



```
end
%FILE: d037.m end
```

Output: