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QUASICONFORMAL MAPPINGS AND INEQUALITIES INVOLVING SPECIAL FUNCTIONS

by

Barkat Ali Bhayo

TURUN YLIOPISTO
UNIVERSITY OF TURKU
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Supervisors

Professor Matti Vuorinen
Department of Mathematics
University of Turku
Turku, Finland

Reviewers

Professor Peter Hästö
Department of Mathematical Sciences
University of Oulu
Oulu, Finland

Dr. Istvan Prause
Department of Mathematics and Statistics
University of Helsinki
Helsinki, Finland

Opponent

Dr. Antti Rasila
Department of Mathematics and Systems Analysis
Aalto University School of Science and Technology
Espoo, Finland

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Abstract

This PhD thesis in Mathematics belongs to the field of Geometric Function Theory. The thesis consists of four original papers. The topic studied deals with quasiconformal mappings and their distortion theory in Euclidean n-dimensional spaces. This theory has its roots in the pioneering papers of F. W. Gehring and J. Väisälä published in the early 1960's and it has been studied by many mathematicians thereafter.

In the first paper we refine the known bounds for the so-called Mori constant and also estimate the distortion in the hyperbolic metric.

The second paper deals with radial functions which are simple examples of quasiconformal mappings. These radial functions lead us to the study of the so-called p-angular distance which has been studied recently e.g. by L. Maligranda and S. Dragomir.

In the third paper we study a class of functions of a real variable studied by P. Lindqvist in an influential paper. This leads one to study parametrized analogues of classical trigonometric and hyperbolic functions which for the parameter value $p = 2$ coincide with the classical functions. Gaussian hypergeometric functions have an important role in the study of these special functions. Several new inequalities and identities involving p-analogues of these functions are also given.

In the fourth paper we study the generalized complete elliptic integrals, modular functions and some related functions. We find the upper and lower bounds of these functions, and those bounds are given in a simple form. This theory has a long history which goes back two centuries and includes names such as A. M. Legendre, C. Jacobi, C. F. Gauss. Modular functions also occur in the study of quasiconformal mappings.

Conformal invariants, such as the modulus of a curve family, are often applied in quasiconformal mapping theory. The invariants can be sometimes expressed in terms of special conformal mappings. This fact explains why special functions often occur in this theory.

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List of original publications

This thesis is based on the following four papers/manuscripts:

I. B. A. Bhayo and M. Vuorinen: *On Mori's theorem for quasiconformal maps in the n-space.*- Trans. Amer. Math. Soc. (to appear) 16pp. arXiv:0906.2853[math.CA].

II. B. A. Bhayo, V. Božin, D. Kalaj, and M. Vuorinen: *Norm inequalities for vector functions.*- J. Math. Anal. Appl. 380 (2011) 768–781, arXiv math 1008.4254 doi: 10.1016/j.jmaa.2011.02.029,14 pp.

III. B. A. Bhayo and M. Vuorinen: *Inequalities for eigenfunctions of the p-Laplacian.*- January 2011, 23 pp. arXiv math.CA 1101.3911.

IV. B. A. Bhayo and M. Vuorinen: *On generalized complete elliptic integrals and modular functions.*- February 2011, 18 pp., arXiv math.CA 1102.1078.

1. Introduction

Classical Analysis is a very wide area of contemporary mathematics and the topics of the papers I-IV may be specified by saying that papers I and II are motivated by geometric function theory whereas papers III and IV deal mainly with mathematical special functions.

We will now make some remarks about the history of these two topics from the point of view of this thesis and list some of the key references. A survey of the topics of geometric function theory discussed below is given in several recent papers, see e.g. F.W. Gehring [19] and M. Vuorinen [32]. The basic references are the monographs of Lehto and Virtanen [26], Väisälä [31] and Vuorinen [33]. The handbook of Kühnau [25] provides a collection surveys of dealing with geometric function theory and quasiconformal mappings in particular. For the theory of special functions our main reference is the monograph of Anderson, Vamanamurthy and Vuorinen [9] and for the most recent results the papers [8], [10].

In the early 1960's, F. W. Gehring and J. Väisälä originated the theory of quasiconformal mappings in the Euclidean n-space. Their work generalized the theory of quasiconformal mappings in the plane due to H. Grötzsch 1928, O. Teichmüller in the period 1938-44, and L. Bers, L. V. Ahlfors from the early 1950's.

The study of extremal problems of geometric function theory leads to the study of the special functions that have crucial role in the distortion theory of two-dimensional quasiconformal mappings. Conformal invariants can often be closely associated with particular conformal mappings. This leads to the connection between conformal invariants and special functions, expressed in terms of a conformal mapping of the upper half plane onto a rectangle.

Quasiconformal maps are parametrized by a number $K \geq 1$, the maximal dilatation, which roughly speaking measures how far the maps are from being conformal: conformal maps are the special case $K = 1$. Because quasiconformal maps are differentiable almost everywhere, off a set Z of a measure zero, the local behavior of the mapping at the points of Z is of particular importance. Another problem of particular importance is to study the closeness of quasiconformal maps to conformal maps. For the study of these two topics special functions have an important role as we will see in this thesis, for instance in papers I and II.

2. Mori's theorem

Many authors have proved distortion theorems for quasiconformal and quasiregular mappings in the plane or in the Euclidean n-space, which deal with the estimates for the modulus of continuity and the ways distances between points are changed under these mappings. The Hölder continuity, the counterpart of the Schwarz lemma for quasiconformal mappings and Mori's theorem are some of the important examples. A. Mori [30] gave a result, known as Mori's theorem. He showed that if f is a K-quasiconformal mapping of the unit disk $\mathbb{B}^2$ onto itself with $f(0) = 0$, then

$$|f(x) - f(y)| \leq 16|x - y|^{1/K}$$

for all $x, y \in \mathbb{B}^2$. Some weaker results of the same type had been proved earlier by L. V. Ahlfors and M. A. Lavrentieff. In the case $n = 3$ F. W. Gehring [18, Theorem 14, p.387] proved that quasiconformal mappings are Hölder-continuous. In 1988 this problem was studied by G. D. Anderson and M. K. Vamanamurthy for the higher dimensional case [6].

In the same year, R. Fehlmann and M. Vuorinen [16] studied the least constant $M(n, K)$ such that for every K-quasiconformal mapping $f : \mathbb{B}^n \to \mathbb{B}^n = f(\mathbb{B}^n)$ with $f(0) = 0$ we have for all $x, y \in \mathbb{B}^n$

$$|f(x) - f(y)| \le M(n, K)|x - y|^\alpha, \quad \alpha = K^{1/(1-n)}. \tag{2.1}$$

They also found concrete upper bounds for $M(n, K)$ and showed that $M(n, K) \to 1$ when $K \to 1$ unlike Mori's constant 16 or the constant in [6]. On the other hand as A. Mori pointed out [30], letting $K \to \infty$ we see that the constant 16 cannot be replaced by a smaller constant. P. Hästö [20] proved a counterpart of the Fehlmann-Vuorinen result for the chordal metric.

A domain D in $\overline{\mathbb{R}}^n$ is called a ring domain or, briefly, a ring, if $\overline{\mathbb{R}}^n \setminus D$ consists of two components C_0 and C_1, and it is denoted by $R(C_0, C_1)$. The Grötzsch ring $R_G(s)$, $s > 1$ is defined by

$$R_G(s) = R(\overline{\mathbb{B}}^n, [s\, e_1, \infty]), \quad s > 1\,.$$

The conformal modulus of the Grötzsch ring is denoted by

$$M_n(r) = \mathrm{mod} R_{G,n}(1/r), \quad 0 < r < 1$$

(see [9, (8.35)]). The capacity of the Grötzsch ring is denoted by γ_n [33, (5.52)]. The Grötzsch ring constant λ_n is defined by

$$\log \lambda_n = \lim_{r \to 0_+} (M_n(r) + \log r)$$

and $\lambda_n \in [4, 2e^{n-1})\,, \lambda_2 = 4$, [4], [33, p.89].

The main results of this paper are

2.2. **Theorem.** *(1) For $n \ge 2, K \ge 1$, let $M(n, K)$ be as in (2.1). Then $M(n, K) \le T(n, K)$*

$$T(n, K) = \inf\{h(t) : t \ge 1\}\,, \quad h(t) = (3 + \lambda_n^{\beta - 1} t^\beta) t^{-\alpha} \lambda_n^{2(1-\alpha)},\ t \ge 1\,,$$

where $\alpha = K^{1/(1-n)} = 1/\beta$, and $\lambda_n \in [4, 2e^{n-1})\,, \lambda_2 = 4$, is the Grötzsch ring constant [4], [33, p.89].

(2) There exists a number $K_1 > 1$ such that for all $K \in (1, K_1)$ the function h has a minimum at a point t_1 with $t_1 > 1$ and

$$T(n, K) \le h(t_1) = \left[\frac{3^{1-\alpha^2}(\beta - \alpha)^{\alpha^2}}{\alpha^{\alpha^2}} \lambda_n^{\alpha - \alpha^2} + \lambda_n^{\beta - 1} \left(\frac{(3\alpha)^\alpha \lambda_n^{\alpha - 1}}{(\beta - \alpha)^\alpha} \right)^{\beta - \alpha} \right] \lambda_n^{2(1-\alpha)}\,.$$

Moreover, for $\beta \in (1, \min\{2, K_1^{1/(n-1)}\})$ *we have*

$$h(t_1) \le 3^{1-\alpha^2} 2^{5(1-\alpha)} K^5 \left(\frac{3}{2} \sqrt[4]{\beta - \alpha} + \exp(\sqrt{\beta^2 - 1}) \right).$$

In particular, $h(t_1) \to 1$ *when* $K \to 1$.

The hyperbolic metric $\rho(x,y)$, $x, y \in \mathbb{B}^n$, of the unit ball is given by (cf. [24], [33])

$$\mathrm{th}^2 \frac{\rho(x,y)}{2} = \frac{|x-y|^2}{|x-y|^2 + t^2}, \quad t^2 = (1-|x|^2)(1-|y|^2).$$

For $n \ge 2$ and $K > 0$, the distortion function $\varphi_{K,n} : [0,1] \to [0,1]$ defined by

$$\varphi_{K,n}(r) = \frac{1}{\gamma_n^{-1}(K\gamma_n((1/r)))}, \quad r \in (0,1),$$

and $\varphi_{K,n}(0) = 0$, $\varphi_{K,n}(1) = 1$ is a homeomorphism. We denote $\varphi_{K,2} = \varphi_K$.

2.3. **Theorem.** *If* $f : \mathbb{B}^2 \to \mathbb{R}^2$ *is a non-constant* K*-quasiregular mapping with* $f\mathbb{B}^2 \subset \mathbb{B}^2$, *and* ρ *is the hyperbolic metric of* $\mathbb{B}^2$, *then*

$$\rho(f(x), f(y)) \le c(K) \max\{\rho(x,y), \rho(x,y)^{1/K}\}$$

for all $x, y \in \mathbb{B}^2$ *where* $c(K) = 2\mathrm{arth}(\varphi_K(\mathrm{th}\frac{1}{2}))$ *and*

$$K \le u(K-1) + 1 \le \log(\mathrm{ch}(K \mathrm{arch}(e))) \le c(K) \le v(K-1) + K$$

with $u = \mathrm{arch}(e)\mathrm{th}(\mathrm{arch}(e)) > 1.5412$ *and* $v = \log(2(1 + \sqrt{1 - 1/e^2})) < 1.3507$. *In particular,* $c(1) = 1$.

The notation ch, th and arch, arth denote the hyperbolic cosine, tangent and their inverse functions, respectively.

Observe that both Theorems 2.2 and 2.3 are asymptotically sharp when $K \to 1$. The proof of sharpness is based on inequalities for special functions.

3. Norm inequalities

A geometric generalization of the inner product spaces was given by Fréchet [17], in 1935. It was proved by P. Jordan and J. von Neumann [23] that normed linear spaces satisfying the parallelogram law. There are interesting norm inequalities connected with characterizations of inner product spaces. In 1936, the concept of angular distance

$$\alpha(x,y) = \left| \frac{x}{|x|} - \frac{y}{|y|} \right|$$

between nonzero elements x and y in the normed space was introduced by J. A. Clarkson [13]. In 2006, L. Maligranda considered the p-angular distance

$$\alpha_p(x,y) = \left| \frac{x}{|x|}|x|^p - \frac{y}{|y|}|y|^p \right|, \ p \in \mathbb{R}$$

as a generalization of the concept of angular distance. He proved in [29, Theorem 2] the following theorem in the context of normed spaces.

3.1. **Theorem.**

$$\alpha_p(x,y) \le \begin{cases} (2-p)\dfrac{|x-y|\max\{|x|^p,|y|^p\}}{\max\{|x|,|y|\}} & \textit{if} \quad p\in(-\infty,0) \quad \textit{and} \quad x,y\neq 0; \\[2ex] (2-p)\dfrac{|x-y|}{(\max\{|x|,|y|\})^{1-p}} & \textit{if} \quad p\in[0,1] \quad \textit{and} \quad x,y\neq 0; \\[2ex] p\,(\max\{|x|,|y|\})^{p-1}|x-y| & \text{if} \quad p\in(1,\infty). \end{cases}$$

Thereafter, S. S. Dragomir [14] proved in 2009 the following upper bound for the p-angular distance for nonzero vectors x, y. Numerical tests reported in paper II show that sometimes his bounds are better than those in Theorem 3.1.

3.2. **Theorem.**

$$\alpha_p(x,y) \le \begin{cases} |x-y|(\max\{|x|,|y|\})^{p-1} + ||x|^{p-1}-|y|^{p-1}|\min\{|x|,|y|\} \\ \textit{if} \quad p\in(1,\infty)\,; \\[2ex] \dfrac{|x-y|}{(\min\{|x|,|y|\})^{1-p}} + \left||x|^{1-p}-|y|^{1-p}\right|\min\left\{\dfrac{|x|^p}{|y|^{1-p}},\dfrac{|y|^p}{|x|^{1-p}}\right\} \\ \textit{if} \quad p\in[0,1]\,; \\[2ex] \dfrac{|x-y|}{(\min\{|x|,|y|\})^{1-p}} + \dfrac{||x|^{1-p}-|y|^{1-p}|}{\max\{|x|^{-p}|y|^{1-p},|y|^{-p}|x|^{1-p}\}}, \\ \textit{if} \quad p\in(-\infty,0)\,. \end{cases}$$

Studying sharp constants connected to the p-Laplace operator J. Byström [12, Lemma 3.3] proved in 2005 the following result.

3.3. **Theorem.** *For $p\in(0,1)$ and $x,y\in\mathbb{R}^n$, we have*

$$\alpha_p(x,y) \le 2^{1-p}|x-y|^p$$

with equality for $x=-y$.

We define the function

$$\mathcal{A}_{a,b}(x) = \begin{cases} |x|^{a-1}x & \textit{if } |x|\le 1 \\ |x|^{b-1}x & \textit{if } |x|\ge 1\,, \end{cases}$$

for $a,b>0$, $x\in\mathbb{R}^n$. The following are the main results of the paper II.

3.4. **Theorem.** *Let $0<a\le 1\le b$ and*

$$C(a,b) = \sup_{|x|\le|y|} Q(x,y),$$

where

$$Q(x,y) = \frac{|\mathcal{A}(x)-\mathcal{A}(y)|}{|\mathcal{A}(x)-\mathcal{A}(z)|}\,, \quad x,y\in\mathbb{R}^n\setminus\{0\} \ \textit{with} \ x\neq y\,,$$

and

$$z = \frac{x}{|x|}(|x| + |x - y|).$$

Then

$$C(a,b) = \frac{2}{3^a - 1} \text{ and } \lim_{a\to 1} C(a,b) = 1.$$

3.5. **Theorem.** *For all $x, y \in \mathbb{R}^n$ and $p \in (0,1)$*

$$\alpha_p(x,y) \le |\mathcal{A}_{p,1/p}(x) - \mathcal{A}_{p,1/p}(y)|,$$

and furthermore, if $|x| \le |y|$, we have also

$$\alpha_p(x,y) \le |\mathcal{A}_{p,1/p}(x) - \mathcal{A}_{p,1/p}(y)| \le \frac{2}{3^p - 1}|\mathcal{A}_{p,1/p}(x) - \mathcal{A}_{p,1/p}(z)| \tag{3.6}$$

where z is as in Theorem 3.4.

Computer tests reported in paper II shows that none of the above bounds in Theorem 3.1-3.3 and 3.5 for $\alpha_p(x,y)$ is better than others. In some cases our bound (3.6) is better than the bounds in Theorems 3.1-3.3.

4. Eigenfunctions

An eigenfunction of a linear operator A, defined on some function space is any nonzero function f in that space which returns from the operator exactly as is, except for a multiplicative scaling factor. A complete set of eigenfunctions is introduced within the Riemann-Hilbert formation for spectral problems associated to some solvable nonlinear evolution equations. The eigenfunctions of one dimensional p-Laplacian operator

$$\begin{cases} -(|u^{'}(x)|^{p-2}u^{'}(x))' = \lambda|u(x)|^{p-2}u(x), \\ u(0) = 0, \quad u(\pi_p) = 0, \quad 0 \le x \le \pi_p \end{cases}$$

are of the form

$$\sin_p(x), \quad \sin_p(2x), \quad \sin_p(3x), \ldots,$$

where $\pi_p = 2\pi/(p\sin(\pi/p))$ and $\sin_p$ is the inverse function of $\arcsin_p$ to be defined below. In a highly cited paper P. Lindqvist [27] studied in 1995 these eigenfunctions and introduced the generalization form of the trigonometric and hyperbolic functions. With J. Peetre [28] he also studied the generalization of Euclidean distance, which is called p-distance(length). Recently P. J. Bushell and D. E. Edmunds studied these p-analogues functions and introduced many relations [11].

Given complex numbers a, b and c with $c \ne 0, -1, -2, \ldots$, the *Gaussian hypergeometric function* is the analytic continuation to the slit place $\mathbb{C} \setminus [1,\infty)$ of the series

$$F(a,b;c;z) = {}_2F_1(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a,n)(b,n)}{(c,n)} \frac{z^n}{n!}, \qquad |z| < 1.$$

Here $(a,0)=1$ for $a\neq 0$, and (a,n) is the *shifted factorial function* or the *Appell symbol*

$$(a,n)=a(a+1)(a+2)\cdots(a+n-1)$$

for $n\in\mathbb{Z}_+$.

We consider the following homeomorphisms

$$\sin_p:(0,a_p)\to I,\ \cos_p:(0,a_p)\to I,\ \tan_p:(0,b_p)\to I,$$
$$\sinh_p:(0,c_p)\to I,\ \tanh_p:(0,\infty)\to I\,,$$

where $I=(0,1)$ and

$$a_p=\frac{\pi_p}{2},\ b_p=\frac{1}{2p}\left(\psi\left(\frac{1+p}{2p}\right)-\psi\left(\frac{1}{2p}\right)\right)=2^{-1/p}F\left(\frac{1}{p},\frac{1}{p};1+\frac{1}{p};\frac{1}{2}\right)\,,$$
$$c_p=\left(\frac{1}{2}\right)^{1/p}F\left(1\,,\frac{1}{p};1+\frac{1}{p}\,;\frac{1}{2}\right)\,.$$

For $x\in I$, their inverse functions are defined as

$$\begin{aligned}
\operatorname{arcsin}_p x &= \int_0^x(1-t^p)^{-1/p}dt=x\,F\left(\frac{1}{p},\frac{1}{p};1+\frac{1}{p};x^p\right)\\
&= x(1-x^p)^{(p-1)/p}F\left(1,1;1+\frac{1}{p};x^p\right)\,,\\
\operatorname{arctan}_p x &= \int_0^x(1+t^p)^{-1}dt=x\,F\left(1,\frac{1}{p};1+\frac{1}{p};-x^p\right)\,,\\
\operatorname{arsinh}_p x &= \int_0^x(1+t^p)^{-1/p}dt=xF\left(\frac{1}{p}\,,\frac{1}{p};1+\frac{1}{p};-x^p\right)\,,\\
\operatorname{artanh}_p x &= \int_0^x(1-t^p)^{-1}dt=xF\left(1\,,\frac{1}{p};1+\frac{1}{p};x^p\right)\,,
\end{aligned}$$

and by [11, Prop 2.2] $\operatorname{arccos}_p x=\operatorname{arcsin}_p((1-x^p)^{1/p})$. For the particular case $p=2$ one obtains the familiar elementary functions [9, 1.20].

Some of the main results of this paper read as follows

4.1. **Theorem.** *For $p>1$ and $x\in(0,1)$, we have*

(1) $\left(1+\dfrac{x^p}{p(1+p)}\right)x<\operatorname{arcsin}_p x<\dfrac{\pi_p}{2}\,x,$

(2) $\left(1+\dfrac{1-x^p}{p(1+p)}\right)(1-x^p)^{1/p}<\operatorname{arccos}_p x<\dfrac{\pi_p}{2}\,(1-x^p)^{1/p},$

(3) $\dfrac{(p(1+p)(1+x^p)+x^p)x}{p(1+p)(1+x^p)^{1+1/p}}<\operatorname{arctan}_p x<2^{1/p}\,b_p\,\left(\dfrac{x^p}{1+x^p}\right)^{1/p}.$

4.2. **Theorem.** *For $p>1$ and $x\in(0,1)$, we have*

(1) $z\left(1+\frac{\log(1+x^p)}{1+p}\right)<\operatorname{arsinh}_p x<z\left(1+\frac{1}{p}\log(1+x^p)\right),\ z=\left(\dfrac{x^p}{1+x^p}\right)^{1/p},$

(2) $x\left(1-\frac{1}{1+p}\log(1-x^p)\right) < \mathrm{artanh}_p\, x < x\left(1-\frac{1}{p}\log(1-x^p)\right)$.

5. Generalized complete elliptic integrals

In 1655, John Wallis first used the term “hypergeometric series”. L. Euler studied hypergeometric series, but the first full systematic treatment was given by J. C. F. Gauss in 1813. Gauss hypergeometric function $F(a,b;c;z)$ is a special function represented by the hypergeometric series. The investigation of integral addition theorems introduced the discovery of elliptic functions. An addition theorem for a function f is a formula expressing $f(u+v)$ in terms of $f(u)$ and $f(v)$. A. M. Legendre investigated elliptic integrals, he showed that integrations of the elliptic integral $\int R(t)/\sqrt{P(t)}\,dt$, where $R(t)$ is a rational function of t and $P(t)$ is a polynomial of fourth degree, can be reduced to the integration of the three integrals

$$\int \frac{dx}{\sqrt{1-x^2}\sqrt{1-l^2x^2}},\ \int \frac{x^2\,dx}{\sqrt{1-x^2}\sqrt{1-l^2x^2}},\ \int \frac{dx}{(x-a)\sqrt{1-x^2}\sqrt{1-l^2x^2}},$$

which he called the elliptic integrals of the first, second, and third kinds, respectively.

The study of the elliptic integrals of the first kind introduces several special functions. In [5], [21], these special functions are generalized, and many results are given there. We introduce some notation here. For $0 < a \le 1/2$ and $a, r \in (0,1)$, the generalized elliptic integrals are defined by

$$\mathcal{K}_a(r) = \frac{\pi}{2}\,F(a,1-a;1;r^2), \qquad \mathcal{E}_a(r) = \frac{\pi}{2}\,,$$

with $\mathcal{K}_{1/2} = \mathcal{K}$ and $\mathcal{E}_{1/2} = \mathcal{E}$. The decreasing homeomorphism $\mu_a : (0,1) \to (0,\infty)$ is defined by

$$\mu_a(r) = \frac{\pi}{2\,\sin(\pi\,a)}\frac{\mathcal{K}_a(r')}{\mathcal{K}_a(r)} \tag{5.1}$$

for $r \in (0,1)$ and $r' = \sqrt{1-r^2}$.

H. Alzer and S.-L. Qiu have given the following bounds for $\mathcal{K}$ in [3, Theorem 18]

$$\frac{\pi}{2}\left(\frac{\mathrm{artanh}(r)}{r}\right)^{3/4} < \mathcal{K}(r) < \frac{\pi}{2}\left(\frac{\mathrm{artanh}(r)}{r}\right)\,. \tag{5.2}$$

In the following theorem we generalize their result. For the case $a = 1/2$ our upper bound is better than their bound.

5.3. **Theorem.** *For $p \ge 2$ and $r \in (0,1)$, we have*

$$\begin{aligned}\frac{\pi}{2}\left(\frac{\mathrm{artanh}_p(r)}{r}\right)^{1/2} &< \frac{\pi}{2}\left(1-\frac{p-1}{p^2}\log(1-r^2)\right)\\ &< \mathcal{K}_a(r) < \frac{\pi}{2}\left(1-\frac{2}{p\,\pi_p}\log(1-r^2)\right)\,,\end{aligned}$$

where $a = 1/p$ and $\pi_p = 2\pi/(p\sin(\pi/p))$, see [27].

5.4. **Theorem.** *The function* $f(x) = 1/\mathcal{K}_a(1/\cosh(x))$ *is increasing and concave from* $(0,\infty)$ *onto* $(0, 2/\pi)$. *In particular,*

$$\frac{\mathcal{K}_a(r)\,\mathcal{K}_a(s)}{\mathcal{K}_a(rs/(1+r's'))} \le \mathcal{K}_a(r) + \mathcal{K}_a(s) \le \frac{2\,\mathcal{K}_a(r)\,\mathcal{K}_a(s)}{\mathcal{K}_a(\sqrt{rs/(1+rs+r's')})} \le \frac{2\,\mathcal{K}_a(r)\,\mathcal{K}_a(s)}{\mathcal{K}_a(rs)}\,,$$

for all $r, s \in (0,1)$, *with equality in the third inequality if and only if* $r = s$.

5.5. **Theorem.** *For* $p \ge 2$ *and* $r \in (0,1)$, *let*

$$l_p(r) = \left(\frac{\pi_p}{2}\right)^2 \left(\frac{p^2 - (p-1)\log r^2}{p\,\pi_p - 2\log r'^2}\right) \quad \textit{and} \quad u_p(r) = \left(\frac{p}{2}\right)^2 \left(\frac{p\,\pi_p - 2\log r^2}{p^2 - (p-1)\log r'^2}\right).$$

(1) The following inequalities hold

$$l_p(r) < \mu_a(r) < u_p(r)\,,$$

where $a = 1/p$.
(2) For $p = 2$ *we have*

$$u_2(r) < \frac{4}{\pi}\, l_2(r)\,.$$

6. Conclusions and open problems

The study of quasiconformal mappings in paper II and IV shows that conformal invariants together with special functions provide a powerful tool when examining the case when mappings have a small maximal dilatation $K > 1$. It is natural to expect that further progress is possible using this approach. This research has led to several open problems and we list here some of them.

1. What is the sharpest constant for the Theorem 2.3 [I, Theorem 1.10] in the higher dimensional case?
2. Do there exist analogues of addition formulas for the p-functions e.g. in the form of an inequality?
3. Let $l_p(r)$ and $u_p(r)$ be as in Theorem 5.5. Is it is true that $u_p(r) < (4/\pi_p) l_p(r)$? For $p = 2$ see [IV, Theorem 1.9].

Also the publications [5], [9] and IV list a few open problems.

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Original publications

Publication I

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I

ON MORI'S THEOREM FOR QUASICONFORMAL MAPS IN THE n-SPACE

IN MEMORIAM: M. K. VAMANAMURTHY, 5 SEPTEMBER 1934 – 6 APRIL 2009

ABSTRACT. R. Fehlmann and M. Vuorinen proved in 1988 that Mori's constant $M(n,K)$ for K-quasiconformal maps of the unit ball in $\mathbb{R}^n$ onto itself keeping the origin fixed satisfies $M(n,K) \to 1$ when $K \to 1$. We give here an alternative proof of this fact, with a quantitative upper bound for the constant in terms of elementary functions. Our proof is based on a refinement of a method due to G. D. Anderson and M. K. Vamanamurthy. We also give an explicit version of the Schwarz lemma for quasiconformal self-maps of the unit disk. Some experimental results are provided to compare the various bounds for the Mori constant when $n = 2$.

1. INTRODUCTION

Distortion theory of quasiconformal and quasiregular mappings in the Euclidean n-space $\mathbb{R}^n$ deals with estimates for the modulus of continuity and change of distances under these mappings. Some of the examples are the Hölder continuity, the quasiconformal counterpart of the Schwarz lemma, and Mori's theorem. The investigation of these topics started in the early 1950's for the case $n = 2$ and ten years later for the case $n \geq 3$. Many authors have contributed to the distortion theory, for some historical remarks see [Vu1, 11.50].

As in [FV] we define Mori's constant $M(n,K)$ in the following way. Let QC_K, $K \geq 1$, stand for the family of all K-quasiconformal maps of the unit ball $\mathbb{B}^n$ onto itself keeping the origin fixed. Note that it is well known that an element in the set QC_K can be extended by reflection to a K-quasiconformal map of the whole space $\overline{\mathbb{R}}^n = \mathbb{R}^n \cup \{\infty\}$ onto itself keeping the point ∞ fixed. Then for all $K \geq 1$, $n \geq 2$, there exists a least constant $M(n,K) \geq 1$ such that

$$(1.1) \qquad |f(x) - f(y)| \leq M(n,K)|x-y|^{\alpha}, \quad \alpha = K^{1/(1-n)},$$

for all $f \in QC_K$ and $x, y \in \mathbb{B}^n$ (see [FV]).

In 1954, L. V. Ahlfors [A1] proved that $M(2,K) \leq 12^{K^2}$ and this property was refined by A. Mori [Mo] in 1956 to the effect that $M(2,K) \leq 16$, and 16 cannot be replaced by a smaller constant independent of K. This result can also be found in [A2], [FM], and [LV]. On the other hand the trivial observation that 16 fails to be

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a sharp constant for $K = 1$ led to the following conjecture, which is still open in 2009.

1.2. **The Mori Conjecture.** $M(2,K) = 16^{1-1/K}$.

O. Lehto and K. I. Virtanen demonstrated in 1973 [LV, pp. 68] that $M(2,K) \geq 16^{1-1/K}$ (this lower bound was not given in the 1965 German edition of the book). It is natural to expect that for a fixed $n \geq 2$, $M(n,K) \to 1$ when $K \to 1$ and this convergence result with an explicit upper bound for $M(n,K)$ was proved by R. Fehlmann and M. Vuorinen [FV]. A counterpart of this result for the chordal metric was proved recently by P. Hästö in [H].

1.3. **Theorem.** [FV, Theorem 1.3] *Let f be a K-quasiconformal mapping of $\mathbb{B}^n$ onto $\mathbb{B}^n$, $n \geq 2$, $f(0) = 0$. Then*

$$|f(x) - f(y)| \leq M(n,K)|x-y|^{\alpha} \tag{1.4}$$

for all $x, y \in \mathbb{B}^n$ where $\alpha = K^{1/(1-n)}$ and the constant $M(n,K)$ has the following three properties:

(1) *$M(n,K) \to 1$ as $K \to 1$, uniformly in n,*
(2) *$M(n,K)$ remains bounded for fixed K and varying n,*
(3) *$M(n,K)$ remains bounded for fixed n and varying K.*

For $n = 2$, the first majorants with the convergence property in 1.3(1) were proved only in the mid 1980s and for $n \geq 3$ in [FV]. In [FV] a survey of the various known bounds for $M(n,K)$ when $n \geq 2$ can be found – that survey reflects what was known at the time of publication of [FV]. Some earlier results on Hölder continuity had been proved in [G], [MRV], [R], [S]. Step by step the bound for Mori's constant was reduced during the past twenty years. As far as we know, the best upper bound known today for $n = 2$ is $M(2,K) \leq 46^{1-1/K}$ due to S.-L. Qiu [Q] (1997). Refining the parallel work [FV], G. D. Anderson and M. K. Vamanamurthy proved the following theorem in [AV].

1.5. **Theorem.** *For $n \geq 2, K \geq 1$,*

$$M(n,K) \leq 4\lambda_n^{2(1-\alpha)},$$

where $\alpha = K^{1/(1-n)}$ and $\lambda_n \in [4, 2e^{n-1})$, $\lambda_2 = 4$, is the Grötzsch ring constant [AN], [Vu1, p.89].

The first main result of this paper is Theorem 1.6 which improves on Theorem 1.5.

1.6. **Theorem.** *(1) For $n \geq 2, K \geq 1$, $M(n,K) \leq T(n,K)$,*

$$T(n,K) = \inf\{h(t) : t \geq 1\}, \quad h(t) = (3 + \lambda_n^{\beta-1} t^{\beta}) t^{-\alpha} \lambda_n^{2(1-\alpha)}, \ t \geq 1, \tag{1.7}$$

where $\alpha = K^{1/(1-n)} = 1/\beta$, and λ_n is as in Theorem 1.5.

(2) There exists a number $K_1 > 1$ such that for all $K \in (1, K_1)$ the function h has a minimum at a point t_1 with $t_1 > 1$, and

(1.8)
$$T(n,K) \le h(t_1) = \left[\frac{3^{1-\alpha^2}(\beta-\alpha)^{\alpha^2}}{\alpha^{\alpha^2}} \lambda_n^{\alpha-\alpha^2} + \lambda_n^{\beta-1} \left(\frac{(3\alpha)^\alpha \lambda_n^{\alpha-1}}{(\beta-\alpha)^\alpha} \right)^{\beta-\alpha} \right] \lambda_n^{2(1-\alpha)} .$$

Moreover, for $\beta \in (1, \min\{2, K_1^{1/(n-1)}\})$ *we have*

$$h(t_1) \le 3^{1-\alpha^2} 2^{5(1-\alpha)} K^5 \left(\frac{3}{2} \sqrt[4]{\beta-\alpha} + \exp(\sqrt{\beta^2-1}) \right) . \tag{1.9}$$

In particular, $h(t_1) \to 1$ *when* $K \to 1$.

The last statement shows that Theorem 1.6 is better than the result of Anderson and Vamanamurthy, Theorem 1.5, at least for values of K close to the critical value 1, because the constant of Theorem 1.5 satisfies $4\lambda_n^{2(1-\alpha)} \ge 4$.

The main method of our proof is to replace the argument of Anderson and Vamanamurthy by a more refined inequality from [Vu2] and to introduce an additional parameter (t in the above theorem) which will be chosen in an optimal way. The fact that this refined inequality is essentially sharp for values of t large enough, was recently proved by V. Heikkala and M. Vuorinen in [HV]. This gave us a hint that the inequality from [Vu2] might lead to an improvement of the results in [AV]. For the case $n = 2$ a numerical comparison of our bound (1.8) to Mori's conjectured bound, to the bound in Theorem 1.5 and to the bound in [FV] is presented in tabular and graphical form at the end of the paper.

We conclude this paper by discussing the Schwarz lemma for plane quasiconformal self-mappings of the unit disk, formulated in terms of the hyperbolic metric. The long history of this result is summarized in [Vu1, p.152, 11.50]. An up-to-date form of the Schwarz lemma was given in [Vu1, Theorem 11.2] and it will be stated for convenient reference also below as Theorem 4.4. A particular case, formula (4.6), was rediscovered by D. B. A. Epstein, A. Marden and V. Markovic [EMM, Thm 5.1].

We use the notations ch, th, arch and arth as in [Vu1], to denote the hyperbolic cosine, tangent and their inverse functions, respectively. The second main result of this paper is an explicit form of the Schwarz lemma for quasiregular mappings, Theorem 1.10. We believe that in this simple form the result is new and perhaps of independent interest. The constant $c(K)$ below involves the transcendental function φ_K defined in Section 4.

1.10. **Theorem.** *If* $f : \mathbb{B}^2 \to \mathbb{R}^2$ *is a non-constant* K*-quasiregular mapping with* $f\mathbb{B}^2 \subset \mathbb{B}^2$*, and* ρ *is the hyperbolic metric of* $\mathbb{B}^2$ *, then*

$$\rho(f(x), f(y)) \le c(K) \max\{\rho(x,y), \rho(x,y)^{1/K}\}$$

for all $x, y \in \mathbb{B}^2$ *where* $c(K) = 2\mathrm{arth}(\varphi_K(\mathrm{th}\frac{1}{2}))$ *and*

$$K \le u(K-1)+1 \le \log(\mathrm{ch}(K\mathrm{arch}(e))) \le c(K) \le v(K-1)+K$$

with $u = \mathrm{arch}(e)\mathrm{th}(\mathrm{arch}(e)) > 1.5412$ *and* $v = \log(2(1+\sqrt{1-1/e^2})) < 1.3507$. *In particular,* $c(1) = 1$.

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2. The main results

We shall follow here the standard notation and terminology for K-quasiconformal and K-quasiregular mappings in the Euclidean n-space $\mathbb{R}^n$, see e.g. [V], [Vu1]. We also recall some basic notation. For the modulus $M(\Gamma)$ of a curve family Γ and its basic properties see [V] and [Vu1].

Let D and D' be domains in $\overline{\mathbb{R}}^n$, $K \geq 1$, and let $f : D \to D'$ be a homeomorphism. Then f is K-quasiconformal if

$$M(\Gamma)/K \leq M(f\Gamma) \leq K M(\Gamma)$$

for every curve family Γ in D [V].

For subsets $E, F, D \subset \overline{\mathbb{R}}^n$ we denote by $\Delta(E,F;D)$ the family of all curves joining E and F in D. For brevity we write $\Delta(E,F) = \Delta(E,F;\mathbb{R}^n)$. A ring is a domain in $\mathbb{R}^n$, whose complement consists of two compact and connected sets. If these sets are E and F, then the ring is denoted by $R(E,F)$. The capacity of a ring $R(E,F)$ is

$$\mathrm{cap} R(E,F) = M(\Delta(E,F)).$$

The complementary components of the Grötzsch ring $R_{G,n}(s)$ are $\overline{\mathbb{B}}^n$ and $[se_1, \infty]$, $s > 1$, while those of the Teichmüller ring $R_{T,n}(t)$ are $[-e_1, 0]$ and $[te_1, \infty]$, $t > 0$. The conformal capacities of $R_{G,n}(s)$ and $R_{T,n}(t)$ are denoted by

$$\begin{cases} \gamma_n(s) = M(\Delta(\overline{\mathbb{B}}^n, [se_1, \infty])), \\ \tau_n(t) = M(\Delta([-e_1, 0], [te_1, \infty])), \end{cases}$$

respectively. Here $\gamma_n : (1,\infty) \to (0,\infty)$ and $\tau_n : (0,\infty) \to (0,\infty)$ are decreasing homeomorphisms and they satisfy the fundamental identity

$$\gamma_n(s) = 2^{n-1}\tau_n(s^2-1), \quad t > 1, \tag{2.1}$$

see e.g. [Vu1, 5.53].

For $n \geq 2$ and $K > 0$, the distortion function $\varphi_{K,n} : [0,1] \to [0,1]$ is a homeomorphism. It is defined by

$$\varphi_{K,n}(t) = \frac{1}{\gamma_n^{-1}(K\gamma_n(1/t))}, \quad t \in (0,1), \tag{2.2}$$

and $\varphi_{K,n}(0) = 0$, $\varphi_{K,n}(1) = 1$. For $n \geq 2, K \geq 1$ and $0 \leq r \leq 1$

$$\varphi_{K,n}(r) \leq \lambda_n^{1-\alpha} r^\alpha, \quad \alpha = K^{1/(1-n)}\,, \tag{2.3}$$

$$\varphi_{1/K,n}(r) \geq \lambda_n^{1-\beta} r^\beta, \quad \beta = K^{1/(n-1)}\,, \tag{2.4}$$

by [Vu1, Theorem 7.47] and where $\lambda_n \geq 4$ is as in Theorem 1.5.

2.5. **Lemma.** *Suppose that $f : \mathbb{B}^n \to \mathbb{B}^n$ is a K-quasiconformal mapping with $f\mathbb{B}^n = \mathbb{B}^n$, $f(0) = 0$, and let $h : \overline{\mathbb{R}}^n \to \overline{\mathbb{R}}^n$ be the inversion $h(x) = x/|x|^2\,, h(\infty) = 0, h(0) = \infty$, and define $g : \overline{\mathbb{R}}^n \to \overline{\mathbb{R}}^n$ by $g(x) = f(x)$ for $x \in \mathbb{B}^n, g(x) = h(f(h(x)))$ for $x \in \mathbb{R}^n \setminus \overline{\mathbb{B}}^n$ and $g(x) = \lim_{z\to x} f(z)$ for $x \in \partial\mathbb{B}^n, g(\infty) = \infty$. Then g is a K-quasiconformal mapping, and we have for $x \in \mathbb{B}^n$*

$$\varphi_{1/K,n}(|x|) \leq |f(x)| \leq \varphi_{K,n}(|x|). \tag{2.6}$$

For $x \in \mathbb{R}^n \setminus \overline{\mathbb{B}}^n$

$$1/\varphi_{K,n}(1/|x|) \leq |g(x)| \leq 1/\varphi_{1/K,n}(1/|x|). \tag{2.7}$$

Proof. It is well-known that the above definition defines g as a K-quasiconformal homeomorphism. The formula (2.6) is well-known (see [AVV2, Theorem 4.2]) and (2.7) follows easily. $\square$

2.8. **Lemma.** [Vu1, Lemma 7.35] *Let $R = R(E, F)$ be a ring in $\overline{\mathbb{R}}^n$ and let $a, b \in E,\ c, d \in F$ be distinct points. Then*

$$\mathrm{cap} R = M(\Delta(E, F)) \geq \tau_n \left(\frac{|a - c||b - d|}{|a - b||c - d|} \right).$$

Equality holds if $b = t_1 e_1, a = t_2 e_1, c = t_3 e_1, d = t_4 e_1$ and $t_1 < t_2 < t_3 < t_4$.

We consider Teichmüller's extremal problem, which will be used to provide a key estimate in what follows. For $x \in \mathbb{R}^n \setminus \{0, e_1\}, n \geq 2$, define

$$p_n(x) = \inf_{E,F} M(\Delta(E, F))$$

where the infimum is taken over all the pairs of continua E and F in $\overline{\mathbb{R}}^n$ with $0, e_1 \in E$ and $x, \infty \in F$. Note that Lemma 2.8 gives the lower bound for $p_n(x)$ in Lemma 2.9.

2.9. **Lemma.** [Vu2, Theorem 1.5] *For $z \in \mathbb{R}^n, |z| > 1$, the following inequalities hold:*

$$\tau_n(|z|) = p_n(-|z|e_1) \leq p_n(z) \leq p_n(|z|e_1) = \tau_n(|z| - 1)$$

where $p_n(z)$ is the Teichmüller function. Furthermore, for $z \in \mathbb{R}^n \setminus [0, e_1]$, there exists a circular arc E with $0, e_1 \in E$ and a ray F with $z, \infty \in F$ such that

$$p_n(z) \leq M(\Delta(E, F)) \leq \tau_n \left(\frac{|z| + |z - e_1| - 1}{2} \right) \tag{2.10}$$

with equality for both $z = -se_1, s > 0$, and for $z = se_1, s > 1$.

2.11. **Notation.** For $t > 0, x, y \in \mathbb{B}^n$, we write

$$D(t,x,y) = \left| x + t\frac{y}{|y|} \right| \ \text{ if } y \neq 0, \quad D(t,x,0) = |x + e_1| \,.$$

By the triangle inequality we have

$$t - |x| \le D(t,x,y) \le t + |x| \,. \tag{2.12}$$

2.13. **Theorem.** *For $n \geq 2, K \geq 1$, let $f : \overline{\mathbb{R}}^n \to \overline{\mathbb{R}}^n$ be a K-quasiconformal mapping, with $f\mathbb{B}^n = \mathbb{B}^n$, $f(0) = 0$ and $f(\infty) = \infty$. Then for $t \geq 1$, $x, y \in \mathbb{B}^n \backslash \{0\}$, we have*

$$\begin{aligned} |f(x) - f(y)| \ &\le \ (3 + \varphi_{1/K,n}(1/t)^{-1})\varphi^2_{K,n}\left(\left(\frac{2|x-y|}{s_1 + |x-y|}\right)^{1/2}\right) \\ &\le \ (3 + \lambda_n^{(\beta-1)}t^{\beta})\lambda_n^{2(1-\alpha)}\left(\frac{2|x-y|}{s_1 + |x-y|}\right)^{\alpha}, \ \alpha = K^{1/(1-n)} = 1/\beta, \end{aligned}$$

where $s_1 = \max\{a, b\}, a = t + |x| + D(t, y, x), b = t + |y| + D(t, x, y)$.

Proof. Let Γ be the family $\Delta(E, F)$ and let E and F be connected sets as in Lemma 2.9 with $x, y \in E, z, \infty \in F$, where $z = -tx/|x|$ and $\Gamma' = f(\Gamma)$. By Lemma 2.8 and (2.10), we have

$$\tau_n\left(\frac{|f(z) - f(x)|}{|f(x) - f(y)|}\right) \le M(\Gamma') \le KM(\Gamma) \le K\tau_n(u-1)\,,$$

$$u = \frac{|x-z| + |z-y| + |x-y|}{2|x-y|}\,.$$

The basic identity (2.1) yields

$$\begin{aligned} \gamma_n\left(\left(\frac{|f(z) - f(y)| + |f(x) - f(y)|}{|f(x) - f(y)|}\right)^{1/2}\right) &\le K\gamma_n\left((u)^{1/2}\right) \\ &= K\gamma_n\left(\left(\frac{t + |x| + D(t,y,x) + |x-y|}{2|x-y|}\right)^{1/2}\right). \end{aligned} \tag{2.14}$$

Applying γ_n^{-1} to (2.14) we have

$$\frac{|f(z) - f(y)| + |f(x) - f(y)|}{|f(x) - f(y)|} \ge \left(\gamma_n^{-1}\left(K\gamma_n\left(\left(\frac{a + |x-y|}{2|x-y|}\right)^{1/2}\right)\right)\right)^2 = v.$$

Because $f\mathbb{B}^n = \mathbb{B}^n$, by (2.6) and (2.4) we know that

$$|f(z) - f(y)| + |f(x) - f(y)| \le 3 + \varphi_{1/K,n}(1/t)^{-1} \le 3 + \lambda_n^{(\beta-1)}t^{\beta},$$

$$\frac{|f(x) - f(y)|}{3 + \varphi_{1/K,n}(1/t)^{-1}} \le \frac{|f(x) - f(y)|}{|f(z) - f(y)| + |f(x) - f(y)|} \le 1/v, \tag{2.15}$$

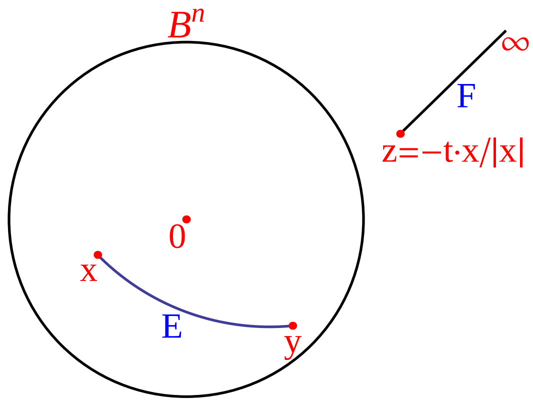

FIGURE 1. Geometrical meaning of the proof of Theorem 2.13.

also

$$\begin{aligned} |f(x)-f(y)| &\le (3+\varphi_{1/K,n}(1/t)^{-1})\varphi^2_{K,n}\left(\left(\frac{2|x-y|}{a+|x-y|}\right)^{1/2}\right) \\ &\le (3+\lambda_n^{(\beta-1)}t^{\beta})\lambda_n^{2(1-\alpha)}\left(\frac{2|x-y|}{a+|x-y|}\right)^{\alpha} \end{aligned}$$

by inequalities (2.2) and (2.3). Exchanging the roles of x and y we see that

$$\begin{aligned} |f(x)-f(y)| &\le (3+\varphi_{1/K,n}(1/t)^{-1})\varphi^2_{K,n}\left(\left(\frac{2|x-y|}{s_1+|x-y|}\right)^{1/2}\right) \\ &\le (3+\lambda_n^{(\beta-1)}t^{\beta})\lambda_n^{2(1-\alpha)}\left(\frac{2|x-y|}{s_1+|x-y|}\right)^{\alpha}. \end{aligned}$$

□

Setting $t=1$, we get the following corollary.

2.16. **Corollary.** *For $n\ge 2, K\ge 1$, let $f:\overline{\mathbb{R}}^n\to\overline{\mathbb{R}}^n$ be a K-quasiconformal mapping, with $f\mathbb{B}^n=\mathbb{B}^n$, $f(0)=0$ and $f(\infty)=\infty$. Then for all $x,y\in\mathbb{B}^n\setminus\{0\}$,*

$$|f(x)-f(y)|\le 4\lambda_n^{2(1-\alpha)}\left(\frac{2|x-y|}{s+|x-y|}\right)^{\alpha},$$

where $\alpha=K^{1/(1-n)}$ and $s=\max\{a,b\}, a=1+|x|+D(1,y,x), b=1+|y|+D(1,x,y)$.

Proof. The proof is similar to the above proof except that here we consider the particular case $t=1$. Because $f\mathbb{B}^n=\mathbb{B}^n$, we know that $|f(z)-f(y)|+|f(x)-f(y)|\le$

4,

$$\begin{aligned}\frac{|f(x)-f(y)|}{4} &\leq \frac{|f(x)-f(y)|}{|f(z)-f(y)|+|f(x)-f(y)|} \\ &\leq \frac{1}{\left(\gamma_n^{-1}\left(K\gamma_n\left(\left(\frac{a+|x-y|}{2|x-y|}\right)^{1/2}\right)\right)\right)^2},\end{aligned}$$

or

$$\begin{aligned}|f(x)-f(y)| &\leq 4\varphi_{K,n}^2\left(\left(\frac{2|x-y|}{a+|x-y|}\right)^{1/2}\right) \\ &\leq 4\lambda_n^{2(1-\alpha)}\left(\frac{2|x-y|}{a+|x-y|}\right)^{\alpha}\end{aligned}$$

by inequalities (2.2) and (2.3). Exchanging the roles of x and y we get

$$|f(x)-f(y)| \leq 4\lambda_n^{2(1-\alpha)}\left(\frac{2|x-y|}{\max\{a,b\}+|x-y|}\right)^{\alpha}.$$

$\square$

2.17. **Corollary.** *For $n \geq 2, K \geq 1, t \geq 1$, let f be as in Theorem 2.13. Then*

$$|f(x)-f(y)| \leq (3+\lambda_n^{(\beta-1)}t^{\beta})\lambda_n^{2(1-\alpha)}\left(\frac{2|x-y|}{2t+||x|-|y||+|x-y|}\right)^{\alpha}, \tag{2.18}$$

for all $x, y \in \mathbb{B}^n$,

$$|f(x)-f(y)| \leq (3+\lambda_n^{\beta-1}t^{\beta})\lambda_n^{2(1-\alpha)}\left(\frac{|x-y|}{\max\{t+|x|,t+|y|\}}\right)^{\alpha}, \tag{2.19}$$

for all $x, y \in \mathbb{B}^n$, and

$$|f(x)-f(y)| \leq (3+\lambda_n^{(\beta-1)}t^{\beta})\lambda_n^{2(1-\alpha)}\left(\frac{|x-y|}{t+|x|+(|x-y|)/2}\right)^{\alpha}, \tag{2.20}$$

if $D(t,y,x) > t+|x|,\ x, y \in \mathbb{B}^n$.

Proof. Inequality (2.18) follows because by (2.11) $D(t,y,x) > t-|y|$ and $D(t,x,y) > t-|x|$ for $x, y \in \mathbb{B}^n$, and hence, in the notation of Theorem 2.13,

$$s_1 \geq \max\{2t+|x|-|y|, 2t+|y|-|x|\} = 2t+||x|-|y||.$$

It is also clear that $D(t,y,x) \geq t+|x|-|x-y|$, and this implies that

$$s_1 \geq \max\{2(t+|x|)-|x-y|, 2(t+|y|)-|x-y|\} = 2\max\{t+|x|,t+|y|\}-|x-y|$$

and hence the inequality (2.19) follows. In the case of (2.20) we have $D(t,y,x) > t+|x|$ and see that, in the notation of Corollary 2.16, $s > 2(t+|x|)$ and (2.20) holds. $\square$

2.21. **Corollary.** *For $n \geq 2, K \geq 1$, let f be as in Theorem 2.13. Then*

$$|f(x)-f(y)| \leq 4\lambda_n^{2(1-\alpha)}\left(\frac{2|x-y|}{2+||x|-|y||+|x-y|}\right)^{\alpha}, \tag{2.22}$$

for all $x, y \in \mathbb{B}^n \setminus \{0\}$.

2.23. **Remark.** (1) In several of the above results we have supposed that $x, y \in \mathbb{B}^n \setminus \{0\}$. If one of the points x, y were equal to 0, then we would have a better result from the Schwarz lemma estimate (4.7).

(2) Corollary 2.21 is an improvement of the Anderson-Vamanamurthy theorem 1.5.

3. Comparison with earlier bounds

3.1. **Proof of Theorem 1.6.** (1) The inequality (1.7) follows easily from the inequality (2.19).

(2) We see that the function h has a local minimum at $t_1 = (3\alpha)^{\alpha}\lambda_n^{\alpha-1}(\beta-\alpha)^{-\alpha}$. If $t_1 \geq 1$, then the inequality (2.19) yields the desired conclusion. The upper bound for $T(n,K)$ follows by substituting the argument t_1 in the expression of h.

We next show that the value $K_1 = 4/3$ will do. Fix $K \in (1, K_1)$. Then $\alpha = K^{1/(1-n)} \geq 3/4$ and $\alpha/(1-\alpha^2) > 1$.

Because $\lambda_n^{\alpha-1} \geq 2^{1/K-1}K^{-1}$ by [Vu1, Lemma 7.50(1)], with $d = (6/K)^{1/K}/2K$ we have

$$t_1 = (3\alpha)^{\alpha}\lambda_n^{\alpha-1}(\beta-\alpha)^{-\alpha} \geq (3/K)^{1/K}2^{1/K-1}K^{-1}\left(\frac{\alpha}{1-\alpha^2}\right)^{\alpha}$$

$$= d\left(\frac{\alpha}{1-\alpha^2}\right)^{\alpha} \geq d\left(\frac{\alpha}{1-\alpha^2}\right)^{3/4}$$

$$= \left(2r(K)\frac{\alpha}{1-\alpha^2}\right)^{3/4} \quad ; \quad r(K) = d^{4/3}/2\,.$$

It suffices to observe that $t_1 > 1$ certainly holds if $2r(K)(\frac{\alpha}{1-\alpha^2}) > 1$ which holds for $\alpha > 1/(r(4/3) + \sqrt{1+r(4/3)^2}) = 0.53\ldots$, in particular, $t_1 > 1$ holds in the present case $\alpha > 3/4$.

For the proof of (1.9) we give the following inequalities

$$\lambda_n^{\alpha-\alpha^2} \leq 2^{\alpha(1-\alpha)}K^{\alpha} \leq 2^{1-\alpha}K^{\alpha}, \quad K \geq 1\,, \tag{3.2}$$

$$\lambda_n^{\beta-\alpha} = \lambda_n^{\beta+1-1-\alpha} = \lambda_n^{\beta(1-\alpha)+1-\alpha} = \lambda_n^{(\beta+1)(1-\alpha)} \leq (2^{1-\alpha}K)^3, \quad \beta \in (1,2)\,, \tag{3.3}$$

see [Vu1, Lemma 7.50(1)]. The formula (1.8) for $h(t_1)$ has two terms. We estimate separately each term as follows

$$\begin{aligned}
\frac{3^{1-\alpha^2}(\beta-\alpha)^{\alpha^2}}{\alpha^{\alpha^2}}\lambda_n^{\alpha-\alpha^2}\lambda_n^{2(1-\alpha)} &\le \frac{3^{(1-\alpha)(1+\alpha)}2^{\alpha(1-\alpha)}2^{2(1-\alpha)}K^2(\beta-\alpha)^{\alpha^2}}{\alpha^{\alpha^2}}K^\alpha\\
&\le \frac{(9\cdot 2\cdot 4)^{1-\alpha}K^2(\beta-\alpha)^{\alpha^2}}{\alpha^{\alpha^2}}K^\alpha\\
&= 72^{1-\alpha}(\beta-\alpha)^{\alpha^2}K^2K^\alpha\exp(-\alpha^2\log\alpha)\\
&\le 72^{1-\alpha}(\beta-\alpha)^{\alpha^2}K^2K^\alpha\exp(-\alpha\log\alpha)\\
&= 72^{1-\alpha}(\beta-\alpha)^{\alpha^2}K^2\exp((\log K-\log\alpha)\alpha)\\
&= 72^{1-\alpha}(\beta-\alpha)^{\alpha^2}K^2\exp\left(\left(1+\frac{1}{n-1}\log K\right)\alpha\right)\\
&= 72^{1-\alpha}(\beta-\alpha)^{\alpha^2}K^2\exp\left(\frac{n}{n-1}\alpha\log K\right)\\
&\le 72^{1-\alpha}(\beta-\alpha)^{\alpha^2}K^2\exp(2\log K)\\
&= 72^{1-\alpha}(\beta-\alpha)^{\alpha^2}K^4
\end{aligned}$$

by inequality (3.2), and

$$\begin{aligned}
\lambda_n^{2(1-\alpha)}\lambda_n^{\beta-1}\left(\frac{(3\alpha)^\alpha\lambda_n^{\alpha-1}}{(\beta-\alpha)^\alpha}\right)^{\beta-\alpha} &= \lambda_n^{2(1-\alpha)}\lambda_n^{\beta-1}\left((3\alpha)^\alpha\lambda_n^{\alpha-1}\right)^{\beta-\alpha}(\beta-\alpha)^{-\alpha(\beta-\alpha)}\\
&\le (2^{1-\alpha}K)^2\lambda_n^{\beta-\alpha}\left((3\alpha)^\alpha\lambda_n^{\alpha-1}\right)^{\beta-\alpha}\left(\frac{\beta^2-1}{\beta}\right)^{-\alpha((\beta^2-1)/\beta)}\\
&\le (2^{1-\alpha}K)^2\left(3^\alpha\lambda_n\right)^{\beta-\alpha}\beta^{\alpha^2}(\beta^2-1)^{-\alpha^2(\beta^2-1)}\\
&\le (2^{1-\alpha}K)^2 3^{\alpha(\beta-\alpha)}\lambda_n^{(\beta+1)(1-\alpha)}\exp\left(\frac{2\alpha^2}{e}\sqrt{\beta^2-1}\right)\\
&\le 3^{1-\alpha^2}(2^{1-\alpha}K)^2(2^{1-\alpha}K)^{(\beta+1)}\exp\left(\frac{2\alpha^2}{e}\sqrt{\beta^2-1}\right)\\
&\le 3^{1-\alpha^2}(2^{1-\alpha}K)^5\exp(\sqrt{\beta^2-1}),
\end{aligned}$$

here we assume that $\beta\in(1,2)$ which implies that $\alpha\in(1/2,1)$. Also the inequalities $(K-1)^{-(K-1)}\le\exp((2/e)\sqrt{K-1})$ and (3.3) were used, and we get

$$h(t_1)\le\left[72^{1-\alpha}(\beta-\alpha)^{\alpha^2}K^4+3^{\beta-\alpha}(2^{1-\alpha}K)^5\exp(\sqrt{\beta^2-1})\right]. \tag{3.4}$$

Because $(\beta-\alpha)\in(0,\frac{3}{2})$ this implies that $\frac{2}{3}(\beta-\alpha)\in(0,1)$ and $\alpha^2\in(\frac{1}{4},1)$ and further $(\frac{2}{3}(\beta-\alpha))^{\alpha^2}\le(\frac{2}{3}(\beta-\alpha))^{1/4}$, and finally

$$(\beta-\alpha)^{\alpha^2}\le(2/3)^{-\alpha^2}\left(\frac{2}{3}(\beta-\alpha)\right)^{1/4}\le(3/2)^{3/4}\sqrt[4]{\beta-\alpha}$$

$$= (3/2)^{3/4} \sqrt[4]{\beta-\alpha} < (3/2)\sqrt[4]{\beta-\alpha}\,.$$

Next we prove that

$$72^{1-\alpha} \le 3^{1-\alpha^2} 2^{5(1-\alpha)} K\,. \tag{3.5}$$

This inequality is equivalent to

$$2^{2(\alpha-1)} 3^{(1-\alpha)^2} \le K \iff -(1-\alpha)\log 4 + (1-\alpha)^2 \log 3 \le \log K\,.$$

This last inequality holds because the left hand side is negative. Now from (3.4) and (3.5) we get the desired inequality (1.9). □

3.6. **Graphical and numerical comparison of various bounds.** The above bounds involve the Grötzsch ring constant λ_n, which is known only for $n = 2, \lambda_2 = 4$. Therefore only for $n = 2$ we can compute the values of the bounds. Solving numerically the equation $4 \cdot 16^{1-1/K} = h(t_1)$ for K we obtain $K = 1.3089$. We give numerical and graphical comparison of the various bounds for the Mori constant.

Tabulation of the various upper bounds for Mori's constant when $n = 2$ and $\lambda_2 = 4$ as a function of K: (a) Mori's conjectured bound $16^{1-1/K}$, (b) the Anderson-Vamanamurthy bound $4 \cdot 16^{1-1/K}$, (c) the bound from (1.8). For $K \in (1, 1.3089)$ the upper bound in (1.8) is better than the Anderson-Vamanamurthy bound. Note that the upper bound $T(n, K) \le h(t_1)$ in (1.8) is proved only for $K \in (1, K_1), K_1 = 4/3$. We do not know whether it holds for larger values of K but just comparing the values of $h(t_1)$ and the bound of Fehlmann and Vuorinen for $K > 1.5946$ we see that $h(t_1)$ is the smaller one of these two. Numerical values of the [FV] bound given in the table were computed with the help of the algorithm for $\varphi_{K,2}(r)$ attached with [AVV1, p. 92, 439].

K	$\log(16^{1-1/K})$	$\log(4 \cdot 16^{1-1/K})$	$\log(FV)$	$\log(h(t_1))$
1.1	0.2521	1.6384	0.7051	1.0188
1.2	0.4621	1.8484	1.2485	1.6058
1.3	0.6398	2.0261	1.7046	2.0107
1.4	0.7922	2.1785	2.0913	2.3061
1.5	0.9242	2.3105	2.4221	2.5296
1.6	1.0397	2.4260	2.7094	2.7031
1.7	1.1417	2.5280	2.9633	2.8409
1.8	1.2323	2.6186	3.1921	2.9521
1.9	1.3133	2.6996	3.4020	3.0433
2.0	1.3863	2.7726	3.5979	3.1192

Note that according to Theorem 1.6 the inequality (1.8) involving $h(t_1)$ holds for $K \in (1, K_1)$ where the number $K_1 > 1$ may be smaller than 2.

For graphing and tabulation purposes we use the logarithmic scale. Note that the upper bound for $M(2, K)$ given in [FV, Theorem 2.29] also has the desirable property that it converges to 1 when $K \to 1$, see Figure 3.

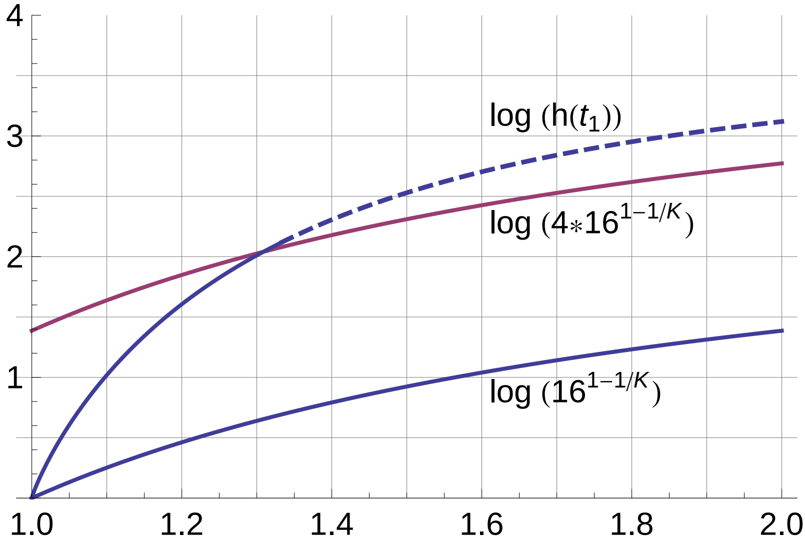

FIGURE 2. Graphical illustration of the various upper bounds for Mori's constant when $n = 2$ and $\lambda_2 = 4$ as a function of K: (a) Mori's conjectured bound $16^{1-1/K}$, (b) the Anderson-Vamanamurthy bound $4 \cdot 16^{1-1/K}$, (c) the bound from (1.8), valid for $K \in (1, K_1), K_1 = 4/3$. For $K \in (1, 1.3089)$ the upper bound in (1.8) is better than the Anderson-Vamanamurthy bound.

3.7. **Comparison of estimates for the Hölder quotient.** For a K-quasiconformal mapping $f : \mathbb{B}^n \to f\mathbb{B}^n = \mathbb{B}^n$, we call the expression

$$HQ(f) = \sup\{|f(x) - f(y)|/|x-y|^\alpha : \; x, y \in \mathbb{B}^n, f(0) = 0 \;\; x \neq y\},$$

the Hölder coefficient of f. Clearly $HQ(f) \le M(n, K)$. Theorem 2.13 yields, after dividing the both sides of the inequality in 2.13 by $|x-y|^\alpha$, the upper bound $HQ(f) \le HQ(K)$ for the Hölder quotient with

$$HQ(K) = \sup\{\inf\{U(t, x, y) : \; t \ge 1\} : \; x, y \in \mathbb{B}^n\}, \tag{3.8}$$

$$U(t, x, y) = (3 + \varphi_{1/K,n}(1/t)^{-1})\varphi^2_{K,n}\left(\left(\frac{2|x-y|}{s_1 + |x-y|}\right)^{1/2}\right)\frac{1}{|x-y|^\alpha}.$$

For $n = 2$ we compare $HQ(K)$ to several other bounds (a) Mori's conjectured bound, (b) the FV bound, (c) the AV bound and give the results as a table and Figure 3. Because the supremum and infimum in (3.8) cannot be explicitly found we use numerical methods that come with Mathematica software. For the numerical tests we used for the supremum a sample of 100,000 random pairs of points of the unit disk.

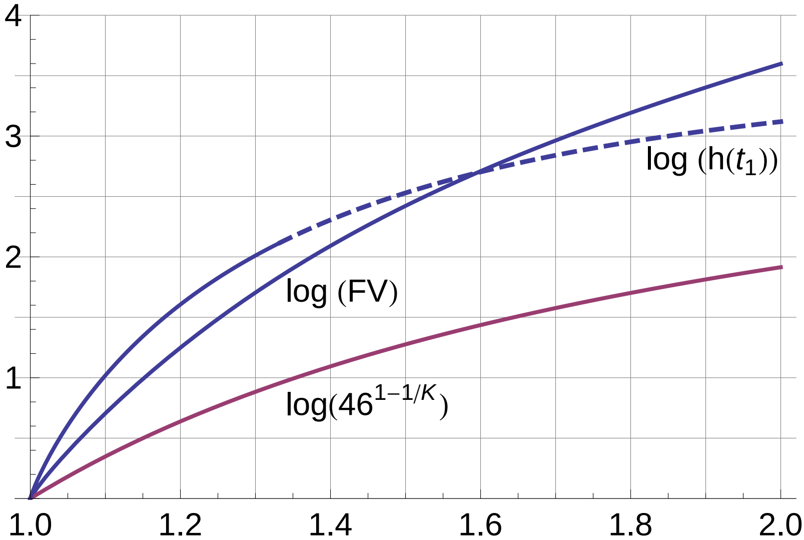

FIGURE 3. Graphical comparison of various bounds when $n = 2$ and $\lambda_2 = 4$, as a function of K: (a) the bound from (1.8), valid for $K \in (1, K_1), K_1 = 4/3$, (b) the Fehlmann and Vuorinen bound [FV]

$$M(2, K) \le \left(1 + \varphi_{K,2}\left(\frac{K^2-1}{K^2+1}\right)\right) 2^{2K-3/K} \frac{(K^2+1)^{(K+1/K)/2}}{(K^2-1)^{(K-1/K)/2}}$$

(c) Qiu's bound $46^{1-1/K}$ [Q].

K	$\log(16^{1-1/K})$	$\log(4 \cdot 16^{1-1/K})$	$\log(FV)$	$\log(HQ(K))$
1.1	0.2521	1.6384	0.7051	1.0171
1.2	0.4621	1.8484	1.2485	1.5940
1.3	0.6398	2.0261	1.7046	1.9712
1.4	0.7922	2.1785	2.0913	2.1668
1.5	0.9242	2.3105	2.4221	2.2928
1.6	1.0397	2.4260	2.7094	2.4003
1.7	1.1417	2.5280	2.9633	2.4922
1.8	1.2323	2.6186	3.1921	2.5706
1.9	1.3133	2.6996	3.4020	2.6371
2.0	1.3863	2.7726	3.5979	2.6934

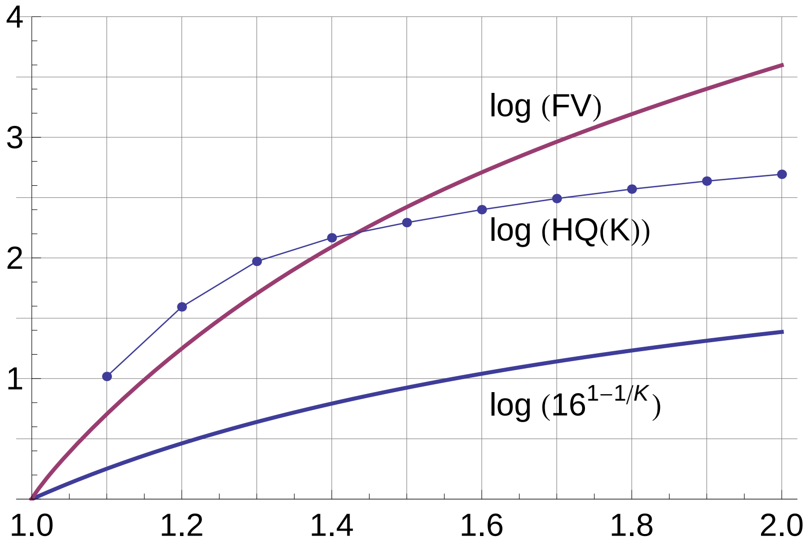

FIGURE 4. Graphical comparison of various bounds when $n = 2$ and $\lambda_2 = 4$, as a function of K: (a) the bound from (3.8), (b) the Fehlmann and Vuorinen bound [FV]

$$M(2,K) \le \left(1 + \varphi_{K,2}\left(\frac{K^2-1}{K^2+1}\right)\right) 2^{2K-3/K} \frac{(K^2+1)^{(K+1/K)/2}}{(K^2-1)^{(K-1/K)/2}}$$

(c) the bound of the Mori conjecture. The bound (3.8) is based on a simulation with 100,000 random pairs of points.

4. An explicit form of the Schwarz lemma

Recall that the hyperbolic metric $\rho(x,y)$, $x, y \in \mathbb{B}^n$, of the unit ball is given by (cf. [KL], [Vu1])

$$\mathrm{th}^2 \frac{\rho(x,y)}{2} = \frac{|x-y|^2}{|x-y|^2 + t^2}, \quad t^2 = (1-|x|^2)(1-|y|^2). \tag{4.1}$$

Next, we consider a decreasing homeomorphism $\mu : (0,1) \longrightarrow (0,\infty)$ defined by

$$\mu(r) = \frac{\pi}{2}\frac{\mathcal{K}(r')}{\mathcal{K}(r)}, \quad \mathcal{K}(r) = \int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-r^2x^2)}}, \tag{4.2}$$

where $\mathcal{K}(r)$ is Legendre's complete elliptic integral of the first kind and $r' = \sqrt{1-r^2}$, for all $r \in (0,1)$.

The Hersch-Pfluger distortion function is an increasing homeomorphism $\varphi_K : (0,1) \longrightarrow (0,1)$ defined by setting

$$\varphi_K(r) = \mu^{-1}(\mu(r)/K), \quad r \in (0,1),\ K > 0. \tag{4.3}$$

Note that with the notation of Section 2, $\gamma_2(1/r) = 2\pi/\mu(r)$ and $\varphi_K(r) = \varphi_{K,2}(r)$ for $r \in (0,1)$.

4.4. **Theorem.** [Vu1, 11.2] *Let $f : \mathbb{B}^n \to \mathbb{R}^n$ be a nonconstant K-quasiregular mapping with $f\mathbb{B}^n \subset \mathbb{B}^n$, $n \geq 2$, and let $\alpha = K^{1/(1-n)}$. Then*

$$\mathrm{th}\frac{\rho(f(x),f(y))}{2} \leq \varphi_{K,n}\left(\mathrm{th}\frac{\rho(x,y)}{2}\right) \leq \lambda_n^{1-\alpha}\left(\mathrm{th}\frac{\rho(x,y)}{2}\right)^{\alpha}, \tag{4.5}$$

$$\rho(f(x),f(y)) \leq K(\rho(x,y)+\log 4), \tag{4.6}$$

for all $x,y \in \mathbb{B}^n$, where λ_n is the same constant as in (1.5). *If $f(0)=0$, then*

$$|f(x)| \leq \lambda_n^{1-\alpha}|x|^{\alpha}, \tag{4.7}$$

for all $x \in \mathbb{B}^n$.

In the case of quasiconformal mappings with $n=2$ formulas (4.5) and (4.7) also occur in [LV, p. 65] and formula (4.6) was rediscovered in [EMM, Theorem 5.1]. Comparing Theorem 4.4 to Theorem 1.10 we see that for $n=2$ the expression $K(\rho(x,y)+\log 4)$ may be replaced with $c(K)\max\{\rho(x,y),\rho(x,y)^{1/K}\}$, which tends to 0 when $x \to y$ and to $\rho(x,y)$ when $K \to 1$, as expected.

4.8. **Lemma.** *For $K>1$ the function*

$$t \mapsto \frac{2\,\mathrm{arth}(\varphi_K(\mathrm{th}\frac{t}{2}))}{\max\{t,t^{1/K}\}},$$

is monotone increasing on $(0,1)$ and decreasing on $(1,\infty)$.

Proof. (1) Fix $K>1$ and consider

$$f(t) = \frac{2\,\mathrm{arth}(\varphi_K(\mathrm{th}\frac{t}{2}))}{t}, \quad t>0.$$

Let $r = \mathrm{th}\frac{t}{2}$. Then $t/2 = \mathrm{arth}(r)$, and t is an increasing function of r for $0<r<1$. Then

$$f(t) = \frac{2\,\mathrm{arth}(\varphi_K(\mathrm{th}\frac{t}{2}))}{t} = \frac{\mathrm{arth}(\varphi_K(r))}{\mathrm{arth}(r)} = F(r).$$

Then by [AVV1, Theorem 10.9(3)], $F(r)$ is strictly decreasing from $(0,1)$ onto (K,∞). Hence $f(t)$ is strictly decreasing from $(0,\infty)$ onto (K,∞).

(2) Next consider

$$g(t) = \frac{2\,\mathrm{arth}(\varphi_K(\mathrm{th}\frac{t}{2}))}{t^{1/K}},$$

and let $r = \mathrm{th}\frac{t}{2}$. Then $t = 2\,\mathrm{arth}(r)$ and

$$g(t) = \frac{2\,\mathrm{arth}(s)}{2^{1/K}(\mathrm{arth}(r))^{1/K}} = \frac{2^{1-1/K}\mathrm{arth}(s)}{(\mathrm{arth}(r))^{1/K}},$$

where $s = \varphi_K(r)$. We next apply [AVV1, Theorem 1.25]. We know $\frac{d}{dr}(\operatorname{arth}(r)) = 1/(1-r^2)$.

Writing $r' = \sqrt{1-r^2}$, $s' = \sqrt{1-s^2}$, we obtain the quotient of the derivatives

$$\begin{aligned}\frac{2^{1-1/K}(1/(1-s^2))\frac{ds}{dr}}{\frac{1}{K}(\operatorname{arth}(r))^{1/K-1}(1/(1-r^2)} &= 2^{1-1/K}\,K\,(\operatorname{arth}(r))^{1-1/K}\frac{r'^2}{s'^2}\frac{1}{K}\frac{ss'^2\,\mathcal{K}(s)^2}{rr'^2\,\mathcal{K}(r)^2}\\ &= 2^{1-1/K}(\operatorname{arth}(r))^{1-1/K}\frac{s\,\mathcal{K}(s)^2}{r\,\mathcal{K}(r)^2}\end{aligned}$$

by [AVV1, appendix E(23)] and l'Hospital rule. By [AVV1, Lemma 10.7(3)], $\mathcal{K}(s)^2/\mathcal{K}(r)^2$ is increasing, since $K>1$, $(\operatorname{arth}(r))^{1-1/K}$ is increasing. Finally, s/r is increasing by [AVV1, Theorem 1.25, E(23)]. So $g(t)$ is increasing in t on $(0,\infty)$.

(3) Fix $K>1$. Clearly

$$\max\{t, t^{1/K}\} = \begin{cases} t^{1/K} & \text{for} \quad 0 \le t \le 1 \\ t \quad \text{for} & 1 \le t < \infty. \end{cases}$$

Thus

$$h(t) = \frac{2\operatorname{arth}(\varphi_K(\operatorname{th}\frac{t}{2}))}{\max\{t,t^{1/K}\}},$$

increases on $(0,1)$ and decreases on $(1,\infty)$. □

4.9**. Proof of Theorem 1.10.** The maximum value of the function considered in Lemma 4.8 is $c(K) = 2\operatorname{arth}(\varphi_K(\operatorname{th}\frac{1}{2}))$. The inequality now follows from Theorem 4.4. □

4.10**. Bounds for the constant** $c(K)$**.** In order to give upper and lower bounds for $c(K)$, we observe that the identity [AVV1, Theorem 10.5(2)] yields the following formula

$$c(K) = 2\operatorname{arth}\left(\varphi_K\left(\frac{1-1/e}{1+1/e}\right)\right) = 2\operatorname{arth}\left(\frac{1-\varphi_{1/K}(1/e)}{1+\varphi_{1/K}(1/e)}\right).$$

A simplification leads to

$$c(K) = -\log\varphi_{1/K}(1/e)\,.$$

Next, from the inequality $\varphi_{1/K}(r) \ge 2^{1-K}(1+r')^{1-K}r^K$ for $K \ge 1, r \in (0,1)$ (cf. [AVV1, Corollary 8.74(2)]) we get with $v = \log(2(1+\sqrt{1-1/e^2})) < 1.3507$

$$\begin{aligned} c(K) &= -\log\varphi_{1/K}(1/e) \le -\log\left(2^{1-K}(1+\sqrt{1-1/e^2})^{1-K}e^{-K}\right)\\ &= v(K-1)+K < 1.3507(K-1)+K. \end{aligned}$$

In order to estimate the constant $c(K)$ from below we need an upper bound for $\varphi_{1/K,2}(r)$, $K>1$, from above. For this purpose we prove the following lemma.

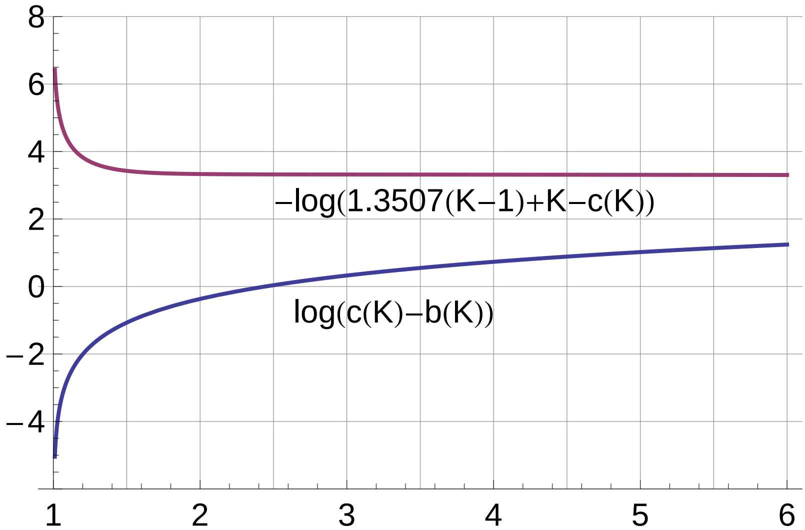

FIGURE 5. Graphical comparison of lower and upper bounds for $c(K)$ with $b(K) = \log(\mathrm{ch}(K\mathrm{arch}(e)))$.

4.11. **Lemma.** *For every integer $n \geq 2$ and each $K > 1$, $r \in (0,1)$, there exists K-quasiconformal maps $g : \mathbb{B}^n \to \mathbb{B}^n$ and $h : \mathbb{B}^n \to \mathbb{B}^n$ with*

(a) $\quad g(0) = 0,\ g(\mathbb{B}^n) = \mathbb{B}^n,\ h(0) = 0,\ h(\mathbb{B}^n) = \mathbb{B}^n$

(b) $\quad g(re_1) = \dfrac{2r^\alpha}{(1+r')^\alpha + (1-r')^\alpha},\ h(re_1) = \dfrac{2r^\beta}{(1+r')^\beta + (1-r')^\beta}$

where $r' = \sqrt{1-r^2}$ and $\alpha = K^{1/(1-n)} = 1/\beta$. In particular, for $n = 2$ and $K > 1$, $r \in (0,1)$

(c) $\quad \varphi_{1/K}(r) \leq \dfrac{2r^K}{(1+r')^K + (1-r')^K}\ ;\ \ \varphi_K(r) \geq \dfrac{2r^{1/K}}{(1+r')^{1/K} + (1-r')^{1/K}}.$

Proof. Fix $r \in (0,1)$. Let $T_a : \mathbb{B}^n \to \mathbb{B}^n$ be a Möbius automorphism with $T_a(a) = 0$ and $T_a(\mathbb{B}^n) = \mathbb{B}^n$. Choose $s \in (0,r)$ such that $T_{se_1}(0) = -T_{se_1}(re_1)$. Then $\rho(0, re_1) = 2\rho(0, se_1)$ [Vu1, (2.17)], or equivalently, $(1+r)/(1-r) = ((1+s)/(1-s))^2$ and hence $s = r/(1+r')$. Consider the K-quasiconformal mapping $f : \mathbb{B}^n \to \mathbb{B}^n$, $f(x) = |x|^{\alpha-1}x$, $\alpha = K^{1/(1-n)}$. Then $f(\pm se_1) = \pm s^\alpha e_1$. The mapping $g = T_{-s^\alpha e_1} \circ f \circ T_{se_1} : \mathbb{B}^n \to \mathbb{B}^n$ satisfies $g(0) = 0$, $g(re_1) = te_1$ where $\rho(-s^\alpha e_1, s^\alpha e_1) = \rho(0, te_1)$ and hence $t = 2r^\alpha/((1+r')^\alpha + (1-r')^\alpha)$ by [Vu1, (2.17)]. The proof for g is complete. For the map h the proof is similar except that we use the K-quasiconformal mapping $m : x \mapsto |x|^{\beta-1}x$, $\beta = 1/\alpha$. Note that $m = f^{-1}$ and $t = 1/\mathrm{ch}(\alpha\ \mathrm{arch}(1/r))$. For the proof of (c) we apply (a), (b) together with [LV, (3.4), p.64]. □

4.12. **Lemma.** *For $K > 1$, $c(K) \geq \log(\mathrm{ch}(K\mathrm{arch}(e))) \geq u(K-1) + 1$, where $u = \mathrm{arch}(e)\mathrm{th}(\mathrm{arch}(e)) > 1.5412$.*

Proof. From Lemma 4.11(c), we know that

$$\begin{aligned}\varphi_{1/K}(1/e) &\le \frac{2/e^K}{(1+\sqrt{1-1/e^2})^K+(1-\sqrt{1-1/e^2})^K}\\ &= \frac{2}{(e+\sqrt{e^2-1})^K+(e-\sqrt{e^2-1})^K},\end{aligned}$$

hence

$$\begin{aligned}c(K) &= -\log\varphi_{1/K}(1/e) \ge -\log\left(\frac{2}{(e+\sqrt{e^2-1})^K+(e-\sqrt{e^2-1})^K}\right)\\ &= \log\left(\frac{(e+\sqrt{e^2-1})^K+(e-\sqrt{e^2-1})^K}{2}\right)\\ &= \log(\mathrm{ch}(K\mathrm{arch}(e))) \ge u(K-1)+1,\end{aligned}$$

where the last inequality follows easily from the mean value theorem, applied to the function $b(K)=\log(\mathrm{ch}(K\mathrm{arch}(e)))$. □

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II

NORM INEQUALITIES FOR VECTOR FUNCTIONS

B. A. BHAYO, V. BOŽIN, D. KALAJ, M. VUORINEN

Abstract. We study vector functions of $\mathbb{R}^n$ into itself, which are of the form $x \mapsto g(|x|)x$, where $g : (0,\infty) \to (0,\infty)$ is a continuous function and call these radial functions. In the case when $g(t) = t^c$ for some $c \in \mathbb{R}$, we find upper bounds for the distance of image points under such a radial function. Some of our results refine recent results of L. Maligranda and S. Dragomir. In particular, we study quasiconformal mappings of this simple type and obtain norm inequalities for such mappings.

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1. Introduction

In 2006 L. Maligranda [M] studied the following function

$$\alpha_p(x,y) = ||x|^{p-1}x - |y|^{p-1}y|\,, \quad p \in \mathbb{R}\,, \tag{1.1}$$

for $x, y \in \mathbb{R}^n \setminus \{0\}$, termed the p-angular distance between x and y. It is clear that α_p satisfies the triangle inequality and thus it defines a metric. Note that $\alpha_0(x,y)$ equals $2\sin(\omega/2)$ where $\omega \in [0,\pi]$ is the angle between the segments $[0,x]$ and $[0,y]$. He proved in [M, Theorem 2] the following theorem in the context of normed spaces.

1.2. **Theorem.**

$$\alpha_p(x,y) \le \begin{cases} (2-p)\dfrac{|x-y|\max\{|x|^p,|y|^p\}}{\max\{|x|,|y|\}} & \text{if} \quad p \in (-\infty,0) \text{ and } x,y \neq 0; \\[2ex] (2-p)\dfrac{|x-y|}{(\max\{|x|,|y|\})^{1-p}} & \text{if} \quad p \in [0,1] \text{ and } x,y \neq 0; \\[2ex] p\,(\max\{|x|,|y|\})^{p-1}|x-y| & \text{if} \quad p \in (1,\infty). \end{cases}$$

Soon thereafter, in 2009, S. Dragomir [D, Theorem 1] refined this result and gave the following upper bound for the p-angular distance for nonzero vectors x, y.

1.3. **Theorem.**

$$\alpha_p(x,y) \le \begin{cases} |x-y|(\max\{|x|,|y|\})^{p-1} + ||x|^{p-1}-|y|^{p-1}|\min\{|x|,|y|\} \\ \quad \textit{if} \quad p\in(1,\infty)\,; \\ \\ \dfrac{|x-y|}{(\min\{|x|,|y|\})^{1-p}} + \left||x|^{1-p}-|y|^{1-p}\right| \min\left\{\dfrac{|x|^p}{|y|^{1-p}}, \dfrac{|y|^p}{|x|^{1-p}}\right\} \\ \quad \textit{if} \quad p\in[0,1]\,; \\ \\ \dfrac{|x-y|}{(\min\{|x|,|y|\})^{1-p}} + \dfrac{||x|^{1-p}-|y|^{1-p}|}{\max\{|x|^{-p}|y|^{1-p}, |y|^{-p}|x|^{1-p}\}} \\ \quad \textit{if} \quad p\in(-\infty,0)\,. \end{cases}$$

Generalizations for operators were discussed very recently in [DFM]. For general information about norm inequalities see [MPF, Chapter XVIII].

Studying sharp constants connected to the p-Laplace operator J. Byström [By, Lemma 3.3] proved in 2005 the following result.

1.4. **Theorem.** *For $p\in(0,1)$ and $x,y\in\mathbb{R}^n$, we have*

$$\alpha_p(x,y) \le 2^{1-p}|x-y|^p$$

with equality for $x=-y$.

In this paper we study a two exponent variant of the function $x\mapsto |x|^{p-1}x$ defined for $a,b>0, x\in\mathbb{R}^n$,

$$\mathcal{A}_{a,b}(x) = \begin{cases} |x|^{a-1}x & \textit{if } |x|\le 1 \\ |x|^{b-1}x & \textit{if } |x|\ge 1. \end{cases} \tag{1.5}$$

This function, like its one exponent version (the special case $a=b$), defines a quasiconformal mapping and it has been used in many examples to illuminate various properties of these maps [Va, p.49]. For instance, if $a\in(0,1)$ the function $\mathcal{A}_{a,b}$ is Hölder-continuous at the origin.

We prove that the change of distance under this function is maximal in the radial direction, up to a constant, in the sense of the next theorem (observe that the points x and z are on the same ray). Note that the result is sharp for $a\to 1$. This result is natural to expect, but the proof is somewhat involved. For brevity we write $\mathcal{A}=\mathcal{A}_{a,b}$ if $0<a\le 1\le b$.

1.6. **Theorem.** *Let $0<a\le 1\le b$ and*

$$C(a,b) = \sup_{|x|\le|y|} Q(x,y),$$

where

$$Q(x,y) = \frac{|\mathcal{A}(x)-\mathcal{A}(y)|}{|\mathcal{A}(x)-\mathcal{A}(z)|}, \quad x,y\in\mathbb{R}^n\setminus\{0\} \ \textit{with} \ x\ne y\,,$$

and

$$z = \frac{x}{|x|}(|x|+|x-y|).$$

Then

$$C(a,b) = \frac{2}{3^a - 1} \ \textit{and} \ \lim_{a\to 1} C(a,b) = 1.$$

Because $\mathcal{A}_{a,b}$ agrees with $x \mapsto |x|^{a-1}x$ in $\mathbb{B}^n$, we can compare Theorem 1.6 to Theorems 1.2, 1.3, and 1.4. We also have the following upper bound for α_p :

1.7. **Theorem.** *For all $x, y \in \mathbb{R}^n$ and $p \in (0,1)$*

$$\alpha_p(x,y) \le |\mathcal{A}_{p,1/p}(x) - \mathcal{A}_{p,1/p}(y)|\,, \tag{1.8}$$

and furthermore, if $|x| \le |y|$, we have also

$$\alpha_p(x,y) \le |\mathcal{A}_{p,1/p}(x) - \mathcal{A}_{p,1/p}(y)| \le \frac{2}{3^p - 1}|\mathcal{A}_{p,1/p}(x) - \mathcal{A}_{p,1/p}(z)| \tag{1.9}$$

where z is as in Theorem 1.6.

For a systematic comparison of the above results, see Section 5 where it is shown that sometimes the bound in Theorem 1.7 is better than the other bounds in Theorems 1.2, 1.3, 1.4.

We also discuss some properties of the distortion function $\varphi_K(r)$ associated with the quasiconformal Schwarz lemma, see [LV].

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2. Preliminary results

We prove here some inequalities for elementary functions that will be applied in later sections. These inequalities deal with the logarithm and some of them may be new results. Note also in the paper [KMV] some elementary Bernoulli type inequalities were proved and used as a key tool. We use the notation sh, ch, th, arsh, arch and arth to denote the hyperbolic sine, cosine, tangent and their inverse functions, respectively.

As well-known, conformal invariants of geometric function theory are on one hand closely linked with function theoretic extremal problems and on the other hand with special functions such as complete elliptic integrals, elliptic functions and hypergeometric functions. The connection between conformal invariants and special functions is provided by conformal maps which can be applied to express maps of quadrilaterals and ring domains onto canonical ring domains such as a rectangle and an annulus.

For example, the quasiconformal version of the Schwarz lemma says that for a K-quasiconformal map of the unit disk $\mathbb{B}^2$ onto itself keeping 0 fixed, we have for all $z \in \mathbb{B}^2$ the sharp bound [LV, p. 64]

$$|f(z)| \le \varphi_K(|z|)\,, \quad \varphi_K(r) = \mu^{-1}(\mu(r)/K) \tag{2.1}$$

where $\mu : (0,1) \longrightarrow (0,\infty)$ is a decreasing homeomorphism defined by

$$\mu(r) = \frac{\pi}{2}\,\frac{\mathcal{K}(r')}{\mathcal{K}(r)}, \quad \mathcal{K}(r) = \int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-r^2x^2)}}\,, \tag{2.2}$$

and where $\mathcal{K}(r)$ is Legendre's complete elliptic integral of the first kind and $r' = \sqrt{1-r^2}$, for all $r \in (0,1)$. The function $\varphi_K(r)$ has numerous applications to quasiconformal mapping theory, see [LV, K, AVV2], which motivates the study of its properties. One of the challenges is to find bounds, in the range $(0,1)$, and yet asymptotically sharp when $K \to 1$. For instance, the change of hyperbolic distances under K-quasiconformal mappings of the unit disk onto itself can be estimated in terms of the function φ_K, see [AVV2, LV].

2.3. **Lemma.** *The following functions are monotone increasing from $(0,\infty)$ onto $(1,\infty)$,*

(1) $\quad f(x) = \dfrac{(1+x)\log(1+x)}{x}\,, \quad$ (2) $\quad g(x) = \dfrac{x}{\log(1+x)}\,,$

(3) *For a fixed $t \in (0,1)$, the function $h(K) = K(1-t^{2/K})$ is monotone increasing on $(1,\infty)$.*

Proof. For the proof of (1) see [KMV, p. 7]. For (2), we get

$$g'(x) = \frac{1}{\log(1+x)} - \frac{x}{(1+x)(\log(1+x))^2} = \frac{(1+x)\log(1+x) - x}{(1+x)(\log(1+x))^2},$$

and $g'(x) > 0$ by (1). Moreover, g tends to 1 and ∞ when x tends 0 and ∞. Proof of (3) follows easily because $x \mapsto (1-a^x)/x$ is decreasing on $(0,1)$ for each $a \in (0,1)$ [AVV2, 1.58(3)]. ☐

2.4. **Corollary.** *For a fixed $x \in (0,1)$, the functions, (1) $f(a) = (1+ax)^{1/a}$, (2) $g(a) = (\log(1+x^a))^{1/a}$ are decreasing and increasing on $(1,\infty)$, respectively. (3) The following inequality holds for $x \geq 0$ and $a \in [0,1]$,*

$$\log(1+x^a) \leq \max\{\log(1+x), \log^a(1+x)\}.$$

2.5. **Lemma.** *For $K > 1\,, r \in (0,1)$, $u = \mathrm{arch}(1/r)/K$, the following functions*

(1) $f(K) = r\,\mathrm{arth}(1/\mathrm{ch(u)})\mathrm{sh}(u)$,
(2) $g(K) = rK\,\mathrm{arth}(1/\mathrm{ch(u)})\mathrm{sh}(u)$

are strictly decreasing and increasing, respectively. Moreover, both functions tend to $\sqrt{1-r^2}\,\mathrm{arth}(r)$ when K tends to 1.

Proof. Differentiating f with respect to K we get

$$f'(K) = -\frac{r\,\mathrm{arth}(1/r)}{K^2}\left(\mathrm{arth}\left(\frac{1}{\mathrm{ch}(u}\right)\mathrm{ch}(u) - 1\right) < 0,$$

$$g'(K) = r\left(\mathrm{ch}\left(\frac{1}{r}\right)\left(1 - \mathrm{arth}\left(\frac{1}{\mathrm{ch}(u)}\right) + K\,\mathrm{arth}\left(\frac{1}{\mathrm{ch}(u)}\right)\mathrm{sh}(u)\right)\right) > 0,$$

respectively. We obtain

$$f(1) = g(1) = r\,\mathrm{arth}(r)\sqrt{(\mathrm{ch}(\mathrm{arch}(1/r)))^2 - 1} = \sqrt{1-r^2}\,\mathrm{arth}(r).$$

□

2.6. **Lemma.** *The following inequality holds for $K \geq 1$ and $t \in [t_0, 1), t_0 = (e-1)/(e+1)$*

$$\log\left(\frac{1+t^{1/K}}{1-t^{1/K}}\right) \leq K \log\left(\frac{1+t}{1-t}\right). \tag{2.7}$$

Proof. Write $h(t) = K\mathrm{arth}(t) - \mathrm{arth}(t^{1/K})$. Differentiating h with respect to t we get,

$$\begin{aligned} h^{'}(t) &= \frac{K}{1-t^2} - \frac{t^{1/K-1}}{K(1-t^{2/K})} = \frac{K^2 t(1-t^{2/K}) - t^{1/K}(1-t^2)}{tK(1-t^2)(1-t^{2/K})} \\ &\geq \frac{Kt(1-t^2) - t^{1/K}(1-t^2)}{tK(1-t^2)(1-t^{2/K})} = \frac{Kt - t^{1/K}}{Kt(1-t^{2/K})} \geq 0. \end{aligned}$$

The first inequality holds by Lemma 2.3(3) and the second one holds when $Kt \geq t^{1/K} \Leftrightarrow t \geq (1/K)^{K/(K-1)} = c_1(K)$. It is easy to see by Lemma 2.3(1) that $c_1(K)$ is decreasing in $(1,\infty)$. We see that $c_1(K) \to 1/e \approx 0.3679\ldots$ and 0 when $K \to 1$ and ∞ respectively, hence $h(t)$ is increasing in $t \geq 1/e$.
We can see that $h(t_0) = K(1 - 2\,\mathrm{arth}(t_0^{1/K})/K)/2$. Now it is enough to prove that $f(K) = 2\,\mathrm{arth}(t_0^{1/K})/K < 1$. Differentiating f with respect to K we get

$$\begin{aligned} f^{'}(K) &= \frac{-2\,\mathrm{arth}(t_0^{1/K})}{K^2} - \frac{2t_0^{1/K}\log(t_0)}{K^3(1-t_0^{2/K})} \\ &= 2(-K(1-t_0^{2/K})\,\mathrm{arth}(t_0^{1/K}) - t_0^{1/K}\log(t_0))/(K^3(1-t_0^{2/K})) \\ &\leq 2(-K(1-t_0^{2/K})\,\mathrm{arth}(t_0) + t_0^{1/K}\log(1/t_0))/(K^3(1-t_0^{2/K})) \\ &= 2(-(K(1-t_0^{2/K})/2)\log\left(\frac{1+t_0}{1-t_0}\right) + t_0^{1/K}\log(1/t_0))/(K^3(1-t_0^{2/K})) \\ &= (t_0^{1/K}\log(1/t_0^2) - K(1-t_0^{2/K}))/(K^3(1-t_0^{2/K})) \\ &= \frac{1}{K^2}\left(\frac{t_0^{1/K}\log(1/t_0^2)}{K(1-t_0^{2/K})} - 1\right) < 0\,, \end{aligned}$$

hence f is a monotone decreasing function from $(1,K)$ onto $(0,1/2)$. This implies the proof. □

2.8. **Lemma.** *The following inequality holds for $K \geq 1$ and $t \in (0, t_0]$,*

$$\log\left(\frac{1+t^{1/K}}{1-t^{1/K}}\right) \leq K\left(\log\left(\frac{1+t}{1-t}\right)\right)^{1/K}. \tag{2.9}$$

Proof. Write

$$F(t)=K-\frac{\log((1+t^{1/K})/(1-t^{1/K}))}{(\log(1+t)/(1-t))^{1/K}}\,.$$

For the proof of (2.9) we show that $F(t)$ is decreasing in t and $F(t_0)\ge 0$. Differentiating F with respect to t we get,

$$F^{'}(t)=\frac{\log\left(\frac{1+t}{1-t}\right)^{(K-1)/K}\left(2t^{1/K}(t^2-1)\log\left(\frac{1+t}{1-t}\right)-2t(t^{2/K}-1)\log\left(\frac{1+t^{1/K}}{1-t^{1/K}}\right)\right)}{Kt(t^2-1)(t^{2/K}-1)}\,.$$

Now we show that

$$t(t^{2/K}-1)\log\left(\frac{1+t^{1/K}}{1-t^{1/K}}\right)\ge t^{1/K}(t^2-1)\log\left(\frac{1+t}{1-t}\right). \tag{2.10}$$

For the proof of (2.10), it is enough to prove that $t(t^{2/K}-1)\ge t^{1/K}(t^2-1)$. We get

$$\begin{aligned} t(t^{2/K}-1)-t^{1/K}(t^2-1) &= (t^{1/K+1}+t)(t^{1/K}-1)-(t^{1/K+1}+t^{1/K})(t-1)\\ &= t^{1/K+1/K+1}+t^{1/K}-t-t^{1/K+1+1}\\ &= t^{1/K}(t^{1/K+1}+1)-t(t^{1/K+1}+1)\\ &= (t^{1/K+1}+1)(t^{1/K}-t)\ge 0, \end{aligned}$$

hence $F(t)$ is decreasing in t, and $F(t_0)$ is positive by the proof of Lemma 2.6. □

2.11. **Corollary.** *The following inequality holds for $K\ge 1$ and $t\in(0,1)$*

$$\log\left(\frac{1+t^{1/K}}{1-t^{1/K}}\right)\le K\max\left\{\left(\log\left(\frac{1+t}{1-t}\right)\right)^{1/K},\ \log\left(\frac{1+t}{1-t}\right)\right\}. \tag{2.12}$$

Proof. The proof follows easily from inequalities (2.7) and (2.9). □

The next function tells us how the hyperbolic distances from the origin are changed under the radial selfmapping of the the unit disk, $z\mapsto|z|^{1/K-1}z, K>1$, which is the restriction of $\mathcal{A}_{1/K,K}(z)$ to the unit disk. See also [BV].

Hyperbolic metric $\rho(x,y),\ x,y\in\mathbb{B}^n$, of the unit ball is define as

$$\mathrm{th}^2\frac{\rho(x,y)}{2}=\frac{|x-y|^2}{|x-y|^2+t^2}\,,\quad t^2=(1-|x|^2)(1-|y|^2)\,,$$

[Vu, pp. 19].

2.13. **Theorem.** *The following inequality holds for $K\ge 1$, $|z|<1$;*

$$\rho(0,\mathcal{A}_{1/K,K}(z))\le K\max\{\rho(0,|z|),\rho(0,|z|)^{1/K}\} \tag{2.14}$$

where ρ is the hyperbolic metric.

Proof. Proof follows easily from inequality (2.12) and the formula $\rho(0,r)=\log((1+r)/(1-r))$. □

2.15. **Remark.** The constant K can not be replaced by $K^{9/10}$ in (2.14), because for $|z| = t_0$, the inequality (2.14) is equivalent to $1 - 2\,\mathrm{arth}(t_0^{1/K})/K^{9/10} \geq 0$. Write $f(K) = 1 - 2\,\mathrm{arth}(t_0^{1/K})/K^{9/10}$, and we get

$$f'(K) = \frac{9\,\mathrm{arth}(t_0^{1/K})}{5K^{19/10}} + \frac{2t^{1/K}\log(t_0)}{K^{29/10}(1-t_0^{2/K})},$$

we see that $f'(1.005) = -0.004 < 0$ and $f(K)$ is not increasing in K.

2.16. **Lemma.** *For $K > 1$ the function*

$$F(r) = \frac{2\mathrm{arth}(1/\mathrm{ch}(\mathrm{arch}(1/r)/K))}{\max\{2\mathrm{arth}(r), (2\mathrm{arth}(r))^{1/K}\}}$$

is monotone increasing in $(0, t_0)$ and decreasing in $(t_0, 1)$.

Proof. (1) Let $u = \mathrm{arch}(1/r)/K$ and

$$f(r) = \frac{\mathrm{arth}(1/\mathrm{ch}(u))}{\mathrm{arth}(r)}.$$

Differentiating f with respect to r we get

$$\begin{aligned} f'(r) &= -\frac{\mathrm{arth}(1/\mathrm{ch}(u))}{(1-r^2)(\mathrm{arth}(r))^2} + \frac{(1/\mathrm{ch}(u))\mathrm{th}(u)}{K\sqrt{1/r-1}\sqrt{1+1/r}\,r^2\,\mathrm{arth}(r)(1-(1/\mathrm{ch}(u))^2)} \\ &= -\frac{Kr\,\mathrm{arth}(1/\mathrm{ch}(u))\mathrm{sh}(u) - \sqrt{1-r^2}\,\mathrm{arth}(r)}{Kr(1-r^2)(\mathrm{arth}(r))^2\mathrm{sh}(u)} \leq 0, \end{aligned}$$

by Lemma 2.5(2), hence f is decreasing in $r \in (0,1)$.
(2) Let

$$g(r) = \frac{2^{1-1/K}\mathrm{arth}(1/\mathrm{ch}(u))}{(\mathrm{arth}(r))^{1/K}}.$$

Differentiating g with respect to r we get

$$g'(r) = \xi\left((1-r^2)\mathrm{arth}(r) - r\sqrt{1-r^2}\,\mathrm{arth}(1/\mathrm{ch}(u))\mathrm{sh}(u)\right) \geq 0$$

by Lemma 2.5(1), here

$$\xi = \frac{2^{1-1/K}(\mathrm{arth}(r))^{-(1+K)/K}}{Kr(1-r^2)^{3/2}\,\mathrm{sh}(u)}.$$

Hence g is increasing in $r \in (0,1)$. We see that $f(t_0) = g(t_0)$. Thus $F(r)$ increases in $r \in (0, t_0)$ and decreases in $t \in (t_0, 1)$. $\square$

It is well-known that for all $K > 1$ *and* $r \in (0,1)$

$$\log\left(\frac{1+\varphi_K(r)}{1-\varphi_K(r)}\right) > K\log\left(\frac{1+r}{1-r}\right) \tag{2.17}$$

[AVV1, (4.5)]. In the next theorem we study a function $p(r)$ which by [AVV2, Theorem 10.14] is a minorant of $\varphi_K(r)$.

2.18. **Theorem.** *The following inequality holds for $K \ge 1$, $r \in (0,1)$,*

$$\log\left(\frac{1+p(r)}{1-p(r)}\right) \le c_3(K)\max\left\{\log\left(\frac{1+r}{1-r}\right), \left(\log\left(\frac{1+r}{1-r}\right)\right)^{1/K}\right\}$$

here $p(r) = 1/\mathrm{ch}(\mathrm{arch}(1/r)/K)$ and $c_3(K) = 2\,\mathrm{arth}(p(t_0))$. Moreover, $c_3(K) \to 1$ when $K \to 1$.

Proof. The inequality follows easily from Lemma 2.16, because the maximum value of the function given in Lemma 2.16 is $c_3(K) = 1/\mathrm{ch}(\mathrm{arch}(1/t_0)/K)$. $\square$

We remark in passing that an inequality similar to (2.18) but with $p(r)$ replaced with $\varphi_K(r)$ and $c_3(K)$ replaced with a constant $c(K)$ was proved in [BV, Lemma 4.8].

3. Quasiinvariance of the distance ratio metric

Our goal in this section is to study how the distances in the j-metric are transformed under the function (1.5) following closely the paper [KMV]. The main result here is Corollary 3.3.

3.1. **Lemma.** *The following inequality holds for $K \ge 1$,*

$$\log\left(1+\frac{|\mathcal{A}_{1/K,K}(x)-\mathcal{A}_{1/K,K}(y)|}{\min\{|\mathcal{A}_{1/K,K}(x)|,|\mathcal{A}_{1/K,K}(y)|\}}\right) \le 2^{1-1/K}\max\{\log^{1/K}(t),\log(t)\} \tag{3.2}$$

here for all $x, y \in \mathbb{B}^n$, here $t = 1+\dfrac{|x-y|}{\min\{|x|,|y|\}}$.

Proof. By Theorem 1.4 and Corollary 2.4(1) we get

$$\begin{aligned} 1+\frac{|\mathcal{A}_{1/K,K}(x)-\mathcal{A}_{1/K,K}(y)|}{\min\{|\mathcal{A}_{1/K,K}(x)|,|\mathcal{A}_{1/K,K}(y)|\}} &= 1+\frac{||x|^{1/K-1}x-|y|^{1/K-1}y|}{\min\{|x|^{1/K},|y|^{1/K}\}} \\ &= 1+\frac{\alpha_{1/K}(x,y)}{\min\{|x|^{1/K},|y|^{1/K}\}} \\ &\le 1+2^{1-1/K}\frac{|x-y|^{1/K}}{\min\{|x|^{1/K},|y|^{1/K}\}} \\ &\le \left(1+\left(\frac{|x-y|}{\min\{|x|,|y|\}}\right)^{1/K}\right)^{2^{1-1/K}}. \end{aligned}$$

Now by Corollary 2.4(3) we get

$$\log\left(1+\frac{|\mathcal{A}_{1/K,K}(x)-\mathcal{A}_{1/K,K}(y)|}{\min\{|\mathcal{A}_{1/K,K}(x)|,|\mathcal{A}_{1/K,K}(y)|\}}\right) \le \log\left(\left(1+\left(\frac{|x-y|}{\min\{|x|,|y|\}}\right)^{1/K}\right)^{2^{1-1/K}}\right)$$

$$\le 2^{1-1/K}\max\left\{\log\left(1+\frac{|x-y|}{\min\{|x|,|y|\}}\right), \left(\log\left(1+\frac{|x-y|}{\min\{|x|,|y|\}}\right)\right)^{1/K}\right\}.$$

□

We denote by ∂G the boundary of a domain G and define
$$d(z) = \min\{|z-m| : m \in \partial G\}.$$
For a domain $G \subset \mathbb{R}^n, G \neq \mathbb{R}^n$, the following formula
$$j(x,y) = \log\left(1 + \frac{|x-y|}{\min\{d(x),d(y)\}}\right), \quad x,y \in G$$
defines j as a metric in G (see [Vu, p.28]).

3.3. **Corollary.** *Let $D = \mathbb{R}^n \setminus \{0\}$, then we have*
$$j_D(\mathcal{A}_{1/K,K}(x), \mathcal{A}_{1/K,K}(y)) \leq 2^{1-1/K} \max\{j_D(x,y), j_D(x,y)^{1/K}\}$$
for all $K \geq 1$, $x,y \in \mathbb{B}^n \cap D$.

Proof. Proof follows from inequality (3.2). □

4. Radial functions

4.1. **Definition.** Let $f : \overline{\mathbb{R}}^n \to \overline{\mathbb{R}}^n$ be a homeomorphism. We say that f is a *radial function* if there exists a homeomorphism $g : (0,\infty) \to (0,\infty)$ such that $f(x) = g(|x|)x$, $x \in \mathbb{R}^n \setminus \{0\}$.

The following functions are examples of the radial functions:

(1) $h(x) = \dfrac{x}{|x|^2}$, $x \in \mathbb{R}^n \setminus \{0\}$, $h(0) = \infty$, $h(\infty) = 0$.

(2) For $a,b > 0$,
$$\mathcal{A}_{a,b}(x) = \begin{cases} |x|^{a-1}x & if\ |x| \leq 1 \\ |x|^{b-1}x & if\ |x| \geq 1. \end{cases}$$

4.2. **Remark.** Properties of $\mathcal{A}$:

(1) For $|x| < 1$ and $a,b,c,d > 0$
$$\begin{aligned}\mathcal{A}_{a,b}(\mathcal{A}_{c,d}(x)) &= \mathcal{A}_{a,b}(|x|^{c-1}x) = ||x|^{c-1}x|^{a-1}|x|^{c-1}x \\ &= |x|^{ac-c}|x|^{c-1}x = |x|^{ac-1}x.\end{aligned}$$

(2) For $|x| > 1$
$$\begin{aligned}\mathcal{A}_{a,b}(\mathcal{A}_{c,d}(x)) &= \mathcal{A}_{a,b}(|x|^{d-1}x) = ||x|^{d-1}x|^{b-1}|x|^{d-1}x \\ &= |x|^{bd-d}|x|^{d-1}x = |x|^{bd-1}x,\end{aligned}$$

(1) and (2) imply that $\mathcal{A}_{a,b}(\mathcal{A}_{c,d}(x)) = \mathcal{A}_{ac,bd}(x)$.

(3) $\mathcal{A}^{-1}_{a,b}(x) = \mathcal{A}_{1/a,1/b}(x)$.

4.3. **Lemma.** [Vu, (1.5)] *An inversion in $S^{n-1}(a,r)$ is defined as,*
$$h(x) = a + \frac{r^2(x-a)}{|x-a|^2}, \ h(a) = \infty, \ h(\infty) = a.$$

Moreover,

$$|h(x)-h(y)| = \frac{r^2|x-y|}{|x-a||y-a|}. \tag{4.4}$$

One of the goals of this section is to find a partial counterpart of the distance formula (4.4) for $\mathcal{A}$ and to prove Theorem 1.6.

4.5. **Lemma.** *Let* $h(w) = r^2 w/|w|^2$, $r > 0$, $w \in \mathbb{R}^n \setminus \{0\}$ *and let* $x, y \in \mathbb{R}^n \setminus \{0\}$ *with* $|x| \le |y|$. *Then with* $\lambda = (|x| + |x-y|)/|x|$ *and* $z = \lambda x$ *we have*

$$|h(x)-h(z)| \le |h(x)-h(y)| \le 3|h(x)-h(z)|.$$

Equality holds in the upper bound for $x = -y$.

Proof. For the proof of first inequality we observe that

$$\begin{aligned} |h(x)-h(z)| &= \left|h(x) - \frac{\lambda}{|\lambda|^2}h(x)\right| = \frac{|\lambda-1|}{\lambda}\frac{r^2}{|x|} \\ &= \frac{r^2|x-y|}{|x|(|x|+|x-y|)} \\ &\le \frac{r^2|x-y|}{|x||y|} = |h(x)-h(y)|, \end{aligned}$$

by triangle inequality.
For the second inequality, we have

$$\begin{aligned} \frac{|h(x)-h(y)|}{|h(x)-h(z)|} &= \frac{|x-y|}{|x||y|}\frac{|x|(|x|+|x-y|)}{|x-y|} \\ &= \frac{|x|}{|y|} + \frac{|x-y|}{|y|} \le 1 + \frac{|x|+|y|}{|y|} \le 3. \end{aligned}$$

Note that here equality holds for $x = -y$. □

4.6. **Lemma.** *The following inequality holds for* $K \ge 1$,

$$||x|^{K-1}x - |y|^{K-1}y| \le e^{\pi(K-1/K)}|x|^{K-1/K}\max\{|x-y|^{1/K}, |x-y|^K\}$$

for all $x, y \in \mathbb{C} \setminus \overline{\mathbb{B}}^2$.

Proof. By [AVV2, Theorem 14.18, (14.4)] we get because $f : x \mapsto |x|^{K-1}x$ is K-quasiconformal [Va, 16.2]

$$\left||x|^{K-1}x, f(0), |y|^{K-1}y, f(\infty)\right| \le \eta^*_{K,2}(|x,0,y,\infty|) = \eta_{K,2}\left(\frac{|x-y|}{|x|}\right).$$

Finally by [AVV2, Theorem 10.24] and [Vu, Remark 10.31] we have

$$\begin{aligned}\left||x|^{K-1}x-|y|^{K-1}y\right| &\le |x|^K\eta_{K,2}\left(\frac{|x-y|}{|x|}\right)\\ &\le \lambda(K)|x|^K\max\{\left(\frac{|x-y|}{|x|}\right)^{1/K},\left(\frac{|x-y|}{|x|}\right)^{K}\}\\ &\le e^{\pi(K-1/K)}|x|^{K-1/K}\max\{|x-y|^{1/K},|x-y|^K\}.\end{aligned}$$

□

4.7. **Lemma.** *The following inequality holds for $K\ge 1$ and for all $x,y\in\mathbb{R}^n\setminus\overline{\mathbb{B}}^n$,*

$$||x|^{\beta-1}x-|y|^{\beta-1}y|\le c(K)|x|^{\beta-\alpha}\max\{|x-y|^{\alpha},|x-y|^{\beta}\}$$

here $c(K)=2^{K-1}K^K\exp(4K(K+1)\sqrt{K-1})$ and $\alpha=K^{1/(1-n)}=1/\beta$.

Proof. Proof follows similarly like Lemma 4.6. □

4.8. **Corollary.** *The following inequalities hold for $K\ge 1$,*

$$\left|\frac{x}{|x|^{1+1/K}}-\frac{y}{|y|^{1+1/K}}\right|\le 2^{1-1/K}\frac{|x-y|^{1/K}}{(|x||y|)^{1/K}} \tag{4.9}$$

for all $x,y\in\mathbb{R}^n\setminus\mathbb{B}^n$,

$$\left|\frac{x}{|x|^{1+\beta}}-\frac{y}{|y|^{1+\beta}}\right|\le\frac{c(K)}{|x|^{\beta-\alpha}}\max\left\{\left(\frac{|x-y|}{|x||y|}\right)^{\alpha},\left(\frac{|x-y|}{|x||y|}\right)^{\beta}\right\} \tag{4.10}$$

for all $x,y\in\mathbb{B}^n$,

$$\left|\frac{x}{|x|^{1+K}}-\frac{y}{|y|^{1+K}}\right|\le\frac{e^{\pi(K-1/K)}}{|x|^{K-1/K}}\max\left\{\left(\frac{|x-y|}{|x||y|}\right)^{1/K},\left(\frac{|x-y|}{|x||y|}\right)^{K}\right\} \tag{4.11}$$

for all $x,y\in\mathbb{B}^2$.

Proof. For the proof of (4.9) we define

$$g(z)=\mathcal{A}_{1/K,K}(h(z))=\frac{z}{|z|^{1+1/K}},\ h(z)=\frac{z}{|z|^2},\ z\in\mathbb{R}^n\setminus\mathbb{B}^n.$$

By Theorem 1.4 and (4.4) we get,

$$|g(x)-g(y)|=\left|\frac{x}{|x|^{1+1/K}}-\frac{y}{|y|^{1+1/K}}\right|\le 2^{1-1/K}|h(x)-h(y)|^{1/K}\le 2^{1-1/K}\frac{|x-y|^{1/K}}{(|x||y|)^{1/K}}.$$

Again for the proof of (4.10) we define

$$g(z)=\mathcal{A}_{\alpha,\beta}(h(z))=\frac{z}{|z|^{1+\beta}},\ h(z)=\frac{z}{|z|^2},\ z\in\mathbb{B}^n.$$

By Lemma 4.7 and (4.4) we get,

$$\begin{aligned}|g(x)-g(y)| &\le c(K)|h(x)|^{\beta-\alpha}\max\{|h(x)-h(y)|^{\alpha},|h(x)-h(y)|^{\beta}\}\\ &= \frac{c(K)}{|x|^{\beta-\alpha}}\max\left\{\left(\frac{|x-y|}{|x||y|}\right)^{\alpha},\left(\frac{|x-y|}{|x||y|}\right)^{\beta}\right\}.\end{aligned}$$

Similarly, inequality (4.11) follows from Lemma 4.6 and (4.4). □

4.12. **Lemma.** *For* $0<a\le 1\le p<\infty$ *and* $0\le s\le 2\pi$ *we have*

$$\frac{(1+p^{2a}-2p^a\cos s)^{1/2}}{(1+X)^a-1}\le\frac{1+p^a}{(2+p)^a-1}\,,\quad X=\sqrt{1+p^2-2p\cos s}\,.$$

Proof. Let

$$f_{p,a}(s)=\frac{1+p^{2a}-2p^a\cos s}{(-1+(1+X)^a)^2}.$$

Then

$$f'_{p,a}(s)=2\frac{(-a(p^{1-a}+p^{a+1}-2p\cos s)/X+(1+X-(1+X)^{1-a}))\sin s}{p^{-a}(1+X)^{1-a}(-1+(1+X)^a)^3}\,.$$

As

$$p^{1-a}+p^{a+1}\le 1+p^2$$

because

$$p^{1+a}(1-p^{1-a})\le 1-p^{1-a}$$

it follows that

$$f'_{p,a}(s)/\sin s\ge 2\frac{-aX+(1+X-(1+X)^{1-a})}{p^{-a}(1+X)^{1-a}(-1+(1+X)^a)^3}.$$

As

$$(1+X)^{1-a}<1+(1-a)X,$$

it follows that

$$f'_{p,a}(s)=0 \text{ if and only } s=0 \text{ or } s=\pi.$$

For $s=0$, the function $f_{p,a}$ achieves its minimum

$$f_{p,a}(0)=\left(\frac{-1+p^a}{-1+p^a}\right)^2=1$$

and for $s=\pi$ its maximum

$$f_{p,a}(\pi)=\left(\frac{1+p^a}{-1+(2+p)^a}\right)^2.$$

□

4.13. **Lemma.** *For* $p\ge 1$, *and* $0<d\le 1$ *we have,*

$$\frac{1+p^d}{(2+p)^d-1}\le\frac{2}{3^d-1}. \tag{4.14}$$

Proof. Let
$$h(p) = (3^d - 1)(1 + p^d) - 2((2 + p)^d - 1).$$
We need to show that $h(p) \le 0$. We get
$$h'(p) = d\left((3^d - 1)p^{d-1} - 2(2 + p)^{d-1}\right).$$
Then
$$h'(p) \le 0 \Leftrightarrow \left(\frac{2}{p} + 1\right)^{1-d} \le \frac{2}{3^d - 1}.$$
Since
$$\left(\frac{2}{p} + 1\right)^{1-d} \le 3^{1-d}.$$
We need to show that
$$3^{1-d} \le \frac{2}{3^d - 1},$$
but this is equivalent to
$$3^d \le 3,$$
which is obviously true. Thus $h'(p) \le 0$, and consequently $h(p) \le h(1) = 0$ and this inequality coincides with (4.14). ☐

4.15. **Proof of Theorem 1.6.** *The case* $1 \le |x| \le |y|$. Let us show that $Q(x, y) \le 1$. Without loss of generality, we can assume that $x = r$ and z are positive real numbers, and $y = Re^{it}$. Then $z = r + |r - Re^{it}|$. Let
$$p = \frac{R}{r},$$
then $p \ge 1$. Next we have:
$$\begin{aligned}
\frac{|\mathcal{A}(x) - \mathcal{A}(y)|}{|\mathcal{A}(x) - \mathcal{A}(z)|} &= \frac{|1 - p^b e^{it}|}{(1 + |1 - pe^{it}|)^b - 1} \\
&\le \frac{|1 - p^b e^{it}|}{(1 + |1 - p|)^{b-1}(1 + |1 - pe^{it}|) - 1} \\
&= \frac{|1 - p^b e^{it}|}{p^{b-1}(1 + |1 - pe^{it}|) - 1} \\
&= \frac{|1 - p^b e^{it}|}{p^{b-1} - 1 + |p^{b-1} - p^b e^{it}|} \\
&= \frac{|1 - p^{b-1} + p^{b-1} - p^b e^{it}|}{p^{b-1} - 1 + |p^{b-1} - p^b e^{it}|} \le 1.
\end{aligned}$$
If $|x| \le |y| \le 1$ and $|z| \le 1$, then by Lemmas 4.12 and 4.13 we get
$$\begin{aligned}
\frac{|\mathcal{A}(x) - \mathcal{A}(y)|}{|\mathcal{A}(x) - \mathcal{A}(z)|} &= \frac{|r^a - R^a e^{it}|}{(r + |r - Re^{it}|)^a - r^a} \\
&\le \frac{1 + p^a}{(2 + p)^a - 1} \le \frac{2}{3^a - 1}.
\end{aligned}$$

If $|x| \leq |y| \leq 1$ and $|z| \geq 1$, then it follows from Lemmas 4.12 and 4.13 and $|z|^b \geq |z|^a$ that

$$\begin{aligned}\frac{|\mathcal{A}(x)-\mathcal{A}(y)|}{|\mathcal{A}(x)-\mathcal{A}(z)|} &\leq \frac{|r^a - R^a e^{it}|}{(r+|r-e^{it}|)^a - r^a}\\ &\leq \frac{1+p^a}{(2+p)^a-1} \leq \frac{2}{3^a-1}.\end{aligned}$$

Next, *The case* $|x| \leq 1 \leq |y|$ *and* $r^{a-1} > R^{b-1}$. Then there holds

$$Q(x,y) \leq \frac{2}{3^a-1}.$$

First of all

$$\begin{aligned}\frac{|\mathcal{A}(x)-\mathcal{A}(y)|}{|\mathcal{A}(x)-\mathcal{A}(z)|} &= \frac{|r^a - R^b e^{it}|}{(r+|r-Re^{it}|)^b - r^a}\\ &= \frac{|\alpha - e^{it}|}{(\beta+|\beta-e^{it}|)^b-\alpha},\end{aligned}$$

where $\alpha = r^a/R^b$ and $\beta = r/R$. Take the continuous function $k(q) = \beta^q$, $a \leq q \leq 1$. Since

$$\beta = k(1) = \frac{r}{R} \leq \alpha = \frac{r^a}{R^b} \leq k(a) = \frac{r^a}{R^a},$$

it follows that there exists a constant c with $a \leq c \leq 1$ such that $k(c) = \beta^c = \alpha$. Then

$$\begin{aligned}\frac{|\alpha - Re^{it}|}{(\beta+|\beta-e^{it}|)^b-\alpha} &= \frac{|\beta^c - e^{it}|}{(\beta+|\beta-e^{it}|)^b-\beta^c}\\ &\leq \frac{|\beta^c - e^{it}|}{(\beta+|\beta-e^{it}|)^c-\beta^c}\\ &\leq \frac{1+\beta^c}{(2+\beta)^c-\beta^c}\\ &\leq \frac{2}{3^c-1} \leq \frac{2}{3^a-1},\end{aligned}$$

the second inequality follows from Lemma 4.12 and the third inequality follows from Lemma 4.13 by taking $p = 1/\beta$ and $c = d$.

The case $|x| \leq 1 \leq |y|$ *and* $r^{a-1} < R^{b-1}$. We get

$$\begin{aligned}\frac{|\mathcal{A}(x)-\mathcal{A}(y)|}{|\mathcal{A}(x)-\mathcal{A}(z)|} &= \frac{|r^a - R^b e^{it}|}{(r+|r-Re^{it}|)^b - r^a}\\ &= \frac{|\alpha - e^{it}|}{(\beta+|\beta-e^{it}|)^b-\alpha}\\ &= \frac{|\alpha - e^{it}|}{(\beta+|\beta-e^{it}|)-\alpha} < 1\end{aligned}$$

because $b > 1$ and $\alpha < \beta$.

Finally, let us show that $C(a,b) \geq 2/(3^a - 1)$. Suppose that $x \in \mathbb{R}^n \setminus \{0\}$ is such that $3|x| < 1$, i.e. $0 < |x| < 1/3$ and $y = -x$. Then $z = x(|x| + |x-y|)/|x| = 3x$ and

$$Q(x,-x) = \frac{2|x|^a}{(3|x|)^a - |x|^a} = \frac{2}{3^a - 1},$$

and hence $C(a,b) \geq 2/(3^a - 1)$. $\square$

5. Refinements

5.1. Proof of Theorem 1.7. For $|x|, |y| < 1$ we have

$$\alpha_p(x,y) = \left||x|^{p-1}x - |y|^{p-1}y\right| = \left|\mathcal{A}_{p,1/p}(x) - \mathcal{A}_{p,1/p}(y)\right|.$$

Consider the case $|x| < 1 < |y|$. It is obvious that

$$\cos\theta \leq 1 < \frac{|x|^{-p}(|y|^{1/p} + |y|^p)}{2},$$

this is equivalent to

$$\cos\theta \leq 1 < \frac{(|y|^{1/p} - |y|^p)(|y|^{1/p} + |y|^p)}{2|x|^p(|y|^{1/p} - |y|^p)}$$

$\Longleftrightarrow$

$$2|x|^p|y|^{1/p}\cos\theta - 2|x|^p|y|^p\cos\theta < |y|^{2/p} - |y|^{2p}$$

$\Longleftrightarrow$

$$|y|^{2p} - 2|x|^{p-1}|y|^{p-1}|x||y|\cos\theta < |y|^{2/p} - 2|x|^{p-1}|y|^{1/p-1}|x||y|\cos\theta$$

$\Longleftrightarrow$

$$||x|^{p-1}x|^2 + ||y|^{p-1}y|^2 - 2|x|^{p-1}|y|^{p-1}x\,y < ||x|^{p-1}x|^2 + ||y|^{1/p-1}y|^2 - 2|x|^{p-1}|y|^{1/p-1}x\,y$$

$\Longleftrightarrow$

$$\left||x|^{p-1}x - |y|^{p-1}y\right|^2 < \left||x|^{p-1}x - |y|^{1/p-1}y\right|^2 = \left|\mathcal{A}_{p,1/p}(x) - \mathcal{A}_{p,1/p}(y)\right|^2.$$

Consider now the case $1 < |x| < |y|$. Starting with the observation that the function $t \mapsto t^{1/p} - t^p$ is increasing for $t > 1$ when $p \in (0,1)$, we see that

$$\frac{|x|^{1/p}}{|x|^p}\left(\left(\frac{|y|}{|x|}\right)^{1/p} - 1\right) > \left(\left(\frac{|y|}{|x|}\right)^p - 1\right) \Longleftrightarrow (|y|^{1/p} - |x|^{1/p})^2 > (|y|^p - |x|^p)^2$$

$\Longleftrightarrow$

$$|x|^{2/p} - |y|^{2p} + |y|^{2/p} - |x|^{2p} > 2|x|^{1/p}|y|^{1/p} - 2|x|^p|y|^p.$$

Now it is clear that

$$\cos\theta \leq 1 < \frac{|x|^{2/p} - |y|^{2p} + |y|^{2/p} - |x|^{2p}}{2|x|^{1/p}|y|^{1/p} - 2|x|^p|y|^p},$$

this is equivalent to

$$|x|^{2p} + |y|^{2p} - 2|x|^p|y|^p\cos\theta < |x|^{2/p} + |y|^{2/p} - 2|x|^{1/p}|y|^{1/p}\cos\theta$$

$$\Longleftrightarrow$$

$$||x|^{p-1}x|^2+||y|^{p-1}y|^2-2|x|^{p-1}|y|^{p-1}x\,y < ||x|^{1/p-1}x|^2+||y|^{1/p-1}y|^2-2|x|^{1/p-1}|y|^{1/p-1}x\,y$$

$$\Longleftrightarrow$$

$$||x|^{p-1}x|^2+||y|^{p-1}y|^2-2|x|^{p-1}|y|^{p-1}x\,y < ||x|^{p-1}x|^2+||y|^{1/p-1}y|^2-2|x|^{p-1}|y|^{1/p-1}x\,y$$

$$\Longleftrightarrow$$

$$\left||x|^{p-1}x-|y|^{p-1}y\right|^2 < \left||x|^{1/p-1}x-|y|^{1/p-1}y\right|^2 = \left|\mathcal{A}_{p,1/p}(x)-\mathcal{A}_{p,1/p}(y)\right|^2. \quad \square$$

5.2. **Comparison of the bounds.** In what follows, we use the symbols M, D, B, K for the bounds given by Theorems 1.2, 1.3, 1.4, 1.6, respectively. In the case of the complex plane, we will show by numerical examples that each of these four bounds can occur as minimal. To this end, for each of the symbols M, D, B, K, we give a table of four x, y pairs and the corresponding upper bound values associated with the four symbols M, D, B, K, such that the bound associated with the symbol in question is the least one. For the computation of the K bound it should be observed that in Theorem 1.7 we have the constraint $|x| \leq |y|$. If this is not the situation to begin with, we have swapped the points for computation. In Tables 1-4 the parameter $p = 0.5$.

TABLE 1. Sample points with $K < \min\{B, D, M\}$.

k	x_k	y_k	B	D	M	K
1	$-2.00-2.65i$	$2.65-2.65i$	3.0496	143.4290	3.6030	2.6591
2	$2.25-0.75i$	$2.65+1.30i$	2.0438	38.9860	1.8236	1.5158
3	$1.35+0.50i$	$1.95-0.65i$	1.6107	14.8000	1.3571	1.2768
4	$1.10+2.30i$	$-2.40+2.10i$	2.6479	82.4142	2.9447	2.3646

TABLE 2. Sample points with $D < \min\{B, K, M\}$.

k	x_k	y_k	B	K	M	D
1	$0.80-0.50i$	$-1.80+1.45i$	3.6968	45.3884	3.2066	2.5495
2	$2.25-0.75i$	$0.00-0.05i$	15.5147	32.3855	2.7931	2.6174
3	$2.55+1.50i$	$-1.10+1.70i$	2.8148	76.9511	3.1879	2.7039
4	$-2.70+3.00i$	$1.50+0.60i$	4.2727	106.6320	3.6118	3.1104

In conclusion, Tables 1-4 demonstrate that each of the above four bounds is sometimes smaller than the minimum of the other three bounds. Some further results, in addition to Theorems 1.2, 1.3, 1.4, 1.6 can be found in the papers [M] and [D]. The tables were compiled with the help of the Mathematica software package.

In Tables 5-7 we compare (4.9), M and D, for $x, y \in \mathbb{R}^n \setminus \mathbb{B}^n$, $p = -0.6$.

TABLE 3. Sample points with $B < \min\{D, K, M\}$.

k	x_k	y_k	D	K	M	B
1	$-2.45 - 2.205i$	$-1.2 + 0.55i$	2.92	43.55	2.42	2.40
2	$-1.65 + 1.45i$	$2.15 + 2.75i$	3.01	92.27	3.22	2.83
3	$-0.2 - 3i$	$-0.4 + 0.2i$	5.21	34.64	2.77	2.53
4	$0.9 - 2.9i$	$-1.4 + 1.35i$	3.74	115.15	4.16	3.11

TABLE 4. Sample points with $M < \min\{B, D, K\}$.

k	x_k	y_k	B	D	K	M
1	$0.30 + 0.50i$	$-0.15 + 2.95i$	2.23	3.69	23.73	2.17
2	$0.95 + 1.85i$	$0.55 + 1.55i$	1.00	0.53	5.18	0.52
3	$1.60 - 0.25i$	$1.10 - 0.35i$	1.01	0.64	3.93	0.60
4	$-0.60 + 0.30$	$-3.00 + 1.95i$	2.41	4.02	32.84	2.31

TABLE 5. Sample points with $M < \min\{(4.9), D\}$.

k	x_k	y_k	(4.9)	D	M
1	$2.25 + 2.45i$	$-0.01 + 2.95i$	0.27	0.27	0.24
2	$-2.60 + 0.40i$	$-0.70 - 0.60i$	1.23	3.30	1.19
3	$0.75 - 0.75i$	$-2.90 - 2.50i$	1.32	4.53	1.23
4	$2.90 + 1.90i$	$1.20 + 0.85i$	0.75	1.67	0.71

TABLE 6. Sample points with $(4.9) < \min\{D, M\}$.

k	x_k	y_k	D	M	(4.9)
1	$-2.60 - 1.05i$	$-1.35 - 1.40i$	0.70	0.65	0.56
2	$-0.45 - 1.05i$	$2.35 + 1.80i$	3.95	1.83	1.46
3	$-1.15 + 2.30i$	$2.70 + 0.65i$	0.99	2.12	0.96
4	$-0.10 + 1.25i$	$2.90 + 2.45i$	0.71	0.94	0.60

TABLE 7. Sample points with $D < \min\{(4.9), M\}$.

k	x_k	y_k	(4.9)	M	D
1	$1.35 + 2.95i$	$-1.35 + 2.90i$	0.59	1.07	0.43
2	$-0.80 + 2.75i$	$-1.85 + 2.40i$	0.38	0.49	0.25
3	$2.65 + 2.20i$	$-2.45 + 2.40i$	0.49	0.64	0.49
4	$1.20 - 0.70i$	$1.30 + 0.70i$	1.05	1.96	0.91

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B. A. Bhayo and M. Vuorinen
Department of Mathematics
University of Turku
20014 Turku
Finland
barbha@utu.fi

V. Božin
Faculty of Mathematics
University of Belgrade
Studentski trg 16, Belgrade
Serbia
bozinv@turing.mi.sanu.ac.rs

D. Kalaj
University of Montenegro
Faculty of Mathematics
Dzordza Vaingtona b.b.
Podgorica

Montenegro
davidk@ac.me

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III

INEQUALITIES FOR EIGENFUNCTIONS OF THE p-LAPLACIAN

BARKAT ALI BHAYO AND MATTI VUORINEN

ABSTRACT. Motivated by the work of P. Lindqvist, we study eigenfunctions of the one-dimensional p-Laplace operator, the $\sin_p$ functions, and prove several inequalities for these and p-analogues of other trigonometric functions and their inverse functions. Similar inequalities are given also for the p-analogues of the hyperbolic functions and their inverses.

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1. INTRODUCTION

In a highly cited paper P. Lindqvist [L] studied generalized trigonometric functions depending on a parameter $p > 1$ which for the case $p = 2$ reduce to the familiar functions. Numerous later authors, see e.g. [BEM1, BEM2, DM, LP] and the bibliographies of these papers, have extended this work in various directions including the study of generalized hyperbolic functions and their inverses. Our goal here is to study these p-trigonometric and p-hyperbolic functions and to prove several inequalities for them.

For the statement of some of our main results we introduce some notation and terminology for classical special functions, such as the classical *gamma function* $\Gamma(x)$, the *psi function* $\psi(x)$ and the *beta function* $B(x, y)$. For $\mathrm{Re}\, x > 0$, $\mathrm{Re}\, y > 0$, these functions are defined by

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1}\, dt,\ \ \psi(x) = \frac{\Gamma'(x)}{\Gamma(x)},\ \ B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x + y)},$$

respectively.

Given complex numbers a, b and c with $c \neq 0, -1, -2, \ldots$, the *Gaussian hypergeometric function* is the analytic continuation to the slit place $\mathbb{C} \setminus [1, \infty)$ of the series

$$F(a, b; c; z) = {}_2F_1(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a, n)(b, n)}{(c, n)} \frac{z^n}{n!}, \qquad |z| < 1.$$

Here $(a, 0) = 1$ for $a \neq 0$, and (a, n) is the *shifted factorial function* or the *Appell symbol*

$$(a, n) = a(a + 1)(a + 2) \cdots (a + n - 1)$$

for $n \in \mathbb{Z}_+$. The hypergeometric function has numerous special functions as its special or limiting cases, see [AS].

We start by discussing eigenfunctions of the so-called one-dimensional p-Laplacian Δ_p on $(0,1),\ p \in (1,\infty)$. The eigenvalue problem [DM]

$$-\Delta_p u = -\left(|u'|^{p-2}u'\right)' = \lambda |u|^{p-2}u, \quad u(0)=u(1)=0,$$

has eigenvalues

$$\lambda_n = (p-1)(n\pi_p)^p,$$

and eigenfunctions

$$\sin_p(n\pi_p\, t), \quad n \in \mathbb{N},$$

where $\sin_p$ is the inverse function of $\arcsin_p$ which is defined below, and

$$\pi_p = \frac{2}{p}\int_0^1 (1-s)^{-1/p} s^{1/p-1} ds = \frac{2}{p}\, B\left(1-\frac{1}{p},\frac{1}{p}\right) = \frac{2\pi}{p\,\sin(\pi/p)}\,.$$

Motivated by P. Lindqvist's work, P. J. Bushell and D. E. Edmunds [BE] found recently many new results for these generalized trigonometric functions. Some authors also considered various other p-analogues of trigonometric and hyperbolic functions and their inverses. In particular, they considered the following homeomorphisms

$$\sin_p : (0,a_p) \to I, \quad \cos_p : (0,a_p) \to I, \quad \tan_p : (0,b_p) \to I,$$
$$\sinh_p : (0,c_p) \to I, \quad \tanh_p : (0,\infty) \to I\,,$$

where $I=(0,1)$ and

$$a_p = \frac{\pi_p}{2},\ b_p = \frac{1}{2p}\left(\psi\left(\frac{1+p}{2p}\right) - \psi\left(\frac{1}{2p}\right)\right) = 2^{-1/p}F\left(\frac{1}{p},\frac{1}{p};1+\frac{1}{p};\frac{1}{2}\right)\,,$$
$$c_p = \left(\frac{1}{2}\right)^{1/p} F\left(1\,,\frac{1}{p};1+\frac{1}{p}\,;\frac{1}{2}\right)\,.$$

For $x \in I$, their inverse functions are defined as

$$\begin{aligned}
\arcsin_p x &= \int_0^x (1-t^p)^{-1/p} dt = x\,F\left(\frac{1}{p},\frac{1}{p};1+\frac{1}{p};x^p\right)\\
&= x(1-x^p)^{(p-1)/p} F\left(1,1;1+\frac{1}{p};x^p\right)\,,\\
\arctan_p x &= \int_0^x (1+t^p)^{-1} dt = x\,F\left(1,\frac{1}{p};1+\frac{1}{p};-x^p\right)\,,\\
\operatorname{arsinh}_p x &= \int_0^x (1+t^p)^{-1/p} dt = xF\left(\frac{1}{p}\,,\frac{1}{p};1+\frac{1}{p};-x^p\right)\,,\\
\operatorname{artanh}_p x &= \int_0^x (1-t^p)^{-1} dt = xF\left(1\,,\frac{1}{p};1+\frac{1}{p};x^p\right)\,,
\end{aligned}$$

and by [BE, Prop 2.2] $\arccos_p x = \arcsin_p((1-x^p)^{1/p})$. For the particular case $p=2$ one obtains the familiar elementary functions.

The paper is organized into sections as follows. Section 1, the introduction, contains the statements of our main results. In Section 2 we give some inequalities for the p-analogues of trigonometric and hyperbolic functions. Section 3 contains the proofs of our main results and some identities. Finally in Section 4 we give some functional inequalities for elementary functions and Section 5 contains two small tables with a few values of the function $\sin_p$ and related functions compiled with the Mathematica® software.

Some of the main results are the following theorems.

1.1. **Theorem.** *For $p > 1$ and $x \in (0, 1)$, we have*

(1) $$\left(1 + \frac{x^p}{p(1+p)}\right) x < \operatorname{arcsin}_p x < \frac{\pi_p}{2}\, x,$$

(2) $$\left(1 + \frac{1-x^p}{p(1+p)}\right) (1-x^p)^{1/p} < \operatorname{arccos}_p x < \frac{\pi_p}{2}\, (1-x^p)^{1/p},$$

(3) $$\frac{(p(1+p)(1+x^p)+x^p)x}{p(1+p)(1+x^p)^{1+1/p}} < \operatorname{arctan}_p x < 2^{1/p}\; b_p \left(\frac{x^p}{1+x^p}\right)^{1/p}.$$

1.2. **Theorem.** *For $p > 1$ and $x \in (0, 1)$, we have*

(1.3)
$$z\left(1 + \frac{\log(1+x^p)}{1+p}\right) < \operatorname{arsinh}_p x < z\left(1 + \frac{1}{p}\log(1+x^p)\right),\ z = \left(\frac{x^p}{1+x^p}\right)^{1/p},$$

(1.4) $$x\left(1 - \frac{1}{1+p}\log(1-x^p)\right) < \operatorname{artanh}_p x < x\left(1 - \frac{1}{p}\log(1-x^p)\right).$$

The next result provides several families of inequalities for elementary functions.

1.5. **Theorem.** *For $x > 0$ and $z = \pi x/2$, the function $g(p) = f(z^p)^{1/p}$ is decreasing in $p \in (0, \infty)$, where $f(z) \in \{\operatorname{arsinh}(z), \operatorname{arcosh}(z), \operatorname{artanh}(2z/\pi)\}$.*

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2. Preliminaries and definitions

For convenience, we use the notation $\mathbb{R}_+ = (0, \infty)$.

2.1. **Lemma.** [N2, Thm 2.1] *Let $f : \mathbb{R}_+ \to \mathbb{R}_+$ be a differentiable, log-convex function and let $a \geq 1$. Then $g(x) = (f(x))^a/f(a\,x)$ decreases on its domain. In particular, if $0 \leq x \leq y$, then the following inequalities*

$$\frac{(f(y))^a}{f(a\,y)} \leq \frac{(f(x))^a}{f(a\,x)} \leq (f(0))^{a-1}$$

hold true. If $0 < a \le 1$*, then the function* g *is an increasing function on* $\mathbb{R}_+$ *and inequalities are reversed.*

We recall the following identity [AS, 15.3.5]:

$$F(a,b;c;z) = (1-z)^{-b} F(b, c-a; c; -z/(1-z))\,. \tag{2.2}$$

For the following lemma see [AVV1, Theorems 1.19(10), 1.52(1), Lemmas, 1.33, 1.35].

2.3. **Lemma.** (1) *For* $a, b, c > 0$*,* $c < a+b$*, and* $|x| < 1$*,*

$$F(a,b;c;x) = (1-x)^{c-a-b} F(c-a, c-b; c; x)\,.$$

(2) *For* $a, x \in (0,1)$*, and* $b, c \in (0,\infty)$

$$F(-a,b;c;x) < 1 - \frac{a\,b}{c}\,x\,.$$

(3) *For* $a, x \in (0,1)$*, and* $b, c \in (0,\infty)$

$$F(a,b;c;x) + F(-a,b;c;x) > 2\,.$$

(4) *Let* $a, b, c \in (0,\infty)$ *and* $c > a+b$*. Then for* $x \in [0,1]$*,*

$$F(a,b;c;x) \le \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\,.$$

(5) *For* $a, b > 0$*, the following function*

$$f(x) = \frac{F(a,b;a+b;x) - 1}{\log(1/(1-x))}$$

is strictly increasing from $(0,1)$ *onto* $(a\,b/(a+b), 1/B(a,b))$*.*

2.4. **Lemma.** *For* $p > 1$ *and* $x \in (0,1)$*, the functions*

$$(\mathrm{arcsin}_p(x^k))^{1/k} \quad \text{and} \quad (\mathrm{artanh}_p(x^k))^{1/k}$$

are decreasing in $k \in (0,\infty)$*, also*

$$(\mathrm{arctan}_p(x^k))^{1/k} \quad \text{and} \quad (\mathrm{arsinh}_p(x^k))^{1/k}$$

are increasing in $k \in (0,\infty)$*.*

In particular, for $k \ge 1$

$$\sqrt[k]{\mathrm{arcsin}_p(x^k)} \le \mathrm{arcsin}_p(x) \le (\mathrm{arcsin}_p \sqrt[k]{x})^k\,,$$

$$\sqrt[k]{\mathrm{artanh}_p(x^k)} \le \mathrm{artanh}_p(x) \le (\mathrm{artanh}_p \sqrt[k]{x})^k\,,$$

$$(\mathrm{arsinh}_p \sqrt[k]{x})^k \le \mathrm{arsinh}_p(x) \le \sqrt[k]{\mathrm{arsinh}_p(x^k)}\,,$$

$$(\mathrm{arctan}_p \sqrt[k]{x})^k \le \mathrm{arctan}_p(x) \le \sqrt[k]{\mathrm{arctan}_p(x^k)}\,.$$

Proof. Let let

$$f(k) = \big(E(x^k)\big)^{1/k}, \qquad E(x) = \int_0^x g(t)\,dt, \quad E = E(x^k).$$

We get

$$f' = -E^{1/k}\log E\frac{1}{k^2} + \frac{1}{k}E^{1/k-1}E'x^k\log x = \frac{E^{1/k}}{k^2}\left(-\log\frac{E}{x^k} - \left(x^k\frac{E'}{E} - 1\right)\log\frac{1}{x^k}\right).$$

If $g \geq 1$, then

$$\frac{E}{x^k} = \frac{1}{x^k}\int_0^{x^k} g(t)\,dt \geq 1.$$

If g is increasing, then

$$E' - \frac{E}{x^k} = g(x^k) - \frac{1}{x^k}\int_0^{x^k} g(t)\,dt \geq 0,$$

so that $x^k\frac{E'}{E} - 1 \geq 0$. Thus $f' \leq 0$ under these assumptions.

For arcsin_p and artanh_p, g is $(1-t^p)^{-1/p}$ and $(1-t^p)^{-1}$, so the conditions are clearly satisfied. Additionally, we see that for arsinh_p and $\arctan_p$ the conditions $g \leq 1$ and g is decreasing and this conclude that $f' \geq 0$. This completes the proof. □

2.5. **Theorem.** *For $p > 1$ and $r, s \in (0,1)$, the following inequalities hold:*

(1) $\mathrm{arcsin}_p(r\,s) \leq \sqrt{\mathrm{arcsin}_p(r^2)\,\mathrm{arcsin}_p(s^2)} \leq \mathrm{arcsin}_p(r)\,\mathrm{arcsin}_p(s)\,,$

(2) $\mathrm{artanh}_p(r\,s) \leq \sqrt{\mathrm{artanh}_p(r^2)\,\mathrm{artanh}_p(s^2)} \leq \mathrm{artanh}_p(r)\,\mathrm{artanh}_p(s)\,,$

(3) $\mathrm{arsinh}_p(r^2)\,\mathrm{arsinh}_p(s^2) \leq \sqrt{\mathrm{arsinh}_p(r^2)\,\mathrm{arsinh}_p(s^2)} \leq \mathrm{arsinh}_p(r\,s)\,,$

(4) $\arctan_p(r)\arctan_p(s) \leq \sqrt{\arctan_p(r^2)\arctan_p(s^2)} \leq \arctan_p(r\,s)\,.$

Proof. Let $h(x) = \log f(e^x)$. Then h is convex (in the C^2 case) when $h'' \geq 0$, i.e. iff

$$\frac{f}{y}(f' + yf'') \geq (f')^2,$$

where $y = e^x$ and the function is evaluated at y. If $f'' \geq 0$, then

$$\frac{f}{y} \geq f'(0)\,,$$

so a sufficient condition for convexity is $f'(0)(f'+yf'') \geq (f')^2$. If $f'' \leq 0$, the reverse holds, so a sufficient condition for concavity is $f'(0)(f'+yf'') \leq (f')^2$. Suppose

$$f(x) = \int_0^x g(t)\,dt.$$

Then $f' = g$ and $f'' = g'$. Then one easily checks that h is convex in case g is $(1-t^p)^{-1/p}$ and $(1-t^p)^{-1}$, and concave for g equal to $(1+t^p)^{-1/p}$ and $(1+t^p)^{-1}$. Now proof follows easily from Lemma 2.4. $\square$

2.6. **Lemma.** *For $k, p > 1$ and $r, s \in (0,1)$ with $r \geq s$, we have*

$$\begin{aligned}
\left(\frac{\operatorname{arcsin}_p(s)}{\operatorname{arcsin}_p(r)}\right)^k &\leq \frac{\operatorname{arcsin}_p(s^k)}{\operatorname{arcsin}_p(r^k)}\,,\\
\left(\frac{\operatorname{artanh}_p(s)}{\operatorname{artanh}_p(r)}\right)^k &\leq \frac{\operatorname{artanh}_p(s^k)}{\operatorname{artanh}_p(r^k)}\,,\\
\frac{\operatorname{arsinh}_p(s^k)}{\operatorname{arsinh}_p(r^k)} &\leq \left(\frac{\operatorname{arsinh}_p(s)}{\operatorname{arsinh}_p(r)}\right)^k\,.
\end{aligned}$$

Proof. For $x > 0$, the following functions

$$u(x) = \operatorname{arcsin}_p(e^{-x})\,, \quad v(x) = \operatorname{artanh}_p(e^{-x})\,,$$
$$w_1(x) = 1/\operatorname{arsinh}_p(e^{-x})$$

are log-convex by the proof of Theorem 2.5. Let $x < y$, $e^{-x} = r \geq s = e^{-y}$, now inequalities follow from Lemma 2.1. $\square$

2.7. **Lemma.** [K, Thm 2, p.151] *Let $J \subset \mathbb{R}$ be an open interval, and let $f : J \to \mathbb{R}$ be strictly monotonic function. Let $f^{-1} : f(J) \to J$ be the inverse to f then*

1. *if f is convex and increasing, then f^{-1} is concave,*
2. *if f is convex and decreasing, then f^{-1} is convex,*
3. *if f is concave and increasing, then f^{-1} is convex,*
4. *if f is concave and decreasing, then f^{-1} is concave.*

2.8. **Lemma.** *For $k, p > 1$ and $r \geq s$, we have*

$$\begin{aligned}
\left(\frac{\sin_p(r)}{\sin_p(s)}\right)^k &\leq \frac{\sin_p(r^k)}{\sin_p(s^k)}, \quad r, s \in (0,1),\\
\left(\frac{\tanh_p(r)}{\tanh_p(s)}\right)^k &\leq \frac{\tanh_p(r^k)}{\tanh_p(s^k)}, \quad r, s \in (0,\infty),\\
\left(\frac{\sinh_p(r)}{\sinh_p(s)}\right)^k &\geq \frac{\sinh_p(r^k)}{\sinh_p(s^k)}, \quad r, s \in (0,1)\,,
\end{aligned}$$

inequalities reverse for $k \in (0,1)$.

Proof. It is clear from the proof of Theorem 2.5 that the functions

$$f(x) = \log(\operatorname{arcsin_p}(e^{-x})),\ g(x) = \log(\operatorname{artanh_p}(e^{-x})),\ h(x) = \log(1/\operatorname{arsinh_p}(e^{x}))$$

are convex and decreasing, then Lemma 2.7(2) implies that

$$f^{-1}(y) = \log(1/\sin_p(e^y)),\ g^{-1}(y) = \log(1/\tanh_p(e^y)),\ h^{-1}(y) = \log(\sinh_p(e^{-y})),$$

are convex, now the result follows from Lemma 2.1. $\square$

2.9. **Lemma.** *For $p>1$, the following inequalities hold*

(1) $\sqrt{\sin_p(r^2)\sin_p(s^2)} \le \sin_p(r\,s)\,, \quad r,s\in(0,\pi_p/2)\,,$

(2) $\sqrt{\tanh_p(r^2)\tanh_p(s^2)} \le \tanh_p(r\,s)\,, \quad r,s\in(0,\infty)\,,$

(3) $\sinh_p(r\,s) \le \sqrt{\sinh_p(r^2)\sinh_p(s^2)}\,, \quad r,s\in(0,\infty)\,.$

Proof. Let $f(z)=\log(\operatorname{arcsin}_p(e^{-z}))$, $z>0$. Then

$$f^{'}(z) = -(1-e^{-pz})^{-1/p}/F(1/p,1/p;1+1/p;e^{-pz})<0,$$

f is decreasing and by the proof of Theorem 2.5 f is convex. By Lemma 2.7(2), $f^{-1}(y)=\log(1/\sin_p(e^y))$ is convex. This implies that

$$\log\left(\frac{1}{\sin_p(e^{x/2}e^{y/2})}\right) \le \frac{1}{2}\left(\log\left(\frac{1}{\sin_p(e^x)}\right)+\log\left(\frac{1}{\sin_p(e^y)}\right)\right),$$

letting $r=e^{x/2}$ and $s=e^{y/2}$, we get the first inequality.

For (2), let $g(z)=\log(\operatorname{artanh}_p(e^{-z}))$, $z>0$ and

$$g^{'}(z) = -1/((1-e^{-pz})F(1,1/p;1+1/p;e^{-pz}))<0,$$

hence g is decreasing and by Theorem 2.5 g is convex. Then $g^{-1}(y)=\log(1/\tanh_p(e^y))$ is convex by Lemma 2.7(2), and (2) follows. Finally, let $h_1(z)=\log(1/\operatorname{arsinh}_p(e^z))$ and

$$h_1^{'}(z) = -1/F\left(1,1/p;1+1/p;\frac{e^{pz}}{1+e^{pz}}\right)<0.$$

Then $h_1^{-1}(y)=\log(\sinh_p(e^{-y}))$ is decreasing and convex by Lemma 2.7(2). This implies that

$$\log(\sinh_p(e^{-x/2}e^{-y/2})) \le (\log(\sinh_p(e^{-x}))+\log(\sinh_p(e^{-y})))/2,$$

and (3) holds for $r,s\in(0,1)$. Again $h_2(z)=\log(1/\operatorname{arsinh}_p(e^{-z}))$ and

$$h_2^{'}(z) = (F(1,1/p;1+1/p;1/(1+e^{pz})))^{-1}>0,$$

similarly proof follows from Lemma 2.7(2), and (3) holds for $r,s\in(1,\infty)$, this completes the proof of (3). $\square$

2.10. **Lemma.** *For $p>1$, the following relations hold*

(1) $\sqrt{\sin_p(r)\sin_p(s)} \le \sin_p((r+s)/2), \quad r,s\in(0,\pi_p/2),$

(2) $\sqrt{\sinh_p(r)\sinh_p(s)} \le \sinh_p((r+s)/2), \quad r,s\in(0,\infty)\,.$

Proof. The proof follows easily from Lemma 2.9 and the inequality $2\sqrt{r\,s}\le r+s$ since the functions are increasing.

$\square$

2.11. **Lemma.** *For $p>1$, the following inequalities hold*

(1) $\sin_p(r+s) \le \sin_p(r)+\sin_p(s)\,, \quad r,s\in(0,\pi_p/4)\,,$

(2) $\tanh_p(r+s) \le \tanh_p(r)+\tanh_p(s)\,, \quad r,s\in(0,b_p/2)\,,$

(3) $\tan_p(r+s) \geq \tan_p(r)+\tan_p(s)\,, \quad r,s\in(0,b_p/2)\,,$
(4) $\sinh_p(r+s) \geq \sinh_p(r)+\sinh_p(s)\,, \quad r,s\in(0,c_p/2)\,.$

Proof. Let $f(x)=\operatorname{arcsin}_p(x),\ x\in(0,1)$. We get

$$f'(x)=(1-x^p)^{-1/p}\,,$$

which is increasing, hence f is convex. Clearly, f is increasing. Therefore

$$f_1=f^{-1}(y)=\sin_p(y)$$

is concave by Lemma 2.7(1). This implies that f_1' is decreasing. Clearly $f_1(0)=0$, and by [AVV1, Theorem 1.25], $f_1(y)/y$ is decreasing. Now it follows from [AVV1, Lemma 1.24] that

$$f_1(r+s)\leq f_1(r)+f_1(s),$$

and (1) follows. The proofs of the remaining claims follow similarly. ☐

3. Proof of main results

3.1. Proof of Theorem 1.1. By Lemma 2.3(3), (2) we get

$$2-\left(1-\frac{x^p}{p\,(1+p)}\right)<F\left(\frac{1}{p},\frac{1}{p};1+\frac{1}{p};x^p\right)\,,$$

and the first inequality of part one holds. For the second one we get

$$\begin{aligned}\operatorname{arcsin}_p x &= x\,F\left(\frac{1}{p},\frac{1}{p};1+\frac{1}{p};x^p\right)\\ &< \frac{x\,\Gamma(1+1/p)\Gamma(1+1/p-1/p-1/p)}{\Gamma(1+1/p-1/p)\Gamma(1+1/p-1/p)}\\ &= x\,\Gamma\left(1+\frac{1}{p}\right)\Gamma\left(1-\frac{1}{p}\right)=x\,\frac{1}{p}B\left(1-\frac{1}{p},\frac{1}{p}\right)=x\,\frac{\pi_p}{2}\end{aligned}$$

by Lemma 2.3(4). By [BE, Prop (2.11)] $\operatorname{arccos}_p x=\operatorname{arcsin}_p\left((1-x^p)^{1/p}\right)$, and (2) follows from (1). For (3), if we replace $b=1, c-a=1/p$, $c=1+1/p$, $x^p=z/(1-z)$ in (2.2) then we get

$$\begin{aligned}\operatorname{arctan}_p x &= x\,F\left(1,\frac{1}{p};1+\frac{1}{p};-x^p\right)\\ &= \left(\frac{x}{1+x^p}\right)F\left(1,1;1+\frac{1}{p};\frac{x^p}{1+x^p}\right)\\ &= \left(\frac{x}{1+x^p}\right)\left(\frac{1}{1+x^p}\right)^{1/p-1}F\left(\frac{1}{p},\frac{1}{p};1+\frac{1}{p};\frac{x^p}{1+x^p}\right)\\ &= \left(\frac{x^p}{1+x^p}\right)^{1/p}F\left(\frac{1}{p},\frac{1}{p};1+\frac{1}{p};\frac{x^p}{1+x^p}\right)\\ &< 2^{1/p}\,b_p\left(\frac{x^p}{1+x^p}\right)^{1/p},\end{aligned}$$

third identity and inequality follow from Lemma 2.3(1), (4). For the lower bound we get

$$\begin{aligned}\operatorname{arctan}_p x &> \left(\frac{x^p}{1+x^p}\right)^{1/p}\left(2-F\left(\frac{1}{p},\frac{1}{p};1+\frac{1}{p};\frac{x^p}{1+x^p}\right)\right)\\ &> \frac{(p(1+p)(1+x^p)+x^p)x}{p(1+p)(1+x^p)^{1+1/p}}\end{aligned}$$

from Lemma 2.3(3), (2).

3.2. **Proof of Theorem 1.2.** For (1.3), we replace $b=1/p$, $c-a=1/p$, $c=1+1/p$ and $x^p=z/(1-z)$ in (2.2) and see that

$$\operatorname{arsinh}_p x = x\,F\left(\frac{1}{p},\frac{1}{p};1+\frac{1}{p};-x^p\right)=\left(\frac{x^p}{1+x^p}\right)^{1/p} F\left(1,\frac{1}{p};1+\frac{1}{p};\frac{x^p}{1+x^p}\right).$$

Now we get

$$\frac{\log(1+x^p)}{1+p}\left(\frac{x^p}{1+x^p}\right)^{1/p} <$$

$$x\,F\left(\frac{1}{p},\frac{1}{p};1+\frac{1}{p};-x^p\right) < \left(1-\frac{1}{p}\log\left(1-\frac{x^p}{1+x^p}\right)\right)\left(\frac{x^p}{1+x^p}\right)^{1/p}$$

from Lemma 2.3(5) and observing that $B(1,1/p)=p$, this implies (1.3).

For (1.4) we get from Lemma 2.3(5)

$$\frac{1}{1+p}\log\left(\frac{1}{1-x^p}\right)+1 < F\left(1,\frac{1}{p};1+\frac{1}{p};x^p\right) < \frac{1}{p}\log\left(\frac{1}{1-x^p}\right)+1\,,$$

which is equivalent to

$$x\left(1-\frac{1}{1+p}\log(1-x^p)\right) < x\,F\left(1,\frac{1}{p};1+\frac{1}{p};x^p\right) < x\left(1-\frac{1}{p}\log(1-x^p)\right)\,,$$

and the result follows.

3.3. **Remark.** For the particular case $p=2$. Zhu [Z] has proved for $x>0$

$$\frac{6\sqrt{2}(\sqrt{1+x^2}-1)^{1/2}}{4+\sqrt{2}(\sqrt{1+x^2}+1)^{1/2}} < \operatorname{arsinh}(x).$$

When $p=2$, our bound in (1.3) differs from this bound roughly 0.01 when $x\in(0,1)$.

3.4. **Lemma.** *For $p>1$ and $x\in(0,1)$, the following inequalities hold:*

(1) $\operatorname{arctan}_p(x) < \operatorname{arsinh}_p(x) < \operatorname{arcsin}_p(x) < \operatorname{artanh}_p(x)\,,$

(2) $\tanh_p(z) < \sin_p(z) < \sinh_p(z) < \tan_p(z)\,,$

the first and the second inequalities hold for $z\in(0,\pi_p/2)$, and the third one holds for $z\in(0,b_p)$.

Proof. From the proof of Theorems 1.1(3), we get

$$\begin{aligned}
\operatorname{arctan}_p(x) &= \left(\frac{x^p}{1+x^p}\right)^{1/p} F\left(\frac{1}{p},\frac{1}{p};1+\frac{1}{p};\frac{x^p}{1+x^p}\right)\\
&< \left(\frac{x^p}{1+x^p}\right)^{1/p} F\left(1,\frac{1}{p};1+\frac{1}{p};\frac{x^p}{1+x^p}\right)\\
&= \operatorname{arsinh}_p(x) < \left(\frac{x^p}{1+x^p}\right)^{1/p}\left(1+\frac{1}{p}\log(1+x^p)\right)\\
&< x\left(1+\frac{x^p}{p(1+p)}\right) < \operatorname{arcsin}_p(x)\\
&= x\,F\left(\frac{1}{p},\frac{1}{p};1+\frac{1}{p};x^p\right)\\
&< x\,F\left(1,\frac{1}{p};1+\frac{1}{p};x^p\right) = \operatorname{artanh}_p(x)\,,
\end{aligned}$$

the first and the fifth inequality follow from the fact that $F(a,b;c;x)$ is increasing in a. For the second and the fourth inequality we use (1.3) and Theorem 1.1(1). The inequalities given in (2) follow from (1), if we take each inequality and apply its inverse function to both sides and use limiting values. □

3.5. **Lemma.** *For $p>2$, we have*

$$\frac{6p^2}{3p^2-2} \le \pi_p \le \frac{12p^2}{6p^2-\pi^2}\,,\ \ \pi_p = \frac{2\pi}{p\,\sin(\pi/p)}\,.$$

Proof. By [KVV, Thm 3.1] we get

$$\frac{\pi}{p}\left(1-\frac{\pi^2}{6p^2}\right) \le \sin\left(\frac{\pi}{p}\right) \le \frac{\pi}{p}\left(1-\frac{2}{3p^2}\right)\,,$$

and the result follows easily. □

3.6. **Lemma.** *For $a\in(0,1)$ and $k,r,s\in(1,\infty)$, the following inequalities hold*

(1) $\pi_{r\,s} \le \sqrt{\pi_{r^2}\,\pi_{s^2}} \le \sqrt{\pi_r\,\pi_s}\,,$
(2) $\pi_{r^a\,s^{1-a}} \le a\,\pi_r + (1-a)\pi_s\,,$
(3) $\displaystyle\left(\frac{\pi_s}{\pi_r}\right)^k \le \frac{\pi_{s^k}}{\pi_{r^k}},\ r\le s\,.$

Proof. Let $f(x)=\log(\pi_{e^x})$, $x>0$. We get

$$f''(x) = e^{-x}\pi(e^{-x}\pi\csc\left(e^{-x}\pi\right)^2 - \cot\left(e^{-x}\pi\right)\,,$$

which is positive, because the function $g(y)=y^2(\csc(y))^2 - y\cot(y)$ is positive. This implies that f is convex. Hence

$$\log(\pi_{e^{(x+y)/2}}) \le \frac{1}{2}\left(\log(\pi_{e^x})+\log(\pi_{e^y})\right)\,,$$

setting $r = e^{x/2}$ and $s = e^{y/2}$, we get the first inequality of (1), and the second one follows from the fact that π_p is decreasing in $p \in (1,\infty)$. Now it is clear that π_{e^x} is convex, and we get

$$\pi_{e^{a\,x+(1-a)y}} \le a\,\pi_{e^x} + (1-a)\pi_{e^y}\,.$$

Let $0 \le x \le y$, then we get

$$\frac{(\pi_{e^y})^k}{\pi_{e^{k\,y}}} \le \frac{(\pi_{e^x})^k}{\pi_{e^{k\,x}}}$$

from Lemma 2.1, and (3) follows if we set $r = e^x$ and $s = e^y$. $\square$

3.7. **Lemma.** *For $p > 1$ and $x \in (0,1)$, we have*

$$\operatorname{arcsin}_p\left(\frac{x}{\sqrt[p]{1+x^p}}\right) = \operatorname{arctan}_p(x)\,,$$

$$\operatorname{arcsin}_p(x) = \operatorname{arctan}_p\left(\frac{x}{\sqrt[p]{1-x^p}}\right)\,,$$

$$\operatorname{arccos}_p(x) = \operatorname{arctan}_p\left(\frac{\sqrt[p]{1-x^p}}{x}\right)\,,$$

$$\operatorname{arccos}_p\left(\frac{1}{\sqrt[p]{1+x^p}}\right) = \operatorname{arctan}_p(x)\,.$$

Proof. We get

$$\begin{aligned}
\operatorname{arctan}_p(x) &= x\,F\left(\frac{1}{p},\frac{1}{p};1+\frac{1}{p};-x^p\right)\\
&= \frac{x}{1+x^p}\,F\left(1,1;1+\frac{1}{p};\frac{x^p}{1+x^p}\right)\\
&= \frac{x}{1+x^p}\left(\frac{1}{1+x^p}\right)^{1/p-1} F\left(\frac{1}{p},\frac{1}{p};1+\frac{1}{p};\frac{x^p}{1+x^p}\right)\\
&= \left(\frac{x^p}{1+x^p}\right)^{1/p} F\left(\frac{1}{p},\frac{1}{p};1+\frac{1}{p};\left(\frac{x}{(1+x^p)^{1/p}}\right)^p\right)\\
&= \operatorname{arcsin}_p\left(\frac{x}{\sqrt[p]{1+x^p}}\right)
\end{aligned}$$

by (2.2) and Lemma 2.3(1). Write $y = x/\sqrt[p]{1-x^p}$, and second follows from first one. For the third identity, we get

$$\begin{aligned}
\operatorname{arctan}_p\left(\frac{\sqrt[p]{1-x^p}}{x}\right) &= \sqrt[p]{1-x^p}\,F\left(\frac{1}{p},\frac{1}{p};1+\frac{1}{p};(1-x^p)\right)\\
&= \operatorname{arcsin}_p((1-x^p)^{1/p}) = \operatorname{arccos}_p(x)
\end{aligned}$$

by (2.2), Lemma 2.3(1) and [BE, Prop 2.2]. Similarly, the fourth identity follows from third one. $\square$

3.8. **Conjecture.** *For a fixed $x \in (0,1)$, the functions*

$$\sin_p(\pi_p\, x/2),\ \tan_p(\pi_p\, x/2),\ \sinh_p(c_p\, x)$$

are monotone in $p \in (1,\infty)$. For fixed $x > 0$, $\tanh_p(x)$ is increasing in $p \in (1,\infty)$.

4. Some relations for elementary functions

4.1. **Lemma.** *For $x \in (0,1)$, the following functions*

$$f_1(k) = \sin(x^k)^{1/k}\,, \quad f_2(k) = \cos(x^k)^{1/k}\,, \quad f_3(k) = \tanh(z^k)^{1/k}\,,$$

are increasing in $(0,\infty)$.

Proof. We get

$$f_1'(k) = (x^k \cot(x^k)\log(x^k) - \log(\sin(x^k)))\sin(x^k)^{1/k}/k^2,$$

which is positive because

$$h_1(y) = y\cot(y)\log(y) - \log(\sin(y)) \geq 0\,.$$

For f_2 we get

$$f_2'(k) = -(x^k\tan(x^k)\log(x^k) + \log(\cos(x^k)))\cos(x^k)^{1/k}/k^2,$$

which is positive because the function $h_2(y) = y\tan(y)\log(y) + \log(\cos(y)) \leq 0$. For f_3 we get

$$f_3'(k) = \frac{\tanh(z^k)^{1/k}}{k^2}(2z^k\log(z^k)/\sinh(2z^k) - \log(\tanh(z^k))).$$

Let

$$h_3(y) = 2y\log(y)/\sinh(2y) - \log(\tanh(y)),\ y = z^k \in (0,\infty).$$

Clearly $h_3(y) > 0$ for $y > 1$. For $y \in (0,1)$ we see that $h_3(y) > 0$ iff

$$\frac{2y}{\sinh(2y)}\frac{\log(y)}{\log(\tanh(y))} \leq 1$$

which holds because $y > \tanh(y)$. In conclusion, $f_3'(k) > 0$ for all $z \in (0,\infty)$. $\square$

4.2. **Lemma.** *The following inequalities hold*

(1) $\sqrt{\operatorname{arccos}(r^2)\operatorname{arccos}(s^2)} < \operatorname{arccos}(r\, s)\,, \quad r, s \in (0,1)$

(2) $\sqrt{\operatorname{arcosh}(r^2)\operatorname{arcosh}(s^2)} < \operatorname{arcosh}(r\, s)\,, \quad r, s \in (1,\infty)\,.$

Proof. For (1) we let $f(x) = \log(\arccos(e^{-x}))\,, x > 0$, and get

$$f''(x) = -\frac{\sqrt{e^{2x}-1} + e^{2x}\arccos(e^{-x})}{(e^{2x}-1)^{3/2}\arccos^2(e^{-x})} \leq 0,$$

hence f is concave, and the inequality follows.

For (2) we define $h(x)=\log(\operatorname{arcosh}(e^x))\,,\ x>0$ and get

$$h^{''}(x)=-\frac{e^x(e^x\sqrt{e^{2x}-1}+\operatorname{arcosh}(e^x))}{(e^{2x}-1)^{3/2}\operatorname{arcosh}(e^x)^2}<0\,.$$

This implies the proof of (2). $\square$

4.3. **Lemma.** *For $r,s\in(0,\infty)$, we have*

(1) $\cosh(r\,s)<\sqrt{\cosh(r^2)\cosh(s^2)}\,,$

(2) $\tanh(r)\tanh(s)<\sqrt{\tanh(r^2)\tanh(s^2)}<\sqrt{\tanh(r^2\,s^2)}\,.$

Proof. For (1) let $g_1(x)=\log(\cosh(e^{-x}))$ and $g_2(x)=\log(\cosh(e^{\,x}))\,,\ x>0$. We get

$$g_1^{''}(x)=e^{-2x}(1/(\cosh(e^{-x})^2)+e^x\tanh(e^{-x}))>0,$$
$$g_2^{''}(x)=e^x(e^x/(\cosh(e^x)^2)+\tanh(e^x))>0,$$

hence g_1 and g_2 are convex, and (1) follows. The first inequality of (2) follows from Lemma 4.1. For the second one let $h_1(x)=\log(\tanh(e^{-x}))$, $x>0$ and get

$$h_1^{''}(x)=e^{-2x}\left(-\operatorname{csch}\left(e^{-x}\right)^2+2e^x\operatorname{csch}\left(2e^{-x}\right)-\operatorname{sech}\left(e^{-x}\right)^2\right)$$

which is negative, hence h_1 is concave. Again, let $h_2(x)=\log(\tanh(e^x))$ and get

$$h_2^{''}(x)=-e^x\left(e^x\operatorname{csch}\left(e^x\right)^2-2\operatorname{csch}\left(2e^x\right)+e^x\operatorname{sech}\left(e^x\right)^2\right)<0\,.$$

This implies that h_2 is also concave, and the second inequality of (2) holds for $r,s\in(0,\infty)$.

$\square$

4.4. **Lemma.** *For $y\in(0,1)$, we have*

$$\frac{\pi}{2}\,y\cot\left(\frac{\pi\,y}{2}\right)\log\,y\leq\log\left(\sin\left(\frac{\pi\,y}{2}\right)\right)\,, \tag{4.5}$$

$$y\coth\left(y\right)\log\,y\leq\log\left(\sinh\left(y\right)\right)\,, \tag{4.6}$$

$$\log\left(\tan\left(\frac{\pi y}{2}\right)\right)\geq\frac{\pi}{2}y\log(y)\csc\left(\frac{\pi y}{2}\right)\sec\left(\frac{\pi y}{2}\right)\,. \tag{4.7}$$

Proof. Let $f(y)=\frac{\pi}{2}\,y\cot\left(\frac{\pi\,y}{2}\right)\log\,y-\log\left(\sin\left(\frac{\pi\,y}{2}\right)\right)$. We get

$$\begin{aligned}
f^{'}(y) &= \frac{\pi}{2}\cot\left(\frac{\pi\,y}{2}\right)\log\,y-\frac{1}{4}\,y\pi^2\csc\left(\frac{\pi\,y}{2}\right)^2\log\,y\\
&= \frac{\pi}{2}\log(y^{-1})\left(\frac{\pi\,y}{2}\frac{1}{\sin^2(\pi\,y/2)}-\frac{\cos(\pi\,y/2)}{\sin(\pi\,y/2)}\right)\\
&= \frac{\pi}{2}\frac{\log(y^{-1})}{\sin(\pi\,y/2)^2}\left(\frac{\pi\,y}{2}-\sin\left(\frac{\pi\,y}{2}\right)\cos\left(\frac{\pi\,y}{2}\right)\right)\\
&= \frac{\pi}{2}\frac{\log(y^{-1})}{\sin(\pi\,y/2)^2}\left(\frac{\pi\,y}{2}-\frac{\sin(\pi\,y)}{2}\right).
\end{aligned}$$

This is positive because $x \geq \sin x$ for $x \in (0, 2\pi)$, and $f(1) = 0$ and (4.5) follows. Next, let

$$g(y) = y \coth(y) \log y - \log(\sinh(y)) \,.$$

We get

$$g'(y) = \frac{\log(1/y)}{\sinh(y)^2}(y - \sinh(y)\cosh(y)) \leq 0,$$

because $\sinh x \geq x/\cosh x$ for $x > 0$. Moreover, g tends to zero when y tends to zero and this implies the proof of (4.6). Finally, let

$$h(y) = \log\left(\tan\left(\frac{\pi y}{2}\right)\right) - \frac{\pi}{2} y \log(y) \csc\left(\frac{\pi y}{2}\right) \sec\left(\frac{\pi y}{2}\right).$$

We see that

$$\begin{aligned} h'(y) &= -\frac{\pi^2}{4} y \log(y) \sec\left(\frac{\pi y}{2}\right)^2 + \frac{1}{4}\pi^2 y \log(y) \csc\left(\frac{\pi y}{2}\right)^2 \\ &\quad -\frac{\pi}{2}\log(y)\csc\left(\frac{\pi y}{2}\right)\sec\left(\frac{\pi y}{2}\right) \\ &= \pi \log\left(\frac{1}{y}\right)\csc(\pi y)^2(\sin(\pi y) - \pi y \cos(\pi y)) \leq 0, \end{aligned}$$

because $x \leq \tan x$ for $x \in (0,1)$. Hence h is increasing and tends to $\log(\pi/2)$ when y tends to zero and this implies the proof. $\square$

4.8. **Lemma.** (1) *The function*

$$H(y) = \frac{1}{2}\pi \log\left(\frac{1}{y^y}\right)\cot\left(\frac{\pi y}{2}\right) - \log\left(\csc\left(\frac{\pi y}{2}\right)\right)$$

is decreasing from $(0,1)$ *onto* $(0, \log(\pi/2))$.

(2) *The function*

$$G(y) = \log\left(\cosh\left(\frac{\pi y}{2}\right)\right) - \frac{1}{2}\pi y \log(y) \tanh\left(\frac{\pi y}{2}\right)$$

is increasing from $(0,1)$ *onto* $(0, \pi \log(\cosh(\pi/2))/2)$.

Proof. We get

$$\begin{aligned} H'(y) &= -\frac{\pi}{4}\csc\left(\frac{\pi y}{2}\right)^2 \left(\pi \log\left(y^{-y}\right) + \log(y)\sin(\pi y)\right) \\ &= -\frac{\pi}{4}\csc\left(\frac{\pi y}{2}\right)^2 (\pi\, y \log(1/y) - \sin(\pi y)\log(1/y)) \,, \end{aligned}$$

which is positive. Next,

$$G'(y) = -\frac{1}{2}\pi \log(y) \tanh\left(\frac{\pi y}{2}\right) - \frac{1}{4}\pi^2 y \log(y) \mathrm{sech}\left(\frac{\pi y}{2}\right)^2 > 0\,,$$

and the limiting values follow easily. $\square$

4.9. **Lemma.** *The following function is increasing from* $(0,1)$ *onto* $(0,\pi(\log(\pi/2))/2)$
$$g(x)=\frac{x}{\sqrt{1-x^2}}\log\left(\frac{1}{x}\right)-\arcsin(x)\log\left(\frac{1}{\arcsin(x)}\right).$$
In particular,
$$x^{x/\sqrt{1-x^2}}<\arcsin(x)^{\arcsin(x)}<\left(\frac{\pi}{2}\right)^{\pi/2}x^{x/\sqrt{1-x^2}}.$$

Proof. We get
$$\begin{aligned} g^{'}(x) &= -\frac{x^2\log(x)}{(1-x^2)^{3/2}}-\frac{\log(x)}{\sqrt{1-x^2}}+\frac{\log\left(\arcsin(x)\right)}{\sqrt{1-x^2}}\\ &= \frac{\log(1/x)-(1-x^2)\log(1/\arcsin(x))}{(1-x^2)^{3/2}}\\ &= \frac{\log(\arcsin(x)^{(1-x^2)}/x)}{(1-x^2)^{3/2}}, \end{aligned}$$
which is clearly positive, and g tends to zero when x tends to zero and 1. $\square$

4.10. **Lemma.** *For* $x\in(0,1)$, *the following functions*
$$f(k)=\sin\left(\frac{\pi}{2}\,x^k\right)^{1/k},\quad g(k)=\tan\left(\frac{\pi}{2}\,x^k\right)^{1/k},\quad h(k)=\sinh\left(x^k\right)^{1/k},$$
are decreasing in $(0,\infty)$. *In particular, for* $k\geq 1$
$$\sqrt[k]{\sin\left(\frac{\pi}{2}\,x^k\right)}\leq\sin\left(\frac{\pi}{2}\,x\right)\leq\sin\left(\frac{\pi}{2}\sqrt[k]{x}\right)^k,$$
$$\sqrt[k]{\tan\left(\frac{\pi}{2}\,x^k\right)}\leq\tan\left(\frac{\pi}{2}\,x\right)\leq\tan\left(\frac{\pi}{2}\sqrt[k]{x}\right)^k,$$
$$\sqrt[k]{\sinh\left(x^k\right)}\leq\sinh\left(x\right)\leq\sinh\left(\sqrt[k]{x}\right)^k.$$

Proof. We get
$$\begin{aligned} f^{'}(k) &= \sqrt[k]{\sin\left(\frac{\pi x^k}{2}\right)}\left(\frac{\pi x^k\log(x)\cot\left(\frac{\pi x^k}{2}\right)}{2k}-\frac{\log\left(\sin\left(\frac{\pi x^k}{2}\right)\right)}{k^2}\right)\\ &= -\frac{1}{2\,k^2}\sqrt[k]{\sin\left(\frac{\pi x^k}{2}\right)}\left(\pi k\,x^k\log(1/x)\cot\left(\frac{\pi x^k}{2}\right)-2\log\left(1/\sin\left(\frac{\pi x^k}{2}\right)\right)\right), \end{aligned}$$

which is negative by Lemma 4.8(1). Next, we get
$$g^{'}(k)=\sqrt[k]{\tan\left(\frac{\pi x^k}{2}\right)}\left(\frac{\pi x^k\log(x)\csc\left(\frac{\pi x^k}{2}\right)\sec\left(\frac{\pi x^k}{2}\right)}{2k}-\frac{\log\left(\tan\left(\frac{\pi x^k}{2}\right)\right)}{k^2}\right)\leq 0,$$

by (4.7). Finally,

$$\begin{aligned} h^{'}(k) &= \sqrt[k]{\sinh\left(x^k\right)}\left(\frac{x^k\log(x)\coth\left(x^k\right)}{k}-\frac{\log\left(\sinh\left(x^k\right)\right)}{k^2}\right) \\ &= \sqrt[k]{\sinh\left(x^k\right)}\left(x^k\log(x^k)\coth\left(x^k\right)-\log\left(\sinh\left(x^k\right)\right)\right)(1/k^2), \end{aligned}$$

which is negative by inequality (4.6), and this completes the proof. □

4.11. **Lemma.** *The following functions*

$$f(k)=\cos\left(\frac{\pi}{2}x^{1/k}\right)^k, \quad x\in(0,1),$$

$$g(k)=\cosh\left(x^k\right)^{1/k}, \quad x\in(0,1),$$

$$h(k)=\operatorname{arcosh}\left(\frac{\pi}{2}\,x^k\right)^{1/k}, \quad x\in(1,\infty),$$

are decreasing in $(0,\infty)$. *In particular, for* $k\geq 1$

$$\cos\left(\frac{\pi}{2}\sqrt[k]{x}\right)^k\leq\cos\left(\frac{\pi}{2}\,x\right)\leq\sqrt[k]{\cos\left(\frac{\pi}{2}\,x^k\right)},$$

$$\sqrt[k]{\cosh\left(x^k\right)}\leq\cosh\left(x\right)\leq\cosh\left(\sqrt[k]{x}\right)^k,$$

$$\sqrt[k]{\operatorname{arcosh}\left(\frac{\pi}{2}\,x^k\right)}\leq\operatorname{arcosh}\left(\frac{\pi}{2}\,x\right)\leq\operatorname{arcosh}\left(\frac{\pi}{2}\sqrt[k]{x}\right)^k.$$

Proof. We get

$$f^{'}(k)=\cos\left(\frac{1}{2}\pi x^{1/k}\right)^k\left(\frac{\pi x^{1/k}\log(x)\tan\left(\pi x^{1/k}/2\right)}{2k}+\log\left(\cos\left(\pi x^{1/k}/2\right)\right)\right)\leq 0$$

and proof of g follows from Lemma 4.2(1).

Finally, for $y\geq\pi/2$, let

$$j(y)=\operatorname{arcosh}(y)\log\left(\operatorname{arcosh}(y)\right)-\frac{y\log(2y/\pi)}{\sqrt{y^2-1}},$$

and

$$j^{'}(y)=\frac{\log\left(2y/\pi\right)}{\left(y^2-1\right)^{3/2}}+\frac{\log\left(\operatorname{arcosh}(y)\right)}{\sqrt{y^2-1}}>0,$$

and

$$j(\pi/2)=\operatorname{arcosh}(\pi/2)\log(\operatorname{arcosh}(\pi/2))\equiv 0.0235\,.$$

With $z=x^k$ we get

$$h^{'}(x)=\frac{\operatorname{arcosh}\left(\pi z/2\right)^{1/k}}{k^2\operatorname{arcosh}\left(\pi z/2\right)}\left(\frac{\pi z\log(z)}{2\sqrt{(\pi z/2)^2-1}}-\operatorname{arcosh}\left(\frac{\pi}{2}z\right)\log\left(\operatorname{arcosh}\left(\frac{\pi}{2}\right)\right)\right).$$

This is negative, because $j(y)>0$ for $y>\pi/2$. □

4.12. **Lemma.** *The following relations hold*

(1) $\sin(r)\sin(s) < \sqrt{\sin(r^2)\sin(s^2)}, \quad r, s \in (0,1)\,,$

(2) $\cos(r)\cos(s) < \sqrt{\cos(r^2)\cos(s^2)} < \cos(r\, s)\,,$

(3) $\tan(r)\tan(s) > \sqrt{\tan(r^2)\tan(s^2)} > \tan(r\, s),$
the first inequalities in (2) and (3) hold for $r, s \in (0, \sqrt{\pi/2})$, *and second ones for* $r, s \in (0,1)$.

Proof. Clearly (1) and the fist inequality of (2) follwos from Lemmas 4.1 and 4.11, respectively. Let $g(x) = \log(\cos(\pi\, e^{-x}/2))$, $x > 0$, we get

$$\begin{aligned} g''(x) &= -\frac{\pi^2}{4}\, e^{-2x} \sec\left(\frac{e^{-x}\pi}{2}\right)^2 - \frac{\pi}{2}\, e^{-x} \tan\left(\frac{e^{-x}\pi}{2}\right) \\ &= -\frac{\pi}{4}\, e^{-2x} \sec\left(\frac{e^{-x}\pi}{2}\right)^2 \left(e^x \sin\left(e^{-x}\pi\right) + \pi\right) \le 0, \end{aligned}$$

and the second inequality of (2) follows.

For (3), we define $h(x) = \log(\tan(\pi\, e^{-x}/2))$, $x > 0$, and we get

$$h''(x) = e^{-x}\pi \left(1 - e^{-x}\pi \cot\left(e^{-x}\pi\right)\right) \csc\left(e^{-x}\pi\right) \ge 0,$$

hence h is convex, and the second inequality follows easily, and the first one follows from Lemma 4.10. $\square$

4.13. **Lemma.** *For a fixed* $x \in (0,1)$, *the function* $g(k) = (\cos\, kx + \sin kx)^{1/k}$ *is decreasing in* $(0,1)$.

Proof. Differentiation yields

$$g^{'}(k) = \frac{(\sin(kx) + \cos(kx))^{1/k}}{k^2}\left(\frac{kx(\cos(kx) - \sin(kx))}{\sin(kx) + \cos(kx)} - \log(\sin(kx) + \cos(kx))\right).$$

To prove that this is positive, we let $z = k\,x\,, y = \cos\, z + \sin\, z \le 1.1442$

$$h(z) = (\cos\, z + \sin\, z)\log(\cos\, z + \sin\, z) - z(\cos\, z - \sin\, z)\,,$$

and observe that

$$\begin{aligned} h^{'}(z) &= z\, \cos\, z + (\cos\, z - \sin\, z)\log(\cos\, z + \sin\, z) + z\, \sin\, z \\ &= zy + \log\, y^{\cos\, z} - \log\, y^{\sin\, z} \ge 0, \end{aligned}$$

because $e^{zy} > y^{\sin\, z}$. This implies that $g^{'}(k) \ge 0$. $\square$

5. Appendix

In the following tables we give the values of p-analogue functions for some specific values of its domain with $p = 3$ computed with Mathematica®. For instance, we can define [Ru]

```
arcsinp[p_, x_] := x *Hypergeometric2F1[1/p, 1/p, 1 + 1/p, x^p]
```

```
sinp[p_, y_] := x /.  FindRoot[ arcsinp[p, x] == y, {x, 0.5 }]
```

x	$\mathrm{arcsin_p}(x)$	$\mathrm{arccos_p}(x)$	$\mathrm{arctan_p}(x)$	$\mathrm{arsinh_p}(x)$	$\mathrm{artanh_p}(x)$
0.00000	0.00000	1.20920	0.00000	0.00000	0.00000
0.25000	0.25033	1.17782	0.24903	0.24968	0.25099
0.50000	0.50547	1.07974	0.48540	0.49502	0.51685
0.75000	0.78196	0.88660	0.68570	0.72710	0.85661
1.00000	1.20920	0.00000	0.83565	0.93771	∞

x	$\sin_{\mathrm{p}}(x)$	$\cos_{\mathrm{p}}(x)$	$\tan_{\mathrm{p}}(x)$	$\sinh_{\mathrm{p}}(x)$	$\tanh_{\mathrm{p}}(x)$
0.00000	0.00000	1.00000	0.00000	0.00000	0.00000
0.25000	0.24967	0.99478	0.25098	0.25033	0.24903
0.50000	0.49476	0.95788	0.51652	0.50518	0.48517
0.75000	0.72304	0.85362	0.84704	0.77588	0.68283
1.00000	0.91139	0.62399	1.46058	1.08009	0.82304

With a normalization different from ours, some eigenvalue problems of the p-Laplacian have been studied in [BR].

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IV

ON GENERALIZED COMPLETE ELLIPTIC INTEGRALS AND MODULAR FUNCTIONS

B. A. BHAYO AND M. VUORINEN

Abstract. This paper deals with generalized elliptic integrals and generalized modular functions. Several new inequalities are given for these and related functions.

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1. Introduction

During the past fifteen years, after the publication of the landmark paper [BBG], numerous papers have been written about generalized elliptic integrals, modular functions and inequalities for them. See e.g. [AQVV, AQ, B1, B2, HLVV, HVV, WZC, WZQC, ZWC1, ZWC2]. Modular equations have a long history, which goes back to the works of A.M. Legendre, K.F. Gauss, C. Jacobi and S. Ramanujan about number theory. Modular equations also occur in geometric function theory as shown in [AQVV, Vu2, K, LV] and in numerical computations of moduli of quadrilaterals [HRV]. For recent surveys of this topic from the point of view of geometric function theory, see [AVV4, AVV5, AV]. The study of these functions is motivated by potential applications to geometric function theory and to number theory.

Given complex numbers a, b and c with $c \neq 0, -1, -2, \ldots$, the *Gaussian hypergeometric function* is the analytic continuation to the slit place $\mathbb{C} \setminus [1, \infty)$ of the series

$$F(a,b;c;z) = {}_2F_1(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a,n)(b,n)}{(c,n)} \frac{z^n}{n!}, \qquad |z| < 1.$$

Here $(a,0) = 1$ for $a \neq 0$, and (a,n) is the *shifted factorial function* or the *Appell symbol*

$$(a,n) = a(a+1)(a+2)\cdots(a+n-1)$$

for $n \in \mathbb{Z}_+$.

For later use we define classical *gamma function* $\Gamma(x)$, and *beta function* $B(x,y)$. For $\mathrm{Re}\, x > 0$, $\mathrm{Re}\, y > 0$, these functions are defined by

$$\Gamma(x) = \int_0^{\infty} e^{-t} t^{x-1}\, dt, \;\; B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)},$$

respectively.

For the formulation of our main results and for later use we introduce some basic notation. The decreasing homeomorphism $\mu_a : (0,1) \to (0,\infty)$ is defined by

$$\mu_a(r) = \frac{\pi}{2\,\sin(\pi\,a)}\frac{F(a,1-a;1;r'^2)}{F(a,1-a;1;r^2)} = \frac{\pi}{2\,\sin(\pi\,a)}\frac{\mathcal{K}_a(r')}{\mathcal{K}_a(r)}$$

for $r \in (0,1)$ and $r' = \sqrt{1-r^2}$. A generalized modular equation with signature $1/a$ and order (or degree) p is

$$\mu_a(s) = p\,\mu_a(r), \quad 0<r<1. \tag{1.1}$$

We denote

$$s = \varphi_K^a \equiv \mu_a^{-1}(\mu_a(r)/K), \quad K \in (0,\infty),\, p = 1/K\,, \tag{1.2}$$

which is the solution of (1.1).

For $a \in (0,1/2]$, $K \in (0,\infty)$, $r \in (0,1)$, we have by [AQVV, Lemma 6.1]

$$\varphi_K^a(r)^2 + \varphi_{1/K}^a(r')^2 = 1. \tag{1.3}$$

For $a \in (0,1/2]\,$, $r \in (0,1)$ and $r' = \sqrt{1-r^2}$, the generalized elliptic integrals are defined by

$$\begin{cases} \mathcal{K}_a(r) = \dfrac{\pi}{2}\,F(a,1-a;1;r^2), \\ \mathcal{K}_a'(r) = \mathcal{K}_a(r'), \\ \mathcal{K}_a(0) = \dfrac{\pi}{2},\ \mathcal{K}_a(1) = \infty, \end{cases} \qquad \begin{cases} \mathcal{E}_a(r) = \dfrac{\pi}{2}\,F(a-1,1-a;1;r^2), \\ \mathcal{E}_a'(r) = \mathcal{E}_a(r'), \\ \mathcal{E}_a(0) = \dfrac{\pi}{2},\ \mathcal{E}_a(1) = \dfrac{\sin(\pi\,a)}{2(1-a)}. \end{cases}$$

In this paper we study the modular function $\varphi_K^a(r)$ for general $a \in (0,\frac{1}{2}]$, as well as related functions μ_a, $\mathcal{K}_a$, η_K^a, λ_a, and their dependency on r and K, where

$$\eta_K^a(x) = \left(\frac{s}{s'}\right)^2,\ s = \varphi_K^a(r),\ r = \sqrt{\frac{x}{1+x}}\,, \quad \text{for } x, K \in (0,\infty),$$

$$\lambda_a(K) = \left(\frac{\varphi_K^a(1/\sqrt{2})}{\varphi_{1/K}^a(1/\sqrt{2})}\right)^2 = \left(\frac{\mu_a^{-1}(\pi/(2K\sin(\pi\,a)))}{\mu_a^{-1}(\pi K/(2\sin(\pi\,a)))}\right)^2 = \eta_K^a(1). \tag{1.4}$$

Motivated by [L] and [BV] we define for $p>1$ and $r \in (0,1)$,

$$\operatorname{artanh}_p(x) = \int_0^x (1-t^p)^{-1}dt = xF\left(1\,,\frac{1}{p};1+\frac{1}{p};x^p\right)\,.$$

Then $\operatorname{artanh}_2(x)$ is the usual inverse hyperbolic tangent (artanh) function.

We give next some of the main results of this paper.

1.5. **Theorem.** *For $a,b,c>0$, and $r \in (0,1)$, the function $g(p) = F(a,b;c;r^p)^{1/p}$ is decreasing in $p \in (0,\infty)$. In particular, for $p \geq 1$*

$$F(a,b;c;r^p)^{1/p} \leq F(a,b;c;r) \leq F(a,b;c;r^{1/p})^p\,, \tag{1}$$

$$\left(\frac{\pi}{2}\right)^{1-1/p} \mathcal{K}_a(r^p)^{1/p} \le \mathcal{K}_a(r) \le \left(\frac{\pi}{2}\right)^{1-p} \mathcal{K}_a(r^{1/p})^p\,, \tag{2}$$

$$\left(\frac{\pi}{2}\right)^{1-p} \mathcal{E}_a(r^{1/p})^p \le \mathcal{E}_a(r) \le \left(\frac{\pi}{2}\right)^{1-1/p} \mathcal{E}_a(r^p)^{1/p}. \tag{3}$$

H. Alzer and S.-L. Qiu have given the following bounds for $\mathcal{K} = \mathcal{K}_{1/2}$ in [AQ, Theorem 18]

$$\frac{\pi}{2}\left(\frac{\operatorname{artanh}(r)}{r}\right)^{3/4} < \mathcal{K}(r) < \frac{\pi}{2}\left(\frac{\operatorname{artanh}(r)}{r}\right). \tag{1.6}$$

In the following theorem we generalize their result to the case of $\mathcal{K}_a$, and for the particular case $a = 1/2$ our upper bound is better than their bound in (1.6). For a graphical comparison of the bounds see Figure 1 below.

1.7. **Theorem.** *For $p \ge 2$ and $r \in (0,1)$, we have*

$$\begin{aligned} \frac{\pi}{2}\left(\frac{\operatorname{artanh}_p(r)}{r}\right)^{1/2} &< \frac{\pi}{2}\left(1 - \frac{p-1}{p^2}\log(1-r^2)\right) \\ &< \mathcal{K}_a(r) < \frac{\pi}{2}\left(1 - \frac{2}{p\,\pi_p}\log(1-r^2)\right), \end{aligned}$$

where $a = 1/p$ and $\pi_p = 2\pi/(p\sin(\pi/p))$.

In [AQVV, Theorem 5.6] it was proved that for $a \in (0,1/2]$ we have

$$\mu_a\left(\frac{r\,s}{1+r's'}\right) \le \mu_a(r) + \mu_a(s) \le 2\mu_a\left(\frac{\sqrt{2rs}}{\sqrt{1+r\,s+r'\,s'}}\right),$$

for all $r, s \in (0,1)$. See also Theorem 4.3 below. In the next theorem we give a similar result for the function $\mathcal{K}_a$.

1.8. **Theorem.** *The function $f(x) = 1/\mathcal{K}_a(1/\cosh(x))$ is increasing and concave from $(0,\infty)$ onto $(0, 2/\pi)$. In particular,*

$$\frac{\mathcal{K}_a(r)\,\mathcal{K}_a(s)}{\mathcal{K}_a(rs/(1+r's'))} \le \mathcal{K}_a(r) + \mathcal{K}_a(s) \le \frac{2\,\mathcal{K}_a(r)\,\mathcal{K}_a(s)}{\mathcal{K}_a(\sqrt{rs/(1+rs+r's')})} \le \frac{2\,\mathcal{K}_a(r)\,\mathcal{K}_a(s)}{\mathcal{K}_a(rs)},$$

for all $r, s \in (0,1)$, with equality in the third inequality if and only if $r = s$.

There are several bounds for the function $\mu_a(r)$ when $a = 1/2$ in [AVV1, Chap.5]. In the next theorem we give a twosided bound for $\mu_a(r)$.

1.9. **Theorem.** *For $p \ge 2$ and $r \in (0,1)$, let*

$$l_p(r) = \left(\frac{\pi_p}{2}\right)^2 \left(\frac{p^2-(p-1)\log r^2}{p\,\pi_p - 2\log r'^2}\right) \quad \textit{and} \quad u_p(r) = \left(\frac{p}{2}\right)^2 \left(\frac{p\,\pi_p - 2\log r^2}{p^2-(p-1)\log r'^2}\right).$$

(1) The following inequalities hold

$$l_p(r) < \mu_a(r) < u_p(r)\,,$$

where $a = 1/p$.
(2) For $p = 2$ *we have*

$$u_2(r) < \frac{4}{\pi}\, l_2(r)\,.$$

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2. Preliminaries

For easy reference we record the next two lemmas from [AVV1] which have found many applications. Some of the applications are reviewed in [AVV5]. The first result sometimes called the *monotone l'Hospital rule.*

2.1. **Lemma.** [AVV1, Theorem 1.25] *For* $-\infty < a < b < \infty$, *let* $f, g : [a, b] \to \mathbb{R}$ *be continuous on* $[a, b]$, *and be differentiable on* (a, b). *Let* $g'(x) \neq 0$ *on* (a, b). *If* $f'(x)/g'(x)$ *is increasing (decreasing) on* (a, b), *then so are*

$$[f(x) - f(a)]/[g(x) - g(a)] \quad \textit{and} \quad [f(x) - f(b)]/[g(x) - g(b)].$$

If $f'(x)/g'(x)$ *is strictly monotone, then the monotonicity in the conclusion is also strict.*

2.2. **Lemma.** [AVV1, Lemma 1.24] *For* $p \in (0, \infty]$, *let* $I = [0, p)$, *and suppose that* $f, g : I \to [0, \infty)$ *are functions such that* $f(x)/g(x)$ *is decreasing on* $I \setminus \{0\}$ *and* $g(0) = 0$ *and* $g(x) > 0$ *for* $x > 0$. *Then*

$$f(x + y)(g(x) + g(y)) \le g(x + y)(f(x) + f(y))\,,$$

for $x, y, x + y \in I$. *Moreover, if the monotonicity of* $f(x)/g(x)$ *is strict then the above inequality is also strict on* $I \setminus \{0\}$.

For easy reference we recall the following lemmas from [AQVV].

2.3. **Lemma.** *For* $a \in (0, 1/2]$, $K \in (1, \infty)$, $r \in (0, 1)$ *and* $s = \varphi^a_K(r)$, *we have*

(1) $f(r) = s'\,\mathcal{K}_a(s)^2/(r'\,\mathcal{K}_a(r)^2)$ *is decreasing from* $(0, 1)$ *onto* $(0, 1)$,
(2) $g(r) = s\,\mathcal{K}'_a(s)^2/(r\,\mathcal{K}'_a(r)^2)$ *is decreasing from* $(0, 1)$ *onto* $(1, \infty)$,
(3) *the function* $r'^c\,\mathcal{K}_a(r)$ *is decreasing if and only if* $c \ge 2a(1 - a)$, *in which case* $r'^c\,\mathcal{K}_a(r)$ *is decreasing from* $(0, 1)$ *onto* $(0, \pi/2)$. *Moreover,* $\sqrt{r'}\,\mathcal{K}_a(r)$ *is decreasing for all* $a \in (0, 1/2]$.

2.4. **Lemma.** *The following formulae hold for* $a \in (0, 1/2]$, $r \in (0, 1)$ *and* $x, y, K \in (0, \infty)$,

(1) $\dfrac{dF}{dr} = \dfrac{a\,b}{c} F(1 + a, 1 + b; 1 + c; r);\quad F = F(a, b; c; r)\,,$

(2) $\dfrac{d\,\mathcal{K}_a(r)}{dr} = \dfrac{2(1-a)(\,\mathcal{E}_a(r) - r'^2\,\mathcal{K}_a(r))}{rr'^2}$,

(3) $\dfrac{d\,\mathcal{E}_a(r)}{dr} = \dfrac{2(a-1)(\,\mathcal{K}_a(r) - \,\mathcal{E}_a(r))}{r}$,

(4) $\dfrac{d\mu_a(r)}{dr} = \dfrac{-\pi^2}{4rr'^2\,\mathcal{K}_a(r)^2}$,

(5) $\dfrac{d\varphi_K^a(r)}{dr} = \dfrac{ss'^2\,\mathcal{K}_a(s)^2}{Krr'^2\,\mathcal{K}_a(r)^2} = \dfrac{ss'^2\,\mathcal{K}_a(s)\,\mathcal{K}'_a(s)}{rr'^2\,\mathcal{K}_a(r)\,\mathcal{K}'_a(r)} = K\dfrac{ss'^2\,\mathcal{K}'_a(s)^2}{rr'^2\,\mathcal{K}'_a(r)^2}$,

(6) $\dfrac{d\varphi_K^a(r)}{dK} = \dfrac{4ss'^2\,\mathcal{K}_a(s)^2\mu_a(r)}{\pi^2K^2}$,
where $s = \varphi_K^a(r)$,

(7) $\dfrac{d\eta_K^a(x)}{dx} = \dfrac{1}{K}\left(\dfrac{r'\,s\,\mathcal{K}_a(s)}{rs'\,\mathcal{K}_a(r)}\right)^2 = K\left(\dfrac{r'\,s\,\mathcal{K}'_a(s)}{rs'\,\mathcal{K}'_a(r)}\right)^2 = \left(\dfrac{r'\,s}{rs'}\right)^2\dfrac{\mathcal{K}_a(s)\,\mathcal{K}'_a(s)}{\mathcal{K}_a(r)\,\mathcal{K}'_a(r)}$,

(8) $\dfrac{d\eta_K^a(x)}{dK} = \dfrac{8\eta_K^a(x)\mu_a(r)K_a(s)^2}{\pi^2K^2}$,
in (7) *and* (8), $r = \sqrt{x/(1+x)}$ *and* $s = \varphi_K^a(r)$.

2.5. **Lemma.** [AVV1, Theorem 1.52(1)] *For* $a, b > 0$, *the function*

$$f(x) = \frac{F(a,b;a+b;x)-1}{\log(1/(1-x))}$$

is strictly increasing from $(0,1)$ *onto* $(ab/(a+b), 1/B(a,b))$.

2.6**. Proof of Theorem 1.5.** With $G(r) = F(a,b;c;r^p)$, and g as in Theorem 1.5 we get by Lemma 2.4(1)

$$g'(p) = -\frac{(G(r))^{1/p-1}}{cp^2}\left(c\,G(r)\log\left(\,G(r)\right) + a\,b\,p\,r^p\,F\left(a+1,b+1;c+1;r^p\right)\log(1/r)\right)$$

which is negative. Hence this implies (1), and (2) follows from (1). For (3), write $F(r) = F(-a,b;c;r^p)$. We define $h(p) = F(r)^{1/p}$ and get

$$h'(p) = \frac{(F(r))^{1/p-1}}{cp^2}\left(c\,F(r)\log\left(1/\,F(r)\right) + a\,b\,p\,r^p\,F\left(a+1,b+1;c+1;r^p\right)\log(1/r)\right)$$

which is positive because $F(r) \in (0,1)$. Hence h is increasing in p, and (3) follows easily. $\square$

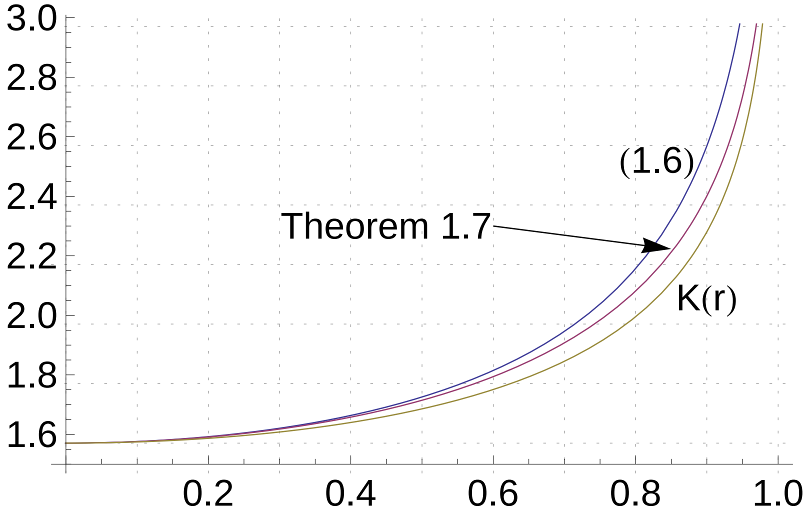

FIGURE 1. Comparison of upper bounds given in Theorem 1.7 and (1.6) for $\mathcal{K}(r)$.

2.7. **Proof of Theorem 1.7.** By the definition of $\mathrm{artanh_p}$, Lemma 2.5 and Bernoulli inequality we get

$$\begin{aligned}
\frac{\pi}{2}\left(\frac{\mathrm{artanh}_p(r)}{r}\right)^{1/2} &= \frac{\pi}{2}\left(F\left(1,\frac{1}{p};1+\frac{1}{p};r^p\right)\right)^{1/2}\\
&< \frac{\pi}{2}\left(1-\frac{1}{p}\log(1-r^p)\right)^{1/2}\\
&\le \frac{\pi}{2}\left(1+\frac{1}{2p}\log\left(\frac{1}{1-r^p}\right)\right)\\
&\le \frac{\pi}{2}\left(1+\frac{p-1}{p^2}\log\left(\frac{1}{1-r^p}\right)\right)\\
&\le \frac{\pi}{2}\left(1-\frac{p-1}{p^2}\log(1-r^2)\right)=\xi.
\end{aligned}$$

Again by Lemma 2.5 and [AS, 6.1.17] we obtain

$$\begin{aligned}
\xi &< \frac{\pi}{2} F\left(\frac{1}{p}, 1-\frac{1}{p}; 1; r^2\right) = \mathcal{K}_{1/p}(r) \\
&< \frac{\pi}{2}\left(1 - \frac{1}{B(1/p, 1-1/p)} \log(1-r^2)\right) \\
&= \frac{\pi}{2}\left(1 - \frac{2}{p\,\pi_p} \log(1-r^2)\right),
\end{aligned}$$

and this completes the proof. $\square$

2.8. **Proof of Theorem 1.8.** Writing $r = 1/\cosh(x)$ we have

$$\frac{dr}{dx} = -(\sinh x)/\cosh^2 x = -r\,r'$$

and

$$\begin{aligned}
f'(x) = -\frac{\mathcal{K}_a'(r)}{\mathcal{K}_a^2(r)} \frac{dr}{dx} &= -\frac{2(1-a)}{\mathcal{K}_a^2(r)} \frac{\mathcal{E}_a(r) - r'^2\,\mathcal{K}_a(r)}{r\,r'^2} (-r\,r') \\
&= 2(1-a) \frac{\mathcal{E}_a(r) - r'^2\,\mathcal{K}_a(r)}{r'\;\mathcal{K}_a(r)^2},
\end{aligned}$$

which is positive and increasing in r by Lemma 2.3(3) and therefore $f'(x)$ is decreasing in x and f is concave. Hence,

$$\frac{1}{2}(f(x)+f(y)) \le f\left(\frac{x+y}{2}\right)$$

$$\iff \frac{1}{2}\left(\frac{1}{\mathcal{K}_a(1/\cosh(x))} + \frac{1}{\mathcal{K}_a(1/\cosh(y))}\right) \le \frac{1}{\mathcal{K}_a(1/\cosh((x+y)/2))}$$

$$\iff \mathcal{K}_a(r) + \mathcal{K}_a(s) \le \frac{2\,\mathcal{K}_a(r)\,\mathcal{K}_a(s)}{\mathcal{K}(\sqrt{rs/(1+rs+r's')})},$$

by using $\cosh^2((x+y)/2) = (1 + rs\,'+'r\,s\,)/(rs)$ and setting $s = 1/\cosh(y)$. Clearly,

$$(r-s)^2 \ge 0 \iff 1 - 2rs + r^2s^2 \ge 1 - r^2 - s^2 + r^2s^2$$

$$\iff 1 - rs \ge r's' \iff 2 \ge 1 + rs + r's' \iff 2rs/(1+rs+r's') \ge rs,$$

and the third inequality follows. Obviously, $f(0+) = 0$, and $f'(x)$ is decreasing in x. Then $f(x)/x$ is decreasing and $f(x+y) \le f(x) + f(y)$ by Lemmas 2.1 and 2.2, respectively. This implies the first inequality.

2.9. **Proof of Theorem 1.9.** By Lemma 2.5 we get

(a) $\displaystyle 1 - \frac{p-1}{p^2} \log r^2 < F\left(\frac{1}{p}, 1-\frac{1}{p}; 1; 1-r^2\right) < 1 - \frac{2}{p\,\pi_p} \log r^2$

(b) $\displaystyle 1 - \frac{p-1}{p^2} \log(1-r^2) < F\left(\frac{1}{p}, 1-\frac{1}{p}; 1; r^2\right) < 1 - \frac{2}{p\,\pi_p} \log(1-r^2).$

By using (a), (b) and the definition of μ_a, we get (1). The claim (2) is equivalent to

$$\frac{2(\pi-\log(r^2))}{4-\log(1-r^2)} < \frac{4}{\pi}\left(\frac{\pi}{2}\right)^2 \frac{4-\log(r^2)}{\pi-\log(1-r^2)}$$

$$\Longleftrightarrow 4(\pi-\log(r^2))(\pi-\log(1-r^2))-(4-\log(r^2))(4-\log(1-r^2))<0$$

$$\Longleftrightarrow (\pi-4)(4\pi-\log(r^2)\log(1-r^2))<(\pi-4)(4\pi-(\log(2))^2)<0\,.$$

For the second last inequality we define $w(x)=\log(x)\log(1-x)$, and get

$$w^{'}(x)=\frac{(1-x)\log(1-x)-x\log(x)}{x(1-x)}=\frac{-g(x)}{x(1-x)}\,,$$

and see that $g(x)=x\log(x)-(1-x)\log(1-x)$ is convex on $(0,1/2)$ and concave on $(1/2,1)$. This implies that $g<0$ for $x\in(0,1/2)$ and $g>0$ for $x\in(1/2,1)$. Therefore w is increasing in $(0,1/2)$ and decreasing in $(1/2,1)$. Hence the function w has a global maximum at $x=1/2$ and this completes the proof.

□

One can obtain the following inequalities by using the proof of Theorem 1.9:

$$\frac{p\,\pi_p}{2\pi}\frac{\mathcal{K}_a(r)}{(1-(2/(p\,\pi_p))\log r^2)}\le\mu_a(r^{'})\le\frac{p\,\pi_p}{2\pi}\frac{\mathcal{K}_a(r)}{(1-((p-1)/p)\log r^2)}\,,$$

with $a=1/p$ and $p\ge 2$.

2.10. **Lemma.** *The following inequalities hold for all $r,s\in(0,1)$ and $a\in(0,1/2]$,*

(1) $\mathcal{K}_a(r\,s)\le\sqrt{\mathcal{K}_a(r^2)\,\mathcal{K}_a(s^2)}\le\frac{2}{\pi}\,\mathcal{K}_a(r)\,\mathcal{K}_a(s)\,,$

(2) $\frac{2}{\pi}\,\mathcal{E}_a(r)\,\mathcal{E}_a(s)\le\sqrt{\mathcal{E}_a(r^2)\,\mathcal{E}_a(s^2)}\le\,\mathcal{E}_a(rs)\,.$

Proof. Define $f(x)=\log(\,\mathcal{K}_a(e^{-x}))$, $x>0$. We get by Lemma 2.4(2)

$$f'(x)=-2(1-a)\frac{\mathcal{E}_a(r)-r'^2\,\mathcal{K}_a(r)}{r'^2\,\mathcal{K}_a(r)}\,,\quad r=e^{-x}\,,$$

and this is negative by the fact that $h(r)=\,\mathcal{E}_a(r)-r'^2\,\mathcal{K}_a(r)>0$ and decreasing in r by [AQVV, Lemma 5.4(1)] and the fact that h is increasing ($h'(r)=2ar\,\mathcal{K}_a(r)>0$). Therefore $f'(x)$ is increasing in x, hence f is convex, and this implies the first inequality of part one. The second inequality follows from Theorem 1.5(2).

The first inequality of part two follows from the Theorem 1.5(3), for the second inequality we define $g(x)=\log(\,\mathcal{E}_a(z))$, $z=e^{-x}$, $x>0$, and get Lemma 2.4(3)

$$g^{'}(x)=2(1-a)\frac{(\,\mathcal{K}_a(z)-\,\mathcal{E}_a(z))}{\mathcal{E}_a(z)}\,,$$

which is positive and increasing in z by [AQVV, Theorem 4.1(3), Lemma 5.2(3)], hence $g^{'}(x)$ is decreasing in x, therefore g is increasing and concave. This implies that

$$\log(\,\mathcal{E}_a(e^{-(x+y)/2}))\ge(\log(\,\mathcal{E}_a(e^{-x}))+\log(\,\mathcal{E}_a(e^{-y})))/2\,,$$

and the second inequality follows if we set $r = e^{-x/2}$ and $s = e^{-y/2}$. $\square$

3. Few remarks on special functions

In this section we generalize some results from [AVV1, Chapter 10].

3.1. **Theorem.** *The function $\mu_a^{-1}(y)$ has exactly one inflection point and it is log-concave from $(0, \infty)$ onto $(0, 1)$. In particular,*

$$(\mu_a^{-1}(x))^p (\mu_a^{-1}(y))^q \le \mu_a^{-1}(p\,x + q\,y)$$

for p, q, x, $y > 0$ with $p + q = 1$.

Proof. Letting $s = \mu_a^{-1}(y)$ we see that $\mu_a(s) = y$. By Lemma 2.4(4) we get

$$\frac{ds}{dy} = -\frac{4}{\pi^2} s s'^2\, \mathcal{K}_a(s)^2\,,$$

$$\begin{aligned}
\frac{d^2 s}{dy^2} &= -\frac{ds}{dy}\frac{4}{\pi^2}(s'^2\, \mathcal{K}_a(s)^2 - 2s^2\, \mathcal{K}_a(s)^2 + 2\, \mathcal{K}_a(s)^2(\, \mathcal{E}_a(s) - s'^2\, \mathcal{K}_a(s))) \\
&= \frac{16}{\pi^4} s s'^2\, \mathcal{K}_a(s)^3 (2\, \mathcal{E}_a(s) - (1 + s^2)\, \mathcal{K}_a(s))\,.
\end{aligned}$$

We see that $2\, \mathcal{E}_a(s) - (1 + s^2)\, \mathcal{K}_a(s)$ is increasing from $(0, \infty)$ onto $(-\infty, \pi/2)$ as a function of y. Hence $d(\mu_a^{-1}(y_0))/dy^2 = 0$, for $y_0 \in (0, \infty)$ and μ_a^{-1} has exactly one inflection point. Let $f(y) = \log(\mu_a^{-1}(y)) = \log\, s$, we get

$$f'(y) = -\frac{4}{\pi^2} s'^2\, \mathcal{K}_a(s)^2$$

which is decreasing as a function of y, by Lemma 2.3(3), hence μ_a^{-1} is log-concave. This completes the proof. $\square$

3.2. **Corollary.** (1) *For $K \ge 1$, the function $f(r) = (\log\, \varphi_K^a(r))/\log r$ is strictly decreasing from $(0, 1)$ onto $(0, 1/K)$.*
(2) *For $K \ge 1\,, r \in (0, 1)$, the function $g(p) = \varphi_K^a(r^p)^{1/p}$ is decreasing from $(0, \infty)$ onto $(r^{1/K}, 1)$. In particular,*

$$r^{p/K} \le \varphi_K^a(r^p) \le \varphi_K^a(r)^p,\ \ p \ge 1\,,$$

and

$$\varphi_K^a(r^p) \ge \varphi_K^a(r)^p, \quad 0 < p \le 1\,.$$

Proof. Let $s = \varphi_K^a(r)$. By Lemma 2.4(5) we get

$$f'(r) = \frac{rss'^2}{srr'^2}\frac{\mathcal{K}_a(s)\, \mathcal{K}_a'(s)}{\mathcal{K}_a(r)\, \mathcal{K}_a'(r)} \log r - \log s,$$

and this is equivalent to

$$r(\log r)^2 f'(r) = s'^2\, \mathcal{K}_a(s)\, \mathcal{K}_a'(s) \left(\frac{\log r}{r'^2\, \mathcal{K}_a(r)\, \mathcal{K}_a'(r)} - \frac{\log s}{s'^2\, \mathcal{K}_a(s)\, \mathcal{K}_a'(s)} \right),$$

which is negative by Lemma 2.3(3). The limiting values follow from l'Hôpital Rule and Lemma 2.3(1). We observe that

$$\log g(p) = \left(\frac{\log \varphi_K^a(r^p)}{\log(r^p)}\right) \log r\,,$$

and (2) follows from (1). $\square$

3.3. **Lemma.** *For* $0 < a \le 1/2$, $K, p \ge 1$ *and* $r, s \in (0,1)$, *the following inequalities hold*

$$\frac{\sqrt[p]{\varphi_K^a(r^p)} + \sqrt[p]{\varphi_K^a(s^p)}}{1 + \sqrt[p]{\varphi_K^a(r^p)\varphi_K^a(s^p)}} \le \frac{\varphi_K^a(r) + \varphi_K^a(s)}{1 + \varphi_K^a(r)\varphi_K^a(s)} \le \frac{\varphi_K^a(\sqrt[p]{r})^p + \varphi_K^a(\sqrt[p]{s})^p}{1 + (\varphi_K^a(\sqrt[p]{r})\varphi_K^a(\sqrt[p]{s}))^p}\,.$$

Proof. It follows from the Corollary 3.2(2) that

$$\varphi_K^a(r^p)^{1/p} \le \varphi_K^a(r)\,.$$

From the fact that artanh is increasing, we conclude that

$$\operatorname{artanh}(\varphi_K^a(r^p)^{1/p}) + \operatorname{artanh}(\varphi_K^a(s^p)^{1/p}) \le \operatorname{artanh}(\varphi_K^a(r)) + \operatorname{artanh}(\varphi_K^a(s))\,.$$

This is equivalent to

$$\operatorname{artanh}\left(\frac{\varphi_K^a(r^p)^{1/p} + \varphi_K^a(s^p)^{1/p}}{1 + (\varphi_K^a(r^p) + \varphi_K^a(s^p))^{1/p}}\right) \le \operatorname{artanh}\left(\frac{\varphi_K^a(r) + \varphi_K^a(s)}{1 + (\varphi_K^a(r) + \varphi_K^a(s))}\right),$$

and the first inequality holds. Similarly, the second inequality follows from $\varphi_K^a(r) \le \varphi_K^a(r^{1/p})^p$. $\square$

For $0 < a \le 1$, $K \ge 1$ and $r, s \in (0,1)$, the following inequality

$$\varphi_K^a\left(\frac{r+s}{1+rs}\right) \le \frac{\varphi_K^a(r) + \varphi_K^a(s)}{1 + \varphi_K^a(r)\varphi_K^a(s)} \tag{3.4}$$

is given in [AQVV, Remark 6.17].

3.5. **Theorem.** *For* $r, s \in (0,1)$, *we have*

(1) $|\varphi_K^a(r) - \varphi_K^a(s)| \le \varphi_K^a(|r-s|) \le e^{(1-1/K)R(a)/2}|r-s|^{1/K}$, $K \ge 1$,
here $R(a)$ *is as in* [AQVV, Theorem 6.7]

(2) $|\varphi_K^a(r) - \varphi_K^a(s)| \ge \varphi_K^a(|r-s|) \ge e^{(1-1/K)R(a)/2}|r-s|^{1/K}$, $0 < K \le 1$.

Proof. It follows from [AQVV, Theorem 6.7] that $r^{-1}\varphi_K^a(r)$ is decreasing on $(0,1)$. If $K > 1$ then by Lemma 2.2 we obtain

$$\varphi_K^a(x+y) \le \varphi_K^a(x) + \varphi_K^a(y),\ \ x, y \in (0,1)\,.$$

Now the first inequality in (1) follows if we take $r = x+y$ and $s = y$, the second one follows from [AQVV, Theorem 6.7]. Next, (2) follows from (1) and the fact that

$$\varphi_{AB}^a(r) = \varphi_A^a(\varphi_B^a(r)),\ \ A, B > 0\,, r \in (0,1)$$

when we replace K, r and s by $1/K$, $\varphi_{1/K}^a(r)$, $\varphi_{1/K}^a(s)$, respectively. $\square$

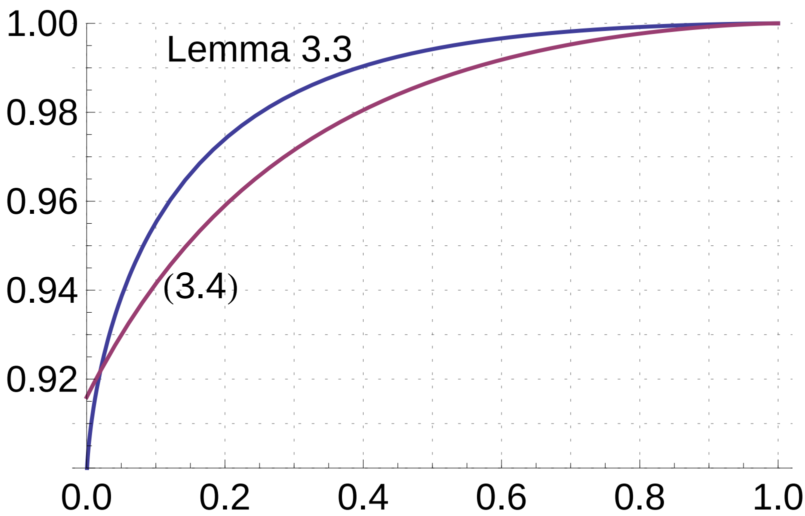

FIGURE 2. Let $g(a,K,p,r,s) = \frac{\sqrt[p]{\varphi_K^a(r^p)}+\sqrt[p]{\varphi_K^a(s^p)}}{1+\sqrt[p]{\varphi_K^a(r^p)\varphi_K^a(s^p)}}$, $h(a,K,r,s) = \varphi_K^a\left(\frac{r+s}{1+rs}\right)$ be the lower bounds in Lemma 3.3 and (3.4), respectively. For $a=0.2$, $K=1.5$, $p=1.3$ and $s=0.5$ the functions g and h are plotted. We see that for $r\in(0.02,1)$ the first lower bound is better.

3.6. **Theorem.** *For $c,r\in(0,1)$ and $K,L\in(0,\infty)$ we have*

(1) *The function $f(K)=\log(\varphi_K^a(r))$ is increasing and concave from $(0,\infty)$ onto $(-\infty,0)$.*
(2) *The function $g(K)=\operatorname{artanh}(\varphi_K^a(r))$ is increasing and convex from $(0,\infty)$ onto $(0,\infty)$.*
(3) $\varphi_K^a(r)^c\varphi_L^a(r)^{1-c}\le\varphi_{cK+(1-c)L}^a(r)\le\tanh(c\operatorname{artanh}(\varphi_K^a(r))+(1-c)\operatorname{artanh}(\varphi_L^a(r)))$.
(4)
$$\sqrt{\varphi_K^a(r)\varphi_L^a(r)}\le\varphi_{(K+L)/2}^a(r)$$
$$\le\frac{\varphi_K^a(r)+\varphi_L^a(r)}{1+\varphi_K^a(r)\varphi_L^a(r)+\varphi_{1/K}^a(r')\varphi_{1/L}^a(r')}.$$

Proof. For (1), by Lemma 2.4(6) we get
$$f'(K)=4s'^2\,\mathcal{K}_a(s)^2\mu_a(r)/(\pi^2K),$$
which is positive and decreasing by Lemma 2.3(3). For (2), we get
$$f'(K)=4s\,\mathcal{K}_a(s)^2\mu_a(r)/(\pi^2K^2)=s\,\mathcal{K}_a'(s)^2/\mu_a(r)$$
by Lemma 2.4(6), which is positive and increasing by Lemma 2.3(3). By (1) and (2) we get
$$c\log(\varphi_K^a(r))+(1-c)\log(\varphi_L^a(r))\le\log(\varphi_{cK+(1-a)L}^a(r)),$$

$$\operatorname{artanh}(\varphi^a_{cK+(1-c)K}(r)) \le a\operatorname{artanh}(\varphi^a_K(r)) + (1-c)\operatorname{artanh}(\varphi^a_L(r)),$$

respectively, and (3) follows. Also

$$(\log(\varphi^a_K(r)) + \log(\varphi^a_L(r)))/2 \le \log(\varphi^a_{(K+L)/2}(r))\,,$$

$$\operatorname{artanh}(\varphi^a_{(K+L)/2}(r)) \le (\operatorname{artanh}(\varphi^a_K(r)) + \operatorname{artanh}(\varphi^a_L))/2\,,$$

follow from (1) and (2), and hence (4) holds. $\square$

3.7. **Theorem.** *For $K \ge 1$ and $0 < m < n$, the following inequalities hold*

(1) $\eta^a_K(m\,n) \le \sqrt{\eta^a_K(m^2)\eta^a_K(n^2)}\,,$

(2) $\left(\dfrac{n}{m}\right)^{1/K} < \dfrac{\eta^a_K(n)}{\eta^a_K(m)} < \left(\dfrac{n}{m}\right)^K\,,$

(3) $\eta^a_K(m)\eta^a_K(n) < \left(\eta^a_K\left(\dfrac{m+n}{2}\right)\right)^2\,,$

(4) $2\dfrac{\eta^a_K(m)\eta^a_K(n)}{\eta^a_K(m)+\eta^a_K(n)} < \eta^a_K(\sqrt{m\,n}) < \sqrt{\eta^a_K(m)\eta^a_K(n)}.$

Proof. We define a function $g(x) = \log\eta^a_K(e^x)$ on $\mathbb{R}$. By [AQVV, Theorem 1.16], g is increasing, convex and satisfies $1/K \le g'(x) \le K$. Then

$$\begin{aligned}\log\eta^a_K(e^{(x+y)/2}) &= g\left(\frac{x+y}{2}\right) \le \frac{g(x)+g(y)}{2}\\ &= \frac{1}{2}\log(\eta^a_K(e^x)) + \frac{1}{2}\log(\eta^a_K(e^y)),\end{aligned}$$

and this is equivalent to

$$\log\left(\eta^a_K(e^{x/2}e^{y/2})\right) \le \log(\eta^a_K(e^{x/2})\eta^a_K(e^{y/2}))\,.$$

Hence (1) follows if we set $e^{x/2} = m$ and $e^{y/2} = n$. For (2), let $x > y$. Then by the inequality $1/K \le g'(x) \le K$ and the mean value theorem we get

$$(x-y)/K \le g(x) - g(y) \le K(x-y),$$

and this is equivalent to

$$(\log(e^x) - \log(e^y))/K \le \log(\eta^a_K(e^x)) - \log(\eta^a_K(e^y)) \le K(\log(e^x) - \log(e^y))\,.$$

By setting $e^{x/2} = m$ and $e^{y/2} = n$ we get the desired inequality. For (3), let $f(x) = \log(\eta^a_K(x))$, $r = \sqrt{x/(1+x)}$ and $s = \varphi^a_K(r)$. Then by Lemma 2.4(7) we get

$$\begin{aligned}f'(x) &= \frac{1}{K}\left(\frac{s'}{s}\right)^2\left(\frac{sr'\,\mathcal{K}_a(s)}{rs'\,\mathcal{K}_a(r)}\right)^2 = \frac{1}{K}\left(\frac{r'}{r}\right)^2\left(\frac{\mathcal{K}_a(s)}{\mathcal{K}_a(r)}\right)^2\\ &= \frac{1}{K}\left(\frac{r'}{s}\right)^2\left(\frac{s\,\mathcal{K}_a(s)}{r\,\mathcal{K}_a(r)}\right)^2,\end{aligned}$$

which is positive and decreasing by Lemma 2.3(2). Hence $(f(x)+f(y))/2 \le f((x+y)/2)$, and the inequality follows.

For (4), letting $h(x) = 1/\eta^a_K(e^x)$, we see that this is log-concave by (1), and we get

$$\frac{\log(1/\eta_K^a(e^x)) + \log(1/\eta_K^a(e^y))}{2} < \log(1/\eta_K^a(e^{(x+y)/2}))\,,$$

Setting $e^x = m$ and $e^y = n$ we get the second inequality. We observe that $h(x) = (s^{'}/s)$, $s = \varphi_K^a(r)$, $r = \sqrt{e^x/(e^x+1)}$. We get

$$-f^{'}(x) = \frac{1}{K}\left(\frac{r^{'}}{s}\right)\left(\frac{s^{'}\,\mathcal{K}_a(s)}{r^{'}\,\mathcal{K}_a(r)}\right)^2,$$

which is positive and decreasing by Lemma 2.3(1), hence h is convex, and the first inequality follows easily. $\square$

3.8. **Theorem.** *For $x \in (0,\infty)$, the function $f : (0,\infty) \to (0,\infty)$ defined by $f(K) = \eta_K^a(x)$ is increasing, convex and log-concave. In particular,*

$$\eta_K^a(x)^c \eta_L^a(x)^{1-c} \le \eta_{cK+(1-c)L}^a(x) \le c\,\eta_K^a(x) + (1-c)\eta_L^a(x)$$

for $K, L, x \in (0,\infty)$ and $c \in (0,1)$, with equality if and only if $K = L$.

Proof. We observe that $f(K) = (s/s^{'})^2$, where $s = \varphi_K^a(r)$ and $r = \sqrt{x/(x+1)}$. We get by Lemma 2.4(8)

$$f^{'}(K) = \frac{8s^2\,\mathcal{K}_a(s)^2}{\pi^2 s^{'2} K^2}\mu_a(r) = \frac{4}{\pi\sin(\pi a)}\frac{\mathcal{K}_a(r)}{\mathcal{K}_a^{'}(r)}\left(\frac{s\,\mathcal{K}_a^{'}(s)}{s^{'}}\right)^2,$$

which is positive and increasing by Lemma 2.3(3), hence f is increasing and convex. For log-concavity, let $g(K) = \log(\eta_K^a(x))$. By Lemma 2.4(8) we get

$$g^{'}(K) = \frac{8\,\mathcal{K}_a(s)^2}{\pi^2 K^2}\mu_a(r) = \frac{4}{\pi\sin(\pi a)}\frac{\mathcal{K}_a(r)}{\mathcal{K}_a^{'}(r)}\,\mathcal{K}_a^{'}(s)^2,$$

which is decreasing, hence f is log-concave. $\square$

3.9. **Theorem.** *The function*

$$f(K) = \frac{\log \eta_K^a(x) - \log(x)}{K-1}$$

is decreasing from $(1,\infty)$ onto

$$\left(\frac{\pi\,\mathcal{K}_a(r)}{\sin(\pi\,a)\,\mathcal{K}_a^{'}(r)}, \frac{4\,\mathcal{K}_a(r)\,\mathcal{K}_a^{'}(r)}{\pi\sin(\pi\,a)}\right),$$

and the function

$$g(K) = \frac{\eta_K^a(x) - (x)}{K-1}$$

is increasing from $(1,\infty)$ onto

$$(4r^2\sin(\pi\,a)\,\mathcal{K}_a(r)\,\mathcal{K}_a^{'}(r)/(\pi r^{'2}), \infty),$$

where $r = \sqrt{x/(x+1)}$.

Proof. It follows from Theorem 3.8 and Lemma 2.1 that f is monotone. Let $s = \varphi_K^a(r)$, by Lemma 2.4(6), the l'Hôpital Rule and definition of μ_a we get

$$\lim_{K\to 1} f(K) = \lim_{K\to 1} \frac{2}{K-1}\log\left(\frac{sr'}{s'r}\right)$$

$$= \lim_{K\to 1} \frac{8\,\mathcal{K}_a(s)^2\mu_a(r)}{K^2\pi^2} = \frac{8}{\pi^2}\,\mathcal{K}_a(r)^2\mu_a(r) = \frac{4\,\mathcal{K}_a(r)\,\mathcal{K}_a'(r)}{\pi\sin(\pi\,a)}.$$

By using the fact that $K = \mu_a(r)/\mu_a(s)$ and the l'Hôpital Rule, we get

$$\lim_{K\to\infty} f(K) = \lim_{K\to\infty} \frac{8\mu_a(s)^2\,\mathcal{K}_a(s)^2}{\pi^2\mu_a(r)}$$

$$= \lim_{K\to\infty} \frac{2\,\mathcal{K}_a'(s)^2}{\sin^2(\pi\,a)\mu_a(r)} = \frac{2\,\mathcal{K}_a(0)^2}{\sin^2(\pi\,a)\mu_a(r)} = \frac{\pi\,\mathcal{K}_a(r)}{\sin(\pi\,a)\,\mathcal{K}_a'(r)}.$$

Next, let $g(K) = G(K)/H(K)$, where $G(K) = (s/s')^2 - (r/r')^2$ and $H(K) = K-1$. We see that $G(1) = H(1) = 0$ and $G(\infty) = H(\infty) = \infty$. We see that

$$G'(K)/H'(K) = 2(s\,\mathcal{K}_a'(s))^2/(s'^2\mu_a(r))\,,$$

and it follows from Lemmas 2.3(3) and 2.1 that $g(K)$ is increasing and the required limiting values follow from $\varphi_K^a(r) = \mu_a^{-1}(\mu_a(r)/K)$. ☐

3.10. **Remark.** If we take $x = 1$ in Theorem 3.9, then

(1) the function $\log(\lambda_a(K))/(K-1)$ is strictly decreasing from $(1,\infty)$ onto $(\pi/\sin(\pi\,a), t)$,
(2) the function $(\lambda_a(K)-1)/(K-1)$ is increasing from $(1,\infty)$ onto $(t\,\sin^2(\pi\,a),\infty)$, where $t = 4\,\mathcal{K}_a(1/\sqrt{2})^2/(\pi\sin(\pi\,a))$.

In particular,

$$e^{\pi(K-1)/\sin(\pi\,a)} < \lambda_a(K) < e^{t(K-1)},$$

$$1 + t(K-1)\sin^2(\pi\,a) < \lambda_a(K) < \infty,$$

respectively, and we get

$$\max\{e^{\pi(K-1)/\sin(\pi\,a)}, 1 + t(K-1)\sin^2(\pi\,a)\} < \lambda_a(K) < e^{t(K-1)}.$$

3.11. **Lemma.** *For $c \in [-3,0)$, the function $f(r) = \mathcal{K}_a(r)^c + \mathcal{K}_a'(r)^c$ is strictly increasing from $(0,1/\sqrt{2})$ onto $((\pi/2)^c, 2\,\mathcal{K}_a(1/\sqrt{2})^c)$.*

Proof. By Lemma 2.4(2) we get

$$f'(r) = \frac{2(1-a)c\,\mathcal{K}_a(r)^{c-1}\left(\mathcal{E}_a(r) - r'^2\,\mathcal{K}_a(r)\right)}{rr'} - \frac{2(1-a)c\,\mathcal{K}_a'(r)^{c-1}(\,\mathcal{E}_a'(r) - r^2\,\mathcal{K}_a'(r))}{rr'}$$

$$= \frac{2(1-a)c(\,\mathcal{K}_a(r)\,\mathcal{K}_a'(r))^{c-1}}{rr'}(h(r) - h(r')),$$

and here $h(r) = \dfrac{r^2\,\mathcal{K}_a'(r)^{1-c}}{r^2}(\,\mathcal{E}_a(r) - r'^2\,\mathcal{K}_a(r))$, which is increasing on $(0,1)$ by [AVV1, Theorem 3.21(1)] and Lemma 2.3(3). Hence $f'(r) < 0$ on $(0, 1/\sqrt{2})$, and the limiting values are clear. $\square$

3.12. **Theorem.** (1) *For* $K > 1$, *the function* $(\log(\lambda_a(K))/(K - 1/K)$ *is strictly increasing from* $(1,\infty)$ *onto* $(2\,\mathcal{K}_a(1/\sqrt{2})/(\pi\,\sin(\pi\,a)), \pi/\sin(\pi\,a))$.
(2) *The function* $\log(\lambda_a(K)+1)$ *is convex on* $(0,\infty)$, *and* $\log(\lambda_a(K))$ *is concave.*
(3) *The function* $g(K) = \log(\lambda_a(K))/\log K$ *is strictly increasing on* $(1,\infty)$. *In particular, for* $c \in (0,1)$

$$\lambda_a(K^c) < (\lambda_a(K))^c\,.$$

Proof. For (1), let $r = \mu_a^{-1}(\pi K/(2\sin(\pi\,a)))$, $0 \le r \le 1/\sqrt{2}$. Then by (1.3)

$$r' = \sqrt{1 - \left(\mu_a^{-1}\left(\frac{\pi K}{2\sin(\pi\,a)}\right)\right)^2}$$

$$= \sqrt{1 - \left(\mu_a^{-1}\left(K\mu_a\left(\frac{1}{\sqrt{2}}\right)\right)\right)^2} = \mu_a^{-1}\left(\frac{\pi}{2K\sin(\pi\,a)}\right),$$

we observe that $K = \mathcal{K}_a'(r)/\mathcal{K}_a(r)$. Now it is enough to prove that the function

$$f(r) = \frac{2\log(r'/r)}{\mathcal{K}_a'(r)/\mathcal{K}_a(r) - \mathcal{K}_a(r)\,\mathcal{K}_a'(r)} = \frac{\pi\log(r'/r)}{\sin(\pi\,a)(\mu_a(r) + \mu_a(r'))},$$

is strictly decreasing on $(0, 1/\sqrt{2})$. Set $f(r) = G(r)/H(r)$. Clearly, $G(1/\sqrt{2}) = H(1/\sqrt{2}) = 0$. By Lemma 2.4(4) we get

$$\frac{G'(K)}{H'(K)} = \frac{4}{\pi\sin(\pi\,a)(\,\mathcal{K}_a(r)^{-2} - \mathcal{K}_a(r')^{-2})},$$

which is strictly decreasing from $(0, 1/\sqrt{2})$ onto

$$(2\,\mathcal{K}_a(1/\sqrt{2})/(\pi\,\sin(\pi\,a)), \pi/\sin(\pi\,a))$$

by Lemma 3.11. Now the proof of (1) follows from Lemma 2.1. For (2), it follows from Theorem 3.8 that $\log(\lambda_a(K))$ is concave. Letting $f(K) = \lambda_a(K) + 1$ we have

$$f(K) = \left(\mu_a^{-1}\left(\frac{\pi K}{2\sin(\pi\,a)}\right)\right)^{-2},$$

by (1.4) and (1.3). Now we have $\log f(K) = -2\log y$, here $\mu_a(y) = \pi K/(2\sin(\pi\,a))$. By Lemma 2.4(4) we get

$$\frac{f'(K)}{f(K)} = -\frac{2}{y}\frac{dy}{dK} = \frac{4}{\pi}\,y'\,\mathcal{K}_a(y),$$

which is decreasing in y by Lemma 2.3(3), and increasing in K. Hence $\log f(K)$ is convex.

For (3), $K>1$, let $h(K)=(K-1/K)/\log K$. We get

$$h'(K)=\frac{(1+K^2)\log K-(K^2-1)}{(K\log K)^2},$$

which is positive because

$$\log K>\frac{2(K-1)}{K+1}>\frac{K^2-1}{K^2+1}$$

by [AVV1, 1.58(4)a], hence h is strictly increasing. Also

$$g(K)=h(K)\frac{\log(\lambda_a(K))}{K-1/K}=\frac{\log(\lambda_a(K))}{\log K}$$

is strictly increasing by (1). This implies that

$$\frac{\log(\lambda_a(K^c))}{c\log K}<\frac{\log(\lambda_a(K))}{\log K},$$

and hence (3) follows. $\square$

3.13. **Corollary.** *For $0<r<1/\sqrt{2}$ and $t=\pi^2/(2\,\mathcal{K}_a(1/\sqrt{2})^2)$, we have*
(1) The function $f(r)=(\mu_a(r)-\mu_a(r'))/\log(r'/r)$ is increasing from $(0,1/\sqrt{2})$ onto $(1,t)$. In particular,

$$\log(r'/r)<\mu_a(r)-\mu_a(r')<\frac{\pi^2}{2\,\mathcal{K}_a(1/\sqrt{2})^2}\log(r'/r).$$

(2) For $g(r)=\log(r'/r)$,

$$g(r)+\sqrt{(\pi/\sin(\pi\,a))^2+g(r)^2}<2\mu_a(r)<t\,g(r)+\sqrt{(\pi/\sin(\pi\,a))^2+t^2\,g(r)^2}.$$

Proof. It follows from the proof of Theorem 3.12(1) that $f(r)$ is increasing, and limiting values follows easily by the l'Hôpital Rule. For (2), from the definition of μ_a we get $\mu_a(r')=\pi^2/((2\sin(\pi\,a))^2\mu_a(r))$, replacing this in (1) we obtain

$$1<\frac{\mu_a(r)^2-\pi^2/(2\sin(\pi\,a))^2}{\mu_a(r)\log(r'/r)}<t=\frac{\pi^2}{2\,\mathcal{K}_a(1\sqrt{2})^2}\,.$$

This implies that

$$\mu_a(r)^2-\mu_a(r)\log(r'/r)>\frac{\pi^2}{(2\sin(\pi\,a))^2} \tag{3.14}$$

and

$$\mu_a(r)^2-t\,\mu_a(r)\log(r'/r)<\frac{\pi^2}{(2\sin(\pi\,a))^2}\,. \tag{3.15}$$

We get the left and right inequalities by solving (3.14) and (3.15) for $\mu_a(r)$, respectively. $\square$

4. Three-parameter complete elliptic integrals

The results in this section have counterpart in [AQVV]. For $a,b,c>0$, $a+b\geq c$, the decreasing homeomorphism $\mu_{a,b,c}:(0,1)\to(0,1)$, defined by

$$\mu_{a,b,c}(r)=\frac{B(a,b)}{2}\frac{F(a,b;c;r'^2)}{F(a,b;c;r^2)},\quad r\in(0,1)$$

The (a,b,c)-modular function is defined by

$$\varphi_K^{a,b,c}(r)=\mu_{a,b,c}^{-1}(\mu_{a,b,c}(r)/K)\,.$$

We denote, in case $a<c$

$$\mu_{a,c}(r)=\mu_{a,c-a,c}(r)\quad\text{and}\quad\varphi_K^{a,c}(r)=\varphi_K^{a,c-a,c}(r)\,.$$

We define the three-parameter complete elliptic integrals of the first and second kinds for $0<a<\min\{c,1\}$ and $0<b<c\leq a+b$, by

$$\mathcal{K}_{a,b,c}(r)=\frac{B(a,b)}{2}F(a,b;c;r^2)$$

$$\mathcal{E}_{a,b,c}(r)=\frac{B(a,b)}{2}F(a-1,b;c;r^2)\,,$$

and denote

$$\mathcal{K}_{a,c}(r)=\mathcal{K}_{a,c-a,c}(r)\quad\text{and}\quad\mathcal{E}_{a,c}(r)=\mathcal{E}_{a,c-a,c}(r)\,.$$

4.1. **Lemma.** [HLVV, Theorem 3.6] *For $0<a<c\leq 1$, the function $f(r)=\mu_{a,c}(r)\operatorname{artanh} r$ is strictly increasing from $(0,1)$ onto $(0,(B/2)^2)$.*

4.2. **Lemma.** [HLVV, Lemma 4.1] *Let $a<c\leq 1$, $K\in(1,\infty)$, $r\in(0,1)$, and let $s=\varphi_K^{a,c}(r)$ and $t=\varphi_{1/K}^{a,c}(r)$. Then the function*

(1) $f_1(r)=\mathcal{K}_{a,c}(s)/\mathcal{K}_{a,c}(r)$ *is increasing from $(0,1)$ onto $(1,K)$,*
(2) $f_2(r)=s'\mathcal{K}_{a,c}(s)^2/(r'\mathcal{K}_{a,c}(r)^2)$ *is decreasing from $(0,1)$ onto $(0,1)$,*
(3) $f_3(r)=s\mathcal{K}'_{a,c}(s)^2/(r\mathcal{K}'_{a,c}(r)^2)$ *is decreasing from $(0,1)$ onto $(1,\infty)$,*
(4) $g_1(r)=\mathcal{K}_{a,c}(t)/\mathcal{K}_{a,c}(r)$ *is decreasing from $(0,1)$ onto $(1/K,1)$,*
(5) $g_2(r)=t'\mathcal{K}_{a,c}(t)^2/(r'\mathcal{K}_{a,c}(r)^2)$ *is increasing from $(0,1)$ onto $(1,\infty)$,*
(6) $g_3(r)=t\mathcal{K}'_{a,c}(t)^2/(r\mathcal{K}'_{a,c}(r)^2)$ *is increasing from $(0,1)$ onto $(0,1)$,*
(7) $g_4(r)=s/r$ *is decreasing from $(0,1)$ onto $(1,\infty)$,*
(8) $g_5(r)=t/r$ *is increasing from $(0,1)$ onto $(0,1)$.*

4.3. **Theorem.** *For $0<a<c\leq 1$, the function $f(x)=\mu_{a,c}(1/\cosh(x))$ is increasing and concave from $(0,\infty)$ onto $(0,\infty)$. In particular,*

$$\mu_{a,c}\left(\frac{rs}{1+r's'}\right)\leq\mu_{a,c}(r)+\mu_{a,c}(s)\leq 2\mu_{a,c}\left(\sqrt{\frac{2rs}{1+rs+r's'}}\right),$$

for all $r,s\in(0,1)$. The second inequality becomes equality if and only if $r=s$.

Proof. Let $r = 1/\cosh(x)$ and (cf. [HLVV])

$$M(r^2) = \left(\frac{2}{B(a,b)}\right)^2 b\,(\,\mathcal{K}_{a,c}(r)\,\mathcal{E}'_{a,c}(r) + \mathcal{K}'_{a,c}(r)\,\mathcal{E}_{a,c}(r) - \mathcal{K}_{a,c}(r)\,\mathcal{K}'_{a,c}(r))\,.$$

We get

$$f^{'}(x) = \frac{B(a,b)}{2}\frac{M(r^2)}{r'^2\,\mathcal{K}(r)^2}\,,$$

which is positive and increasing in r by [HLVV, Lemma 3.4(1), Theorem 3.12(2)], and f is decreasing in x. Hence f is concave. This implies that

$$\frac{1}{2}\left(\mu_{a,c}\left(\frac{1}{\cosh(x)}\right) + \mu_{a,c}\left(\frac{1}{\cosh(y)}\right)\right) \le \mu_{a,c}\left(\frac{1}{\cosh((x+y)/2)}\right)\,,$$

and we get the second inequality by using the formula

$$\left(\cosh\left(\frac{x+y}{2}\right)\right)^2 = \frac{1+rs+r^{'}s^{'}}{2rs}$$

and setting $s = 1/\cosh(y)$. Next, $f^{'}(x)$ is decreasing in x, and $f(0) = 0$. Then $f(x)/x$ is decreasing on $(0,\infty)$ and $f(x+y) \le f(x) + f(y)$ by Lemmas 2.1 and 2.2, respectively. Hence the first inequality follows. $\square$

4.4. **Lemma.** *For $0 < a < c \le 1$, we have*

$$\mu_{a,c}(r) + \mu_{a,c}(s) \le 2\mu_{a,c}(\sqrt{rs})\,,$$

for all $r, s \in (0,1)$, with equality if and only if $r = s$.

Proof. Clearly,

$$(r-s)^2 \ge 0 \Longleftrightarrow 1 + r^2s^2 \ge 1 - (r-s)^2 + r^2s^2$$

$$\Longleftrightarrow (1-rs)^2 \ge 1 - r^2 - s^2 + r^2s^2 \Longleftrightarrow 1 - rs \ge r^{'}s^{'}$$

$$\Longleftrightarrow 2 \ge 1 + rs + r^{'}s^{'} \Longleftrightarrow 1/(rs) \ge (1 + rs + r^{'}s')/(2rs)\,.$$

By using the fact that $\mu_{a,c}$ is decreasing, we get

$$\mu_{a,c}\left(\sqrt{\frac{2rs}{1+rs+r^{'}s^{'}}}\right) \le \mu_{a,c}(\sqrt{r\,s})\,,$$

and the result follows from Theorem 4.3. $\square$

4.5. **Theorem.** *For $K > 1$, $0 < a < c$ and $r, s \in (0,1)$,*

$$\tanh(K\operatorname{artanh} r) < \varphi_K^{a,c}(r).$$

The inequality is reversed if we replace K by $1/K$.

Proof. Let $s = \varphi_K^{a,c}(r)$. Then $s > r$, and by equality $\varphi_K^{a,c}(r) = \mu_{a,c}^{-1}(\mu_{a,c}(r)/K)$ and Lemma 4.1 we get

$$\frac{1}{K}\mu_{a,c}(r)\mathrm{artanh}\, s = \mu_{a,c}(s)\mathrm{artanh}\, s > \mu_{a,c}(r)\mathrm{artanh}\, r,$$

this is equivalent to the required inequality. For the case $1/K$ let $x = \varphi_{1/K}^{a,c}(r)$. Then $x < r$, similarly we get

$$K\mu_{a,c}(r)\mathrm{artanh}\, x = \mu_{a,c}(x)\mathrm{artanh}\, x < \mu_{a,c}(r)\mathrm{artanh}\, r\,,$$

and this is equivalent to $\tanh((\mathrm{artanh}\, r)/K) > \varphi_{1/K}^{a,c}(r)$. $\square$

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Department of Mathematics, University of Turku, FI-20014 Turku, Finland
E-mail address: barbha@utu.fi

Department of Mathematics, University of Turku, FI-20014 Turku, Finland
E-mail address: vuorinen@utu.fi